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APPROXIMATION BY q-PHILLIPS OPERATORS

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Abstract

In this study, we introduce a q-analogue of the Phillips operators and investigate approximation properties. We establish direct and local approximation theorems. We give a weighted approximation theorem. We estimate the rate of convergence of these operators for functions of polynomial growth on the interval $[0, \infty)$.

Keywords: Phillips operators, q-type operators, Rate of convergence, Weighted approximation, q-integral.

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1. Introduction

Phillips firstly introduced the q-analogue of Bernstein polynomials based on q-integer and q-binomial coefficients in [12]. Gupta and Finta obtain some direct results on certain q-Durrmeyer type operators in [6]. Recently, Aral and Gupta introduced Durrmeyer type modification of the q-Baskakov type operators in [1]. We aim to introduce a q-analogue of Phillips operators and to study approximation properties. Before this, we mention the following notations and formulas, which can be founded in [2, 8, 9] and [10]: For $n \in \mathbb{N}$, $0 < q < 1$ and $a, b \in \mathbb{R}$,

(1.1) $$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},\ n \in \mathbb{N}\backslash\{0\};\ [0]_q = 0,$$

(1.2) $$[n]_q! = [1]_q[2]_q \cdots [n]_q,\ n \in \mathbb{N}\backslash\{0\};\ [0]_q! = 1,$$

(1.3) $$(a+b)_q^n = \prod_{j=1}^{n} (a + q^{j-1}b),$$

and

(1.4) $$(1+a)_q^\infty = \prod_{j=1}^{\infty} (1 + q^{j-1}a).$$

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The q-binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \; 0 \le k \le n. \tag{1.5}$$

The q- derivative $D_q f$ of the function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \text{ for } x \neq 0 \tag{1.6}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The two q-analogues of the exponential function are defined by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}} \tag{1.7}$$

and

$$E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^{\infty}. \tag{1.8}$$

The q-Jackson integrals and the q-improper integrals are defined as

$$\int_0^a f(x)\, d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \; a > 0 \tag{1.9}$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n \in \mathbb{Z}} f(\frac{q^n}{A}) \frac{q^n}{A}, \; A > 0, \tag{1.10}$$

respectively. The q-Gamma function is given by

$$\Gamma_q(s) = K(A,s) \int_0^{\infty/A(1-q)} t^{s-1} e_q^{-t}\, d_q t, \tag{1.11}$$

where

$$K(A,s) = \frac{A^s}{1+A} \left(1 + \frac{1}{A}\right)_q^s (1+A)_q^{1-s}. \tag{1.12}$$

In particular, for $s \in \mathbb{Z}$, $K(A,s) = q^{s(s-1)/2}$ and $K(A,0) = 1$.

2. Construction of the operators

Let f be a real valued continuous function on the interval $[0, \infty)$. Using the formulas and notations in (1.1)–(1.12), we now define the q-Phillips operators as

$$\mathcal{P}_n^q(f;x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}(x;q) \int_0^{\infty/A(1-q)} q^{k(k-1)} f(t) p_{n,k-1}(t;q)\, d_q t + e_q^{-[n]_q x} f(0), \tag{2.1}$$

where

$$p_{n,k}(x;q) := \frac{([n]_q x)^k}{[k]_q!} e_q^{-[n]_q x}. \tag{2.2}$$

In the case $q = 1$, these operators are reduced to the Phillips operators studied in [11] and [13].

Now we give an auxiliary lemma for the Korovkin monomial functions.

2.1. Lemma. *Let* $e_m(t) = t^m$, $m = 0, 1, 2, 3, 4$. *We have*

(i) $\mathcal{P}_n^q(e_0; x) = 1$,

(ii) $\mathcal{P}_n^q(e_1; x) = \dfrac{x}{q}$,

(iii) $\mathcal{P}_n^q(e_2; x) = \dfrac{x^2}{q^4} + \dfrac{[2]_q}{q^3[n]_q}x$,

(iv) $\mathcal{P}_n^q(e_3; x) = \dfrac{x^3}{q^9} + \dfrac{([2]_q q + [4]_q)}{q^8[n]_q}x^2 + \dfrac{[2]_q[3]_q}{q^6[n]_q^2}x$,

(v) $\mathcal{P}_n^q(e_4; x) = \dfrac{x^4}{q^{16}} + \dfrac{\left([2]_q q^2 + [4]_q q + [6]_q\right)}{q^{15}[n]_q}x^3 + \dfrac{([2]_q[3]_q q^2 + [2]_q[5]_q q + [4]_q[5]_q}{q^{13}[n]_q^2}x^2 + \dfrac{[2]_q[3]_q[4]_q}{q^{10}[n]_q^3}x.$

Proof. (i) Using the formulas (1.10), (1.11), (1.12) and (2.2), we can calculate the following integral:

$$\begin{aligned}
\int_0^{\infty/A(1-q)} t^m p_{n,k}(t;q)d_qt &= \int_0^{\infty/A(1-q)} t^m \frac{([n]_qt)^k}{[k]_q!}e_q^{-[n]_qt}\,d_qt \\
&= \frac{1}{[n]_q^{m+1}[k]_q!}\int_0^{\infty/(A/[n]_q)(1-q)} u^{k+m}e_q^{-u}d_qu \\
&= \frac{\Gamma_q(k+m+1)}{[n]_q^{m+1}[k]_q!K(A/[n]_q,k+m+1)} \\
&= \frac{[k+m]_q!}{[n]_q^{m+1}[k]_q!q^{(k+m+1)(k+m)/2}}. \qquad (2.3)
\end{aligned}$$

Using (1.7), (1.8) and (2.3) we obtain

$$\begin{aligned}
\mathcal{P}_n^q(e_0; x) &= \sum_{k=1}^{\infty} q^{k(k-1)/2}p_{n,k}(x;q) + e_q^{-[n]_qx} \\
&= \left(\sum_{k=1}^{\infty} q^{k(k-1)/2}p_{n,k}(x;q) + 1\right)e_q^{-[n]_qx} \\
&= E_q^{[n]_qx}e_q^{-[n]_qx} \\
&= 1,
\end{aligned}$$

which completes the proof of (i).

(ii) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_1; x) = \sum_{k=1}^{\infty} \frac{[k]_q}{[n]_q}q^{(k^2-3k)/2}p_{n,k}(x;q).$$

Thus, we obtain

$$\begin{aligned}
\mathcal{P}_n^q(e_1; x) &= \frac{1}{[n]_q}\sum_{k=1}^{\infty} q^{(k^2-3k)/2}\frac{([n]_qx)^k}{[k-1]_q!}e_q^{-[n]_qx} \\
&= \frac{x}{q}\mathcal{P}_n^q(e_0; x),
\end{aligned}$$

as required.

(iii) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_2;x)=\sum_{k=1}^{\infty}\frac{[k]_q[k+1]_q}{[n]_q^2}q^{(k^2-5k-2)/2}p_{n,k}(x;q).$$

Using the equality $[k]_q[k+1]_q=[k]_q[k-1]_q+[2]_q q^{k-1}[k]_q$, we obtain

$$\begin{aligned}\mathcal{P}_n^q(e_2;x)&=\frac{1}{[n]_q^2}\sum_{k=2}^{\infty}q^{(k^2-5k-2)/2}\frac{([n]_qx)^k}{[k-2]_q!}e_q^{-[n]_qx}\\&\qquad+\frac{[2]_q}{[n]_q^2}\sum_{k=1}^{\infty}q^{(k^2-3k-4)/2}\frac{([n]_qx)^k}{[k-1]_q!}e_q^{-[n]_qx}\\&=\frac{x^2}{q^4}\mathcal{P}_n^q(e_0;x)+\frac{[2]_qx}{q^3[n]_q}\mathcal{P}_n^q(e_0;x),\end{aligned}$$

which is the required result.

(iv) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_3;x)=\sum_{k=1}^{\infty}\frac{[k]_q[k+1]_q[k+2]_q}{[n]_q^3}q^{(k^2-7k-6)/2}p_{n,k}(x;q).$$

Using the equality

$$\begin{aligned}[k]_q[k+1]_q[k+2]_q=[k]_q[k-1]_q[k-2]_q+([2]_qq+[4]_q)q^{k-2}[k]_q[k-1]_q\\+[2]_q[3]_qq^{2k-2}[k]_q,\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{P}_n^q(e_3;x)&=\frac{1}{[n]_q^3}\sum_{k=3}^{\infty}q^{(k^2-7k-6)/2}\frac{([n]_qx)^k}{[k-3]_q!}e_q^{-[n]_qx}\\&\qquad+\frac{([2]_qq+[4]_q)}{[n]_q^3}\sum_{k=2}^{\infty}q^{(k^2-5k-10)/2}\frac{([n]_qx)^k}{[k-2]_q!}e_q^{-[n]_qx}\\&\qquad+\frac{[2]_q[3]_q}{[n]_q^3}\sum_{k=1}^{\infty}q^{(k^2-3k-10)/2}\frac{([n]_qx)^k}{[k-1]_q!}e_q^{-[n]_qx}\\&=\frac{x^3}{q^9}\mathcal{P}_n^q(e_0;x)+\frac{([2]_qq+[4]_q)x^2}{q^8[n]_q}\mathcal{P}_n^q(e_0;x)+\frac{[2]_q[3]_qx}{q^6[n]_q^2}\mathcal{P}_n^q(e_0;x),\end{aligned}$$

which is the required result.

(v) From (2.3), we have the equality

$$\mathcal{P}_n^q(e_4;x)=\sum_{k=1}^{\infty}\frac{[k]_q[k+1]_q[k+2]_q[k+3]_q}{[n]_q^4}q^{(k^2-9k-12)/2}p_{n,k}(x;q).$$

Using the equality

$$\begin{aligned}[k]_q[k+1]_q[k+2]_q[k+3]_q&=[k]_q[k-1]_q[k-2]_q[k-3]_q\\&\quad+([2]_qq^2+[4]_qq+[6]_q)q^{k-3}[k]_q[k-1]_q[k-2]_q\\&\quad+([2]_q[3]_qq^2+[2]_q[5]_qq+[4]_q[5]_q)q^{2k-4}[k]_q[k-1]_q\\&\quad+[2]_q[3]_q[4]_qq^{3k-3}[k]_q,\end{aligned}$$

so we can write,

$$\begin{aligned}\mathcal{P}_n^q(e_4;x) &= \frac{1}{[n]_q^4}\sum_{k=4}^{\infty} q^{(k^2-9k-12)/2}\frac{([n]_q x)^k}{[k-4]_q!}e_q^{-[n]_q x}\\ &+ \frac{[2]_q q^2+[4]_q q+[6]_q}{[n]_q^4}\sum_{k=3}^{\infty} q^{(k^2-7k-18)/2}\frac{([n]_q x)^k}{[k-3]_q!}e_q^{-[n]_q x}\\ &+ \frac{[2]_q[3]_q q^2+[2]_q[5]_q q+[4]_q[5]_q}{[n]_q^4}\sum_{k=2}^{\infty} q^{(k^2-5k-20)/2}\frac{([n]_q x)^k}{[k-2]_q!}e_q^{-[n]_q x}\\ &+ \frac{[2]_q[3]_q[4]_q}{[n]_q^4}\sum_{k=1}^{\infty} q^{(k^2-3k-18)/2}\frac{([n]_q x)^k}{[k-1]_q!}e_q^{-[n]_q x}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{P}_n^q(e_4;x) &= \frac{x^4}{q^{16}}\mathcal{P}_n^q(e_0;x) + \frac{\left([2]_q q^2+[4]_q q+[6]_q\right)x^3}{q^{15}[n]_q}\mathcal{P}_n^q(e_0;x)\\ &+ \frac{([2]_q[3]_q q^2+[2]_q[5]_q q+[4]_q[5]_q)x^2}{q^{13}[n]_q^2}\mathcal{P}_n^q(e_0;x)\\ &+ \frac{([2]_q[3]_q[4]_q)x}{q^{10}[n]_q^3}\mathcal{P}_n^q(e_0;x),\end{aligned}$$

as required. □

2.2. Remark. Take a fixed number $q \in (0,1)$. Since

$$\lim_{n\to\infty}[n]_q = \frac{1}{1-q},$$

in Lemma 2.1, $\mathcal{P}_n^q(t^m,x)$, $m \in \mathbb{N}$, does not tend to x^m as $n \to \infty$. From this result, we have to consider the condition $q := (q_n)$ as a sequence with $\lim_{n\to\infty} q_n = 1$ for approximation properties of the operators $\mathcal{P}_n^q(f,x)$ defined by (2.1)

For shortness, q denotes the n^{th} term of the sequence $(q_n) \subset (0,1)$ with $\lim_{n\to\infty} q_n = 1$ after this section.

2.3. Lemma. *For the operators $\mathcal{P}_n^q(f,x)$ defined by* (2.1)*, we have the inequality*

$$\mathcal{P}_n^q((t-x)^2;x) \le \frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x).$$

Proof. From the linearity of the $\mathcal{P}_n^q$ operators, and Lemma 2.1, we have the second moment

$$\begin{aligned}\mathcal{P}_n^q((t-x)^2;x) &= \frac{x^2}{q^4}+\frac{[2]_q}{q^3[n]_q}x-2x\frac{x}{q}+x^2\\ &= \left(\frac{1}{q^4}-\frac{2}{q}+1\right)x^2+\frac{[2]_q}{q^3[n]_q}x\\ &\le \frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x).\end{aligned}$$

Therefore, The proof is completed. □

2.4. Lemma. *If we make a slight modification to the operators $\mathcal{P}_n^q(f;x)$ defined in* (2.1) *as follows:*

$$\overline{\mathcal{P}}_n^q(f;x) = \mathcal{P}_n^q(f;x) - f\left(\frac{x}{q}\right) + f(x), \tag{2.4}$$

then we have

$$\overline{\mathcal{P}}_n^q(t-x;x)=0.$$

Proof. From Lemma 2.1,

$$\overline{\mathcal{P}}_n^q(1;x)=\mathcal{P}_n^q(1;x)$$

and

$$\overline{\mathcal{P}}_n^q(t;x)=\mathcal{P}_n^q(t;x)-\frac{x}{q}+x=x.$$

Therefore, we obtain the result stated in the Lemma. □

3. Local approximation

In this section, let $C_B[0,\infty)$ be the space of all real valued continuous bounded functions on $[0,\infty)$ and let $f\in C_B[0,\infty)$ be equipped with the norm $\|f\|=\sup\limits_{x\in[0,\infty)}|f(t)|$. We denote the first modulus of continuity on the finite interval $[0,a]$, $a>0$, by

$$\omega_{[0,a]}(f;\delta)=\sup_{0<h\leq\delta,x\in[0,a]}|f(x+h)-f(x)|\,. \tag{3.1}$$

Peetre's K-functional is defined by

$$K_2(f;\delta)=\inf\left\{\|f-g\|+\delta\left\|g''\right\|:g\in W_\infty^2\right\},\ \delta>0$$

where $W_\infty^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [3, Theorem 2.4, p. 177] there exists a positive constant M such that

$$K_2(f;\delta)\leq M\omega_2(f;\sqrt{\delta}), \tag{3.2}$$

where

$$\omega_2(f;\sqrt{\delta})=\sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)-f(x)|\,.$$

3.1. Theorem. *For every $x\in[0,\infty)$ and $f\in C_B[0,\infty)$ we have the inequality*

$$|\mathcal{P}_n^q(f;x)-f(x)|\leq M\omega_2\left(f;\sqrt{\delta_{n,q}(x)}\right)+\omega_{[0,a]}\left(f;\frac{1-q}{q}x\right),$$

where

$$\delta_{n,q}(x):=\frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x).$$

Proof. Let $g\in W_\infty^2$ and $x\in[0,\infty)$. Using Taylor's expansion

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)\,du,$$

and from Lemma 2.4, we have

$$\overline{\mathcal{P}}_n^q(g;x)=g(x)+\overline{\mathcal{P}}_n^q\left(\int_x^t(t-u)g''(u)\,du;x\right).$$

Then, we get

$$\left|\overline{\mathcal{P}}_n^q(g;x)-g(x)\right| = \left|\mathcal{P}_n^q\left(\int_x^t (t-u)g''(u)\,du;x\right) - \int_x^{x/q}\left(\frac{x}{q}-u\right)g''(u)\,du\right|$$
$$\leq \mathcal{P}_n^q\left(\left|\int_x^t (t-u)g''(u)\,du\right|;x\right) + \int_x^{x/q}\left|\frac{x}{q}-u\right|\left|g''(u)\right|\,du.$$

Using the inequality

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \|g''\|\,\frac{(t-x)^2}{2},$$

and from Lemma 2.3, we write

$$\left|\overline{\mathcal{P}}_n^q(g;x)-g(x)\right| \leq \|g''\|\mathcal{P}_n^q\left(\frac{(t-x)^2}{2};x\right) + \|g''\|\frac{\left(\frac{x}{q}-x\right)^2}{2}$$
$$\leq \frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x)\|g''\|. \tag{3.3}$$

The operators $\overline{\mathcal{P}}_n^q(f,x)$ are bounded, that is

$$\left|\overline{\mathcal{P}}_n^q(f;x)\right| = \left|\mathcal{P}_n^q(f;x) - f\left(\frac{x}{q}\right)+f(x)\right| \leq \|f\|\mathcal{P}_n^q(1;x)+2\|f\| \leq 3\|f\|. \tag{3.4}$$

From (2.4), (3.3) and (3.4), we get

$$|\mathcal{P}_n^q(f;x)-f(x)| = |\mathcal{P}_n^q(f-g;x)-(f-g)(x)+\mathcal{P}_n^q(g;x)-g(x)|$$
$$\leq \left|\overline{\mathcal{P}}_n^q(f-g;x)-(f-g)(x)\right| + \left|\overline{\mathcal{P}}_n^q(g;x)-g(x)\right|$$
$$+\left|f\left(\frac{x}{q}\right)-f(x)\right|$$
$$\leq 4\,||f-g|| + \frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x)\|g''\|$$
$$+\left|f\left(x+\frac{1-q}{q}x\right)-f(x)\right|.$$

Now taking the infimum over $g \in W_\infty^2$ on the right hand side of the above inequality, and using the inequalities (3.1) and (3.2), we get the desired result. □

4. Weighted approximation

Weighted Korovkin-type theorems were proved by Gadzhiev [4] and [5]. Now, we give Gadzhiev's results in weighted spaces. Let $\rho(x)=1+\varphi^2(x)$, where $\varphi(x)$ is a monotone increasing continuous function on the real axis and B_ρ is the set of all functions f defined on the real axis satisfying the growth condition $|f(x)| \leq M_f\rho(x)$, where M_f is a constant depending only on f. Then B_ρ is a normed space with norm

$$\|f\|_\rho = \sup\{|f(x)|/\rho(x) : x\in\mathbb{R}\}$$

for any $f\in B_\rho$. Let C_ρ denote the subspace of all continuous functions in B_ρ, and C_ρ^* the subspace of all functions $f\in C_\rho$ for which $\lim_{|x|\to\infty}(f(x)/\rho(x))$ exists finitely.

4.1. Theorem. (See [4] and [5])

(a) *There exists a sequence of linear positive operators* $A_n(C_\rho \to B_\rho)$ *such that*

$$\lim_{n\to\infty} \|A_n(\varphi^\upsilon) - \varphi^\nu\|_\rho = 0,\ \ \nu = 0,1,2, \tag{4.1}$$

and a function $f^* \in C_\rho \backslash C_\rho^*$ *with* $\lim_{n\to\infty} \|A_n(f^*) - f^*\|_\rho \geq 1$.

(b) *If a sequence of linear positive operators* $A_n(C_\rho \to B_\rho)$ *satisfies conditions (4.1), then*

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0,$$

for every $f \in C_\rho^*$. □

Throughout this paper we take the growth condition as $\rho(x) = 1 + x^2$.

4.2. Theorem. *For every* $f \in C_B[0,\infty)$ *we have the following limit*

$$\lim_{n\to\infty} \|\mathcal{P}_n^q(f) - f\|_\rho = 0.$$

Proof. Since $\mathcal{P}_n^q(e_0;x) = 1$, it is obvious that

$$\|\mathcal{P}_n^q(e_0) - e_0\|_\rho = 0.$$

Considering Lemma 2.1 (ii), we get

$$\begin{aligned}
\|\mathcal{P}_n^q(e_1) - e_1\|_\rho &= \sup_{x\in[0,\infty)} \frac{|\mathcal{P}_n^q(e_1;x) - x|}{1+x^2} \\
&\leq \sup_{x\in[0,\infty)} \frac{\left|\dfrac{x}{q} - x\right|}{1+x^2} \\
&\leq \left(\frac{1}{q} - 1\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
&= o(1).
\end{aligned}$$

Similarly, from Lemma 2.1 (iii) we get

$$\begin{aligned}
\|\mathcal{P}_n^q(e_2) - e_2\|_\rho &= \sup_{x\in[0,\infty)} \frac{\left|\mathcal{P}_n^q(e_2;x) - x^2\right|}{1+x^2} \\
&\leq \sup_{x\in[0,\infty)} \frac{\left|\dfrac{x^2}{q^4} + \dfrac{[2]_q}{q^3[n]_q}x - x^2\right|}{1+x^2} \\
&= \left(\frac{1-q^4}{q^4} + \frac{[2]_q}{q^3[n]_q}\right) \sup_{x\in[0,\infty)} \frac{x+x^2}{1+x^2} \\
&= o(1).
\end{aligned}$$

Thus, from Theorem 4.1, we obtain the desired result. □

5. Rate of convergence

In this section, we want to estimate the rate of convergence for the sequence of the $\mathcal{P}_n^q$ operators. As is known, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f;\delta)$ does not tend to zero, as $\delta \to 0$. For every $f \in C_\rho^*[0,\infty)$, we would like to take a weighted modulus of continuity $\Omega(f;\delta)$ which tends to zero as $\delta \to 0$.

Let

$$\Omega(f;\delta) = \sup_{0<h\leq\delta, x\geq 0} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}, \text{ for each every } f \in C^*_\rho[0,\infty). \tag{5.1}$$

The weighed modulus of continuity $\Omega(f;\delta)$ was defined by Ispir in [7]. It is known that $\Omega(f;\delta)$ has the following properties.

5.1. Lemma. [7] *Let $f \in C^*_\rho[0,\infty)$. Then:*

(i) *$\Omega(f;\delta)$ is a monotone increasing function of δ,*
(ii) *For each $f \in C^*_\rho[0,\infty)$, $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$,*
(iii) *For each $m \in \mathbb{N}\setminus\{0\}$, $\Omega(f;m\delta) \leq m\Omega(f;\delta)$,*
(iv) *For each $\lambda \in \mathbb{R}^+$, $\Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.* □

Now we obtain a rate of convergence for the operators $\mathcal{P}^q_n$.

5.2. Theorem. *Let $f \in C^*_\rho[0,\infty)$. Then we have the inequality*

$$\|\mathcal{P}^q_n(f) - f\|_{\overline{\rho}} \leq M(q)\Omega\left(f; \sqrt{1-q^3+\frac{1}{[n]_q}}\right),$$

where $\overline{\rho}(x) = 1+x^5$ and $M(q)$ is a positive real number dependent on q.

Proof. From the definition of $\Omega(f;\delta)$, and Lemma 5.1 (iv), we can write

$$|f(t)-f(x)| \leq (1+(t-x)^2)(1+x^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta).$$

Then, we have the inequality

$$\begin{aligned} |\mathcal{P}^q_n(f;x) - f(x)| &\leq (1+x^2)\Omega(f;\delta)\mathcal{P}^q_n\left((1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right);x\right) \\ &\leq (1+x^2)\Omega(f;\delta)\left\{\mathcal{P}^q_n\left((1+(t-x)^2);x\right)\right. \\ &\qquad \left. +\mathcal{P}^q_n\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right\}. \end{aligned} \tag{5.2}$$

Applying the Cauchy-Schwarz inequality to the second term, we get

$$\begin{aligned} &\mathcal{P}^q_n\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) \\ &\qquad \leq \left\{\mathcal{P}^q_n\left((1+(t-x)^2)^2;x\right)\right\}^{1/2}\left\{\mathcal{P}^q_n\left(\frac{|t-x|^2}{\delta^2};x\right)\right\}^{1/2}. \end{aligned} \tag{5.3}$$

From Lemma 2.1 and Lemma 2.3, we get the following estimates

$$\begin{aligned} \left(\mathcal{P}^q_n\left(1+(t-x)^2;x\right)\right) &\leq 1+\frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x) \\ &\leq \frac{2}{q^4}\left(2-q^3+\frac{1}{[n]_q}\right)(1+x)^2 \\ &\leq M_1(q)(1+x)^2, \end{aligned} \tag{5.4}$$

$$
\begin{aligned}
\mathcal{P}_n^q &\left((1+(t-x)^2)^2;x\right) \\
&= 1+2\mathcal{P}_n^q((t-x)^2;x)+\mathcal{P}_n^q((t^4;x)-4x\mathcal{P}_n^q((t^3;x) \\
&\qquad +6x^2\mathcal{P}_n^q(t^2;x)-4x^3\mathcal{P}_n^q(t;x)+x^4\mathcal{P}_n^q(1;x) \\
&= x^4\left(\frac{1}{q^{16}}-\frac{4}{q^9}+\frac{6}{q^4}-\frac{4}{q}+1\right) \\
&\quad +x^3\left(\frac{([2]_q q^2+[4]_q q+[6]_q)}{q^{15}[n]_q}-4\frac{([2]_q q+[4]_q)}{q^8[n]_q}+6\frac{[2]_q}{q^3[n]_q}\right) \\
&\quad +x^2\left(\frac{([2]_q[3]_q q^2+[2]_q[5]_q q+[4]_q[5]_q}{q^{13}[n]_q^2}-4\frac{[2]_q[3]_q}{q^6[n]_q^2}+\frac{2}{q^4}-\frac{4}{q}+2\right) \\
&\quad +x\left(\frac{[2]_q[3]_q[4]_q}{q^{10}[n]_q^3}+2\frac{[2]_q}{q^3[n]_q}\right)+1 \\
&\le 8\left(\frac{1-q^{15}}{q^{16}}\right)x^4+\frac{48}{q^{15}[n]_q}x^3+\frac{68}{q^{13}[n]_q^2}x^2+\frac{28}{q^{10}[n]_q^3}x+1 \\
&\le M_2(q)(1+x^2)^2
\end{aligned}
\tag{5.5, 5.6}
$$

and

$$
\begin{aligned}
\left\{\mathcal{P}_n^q\left(\frac{|t-x|^2}{\delta^2};x\right)\right\}^{1/2} &\le \frac{1}{\delta}\sqrt{\frac{2}{q^4}\left(1-q^3+\frac{1}{[n]_q}\right)x(1+x)} \\
&\le \frac{M_3(q)}{\delta}\sqrt{1-q^3+\frac{1}{[n]_q}}(1+x).
\end{aligned}
\tag{5.7}
$$

Choosing $M(q)=(M_1(q)+\sqrt{M_2(q)}M_3(q))M_4$, where $M_4=\sup_{x\ge 0}(1+x^2)^2(1+x)/(1+x^5)$ and $\delta=\sqrt{1-q^3+\dfrac{1}{[n]_q}}$, and combining the estimates between (5.2) and (5.7), we end up with

$$
|\mathcal{P}_n^q(f,x)-f(x)|\le(1+x^5)M(q)\Omega\left(f;\sqrt{1-q^3+\frac{1}{q^3[n]_q}}\right),
$$

as required. □

5.3. Remark. The weighted approximation theorem, Theorem 4.2, is obtained for the norm $\|\cdot\|_\rho$, where $\rho(x)=1+x^2$. In Theorem 5.2, we estimated the rate of convergence for the operators $\mathcal{P}_n^q$ for the norm $\|\cdot\|_{\overline{\rho}}$, where $\overline{\rho}(x)=1+x^5$. It is an open problem to obtain the rate of convergence for the operators $\mathcal{P}_n^q$ in the norm $\|\cdot\|_\rho$, where $\rho(x)=1+x^2$, without adding an extra condition to the function $f\in C_\rho^*$.

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