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SOME NEW HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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Abstract

In this paper, we establish some new Hermite-Hadamard type inequalities for m-convex and (α, m)-convex functions of 2-variables on the co-ordinates.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers, and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [5]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a)+f(b)}{2}.$$

In [8], the notion of m-convexity was introduced by G.Toader as the following:

1.1. Definition. The function $f : [0, b] \to R$, $b > 0$ is said to be *m-convex*, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is *m-concave* if $-f$ is m-convex.

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Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \leq 0$. Obviously, if we choose $m = 1$, Definition 1.1 recaptures the concept of standard convex functions on $[0, b]$.

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequalities for m-convex functions.

1.2. Theorem. *Let $f : [0, \infty) \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.1}$$ □

Some generalizations of this result can be found in [2, 3].

1.3. Theorem. *Let $f : [0, \infty) \to \mathbb{R}$ be an m-convex differentiable function with $m \in (0, 1]$. Then for all $0 \leq a < b$ the following inequality holds:*

$$\begin{aligned} \frac{f(mb)}{m} - \frac{b-a}{2}f'(mb) &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \frac{(b-ma)f(b) - (a-mb)f(a)}{2(b-a)}. \end{aligned} \tag{1.2}$$ □

Also, in [5], Dragomir and Pearce proved the following Hadamard type inequality for m-convex functions.

1.4. Theorem. *Let $f : [0, \infty) \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$ and $0 \leq a < b$. If $f \in L_1[a, b]$, then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2}\,dx. \tag{1.3}$$ □

In [7], the definition of (α, m)-convexity was introduced by V. G. Miheşan as the following:

1.5. Definition. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$. If we take $(\alpha, m) = (1, m)$, it can be easily seen that (α, m)-convexity reduces to m-convexity, and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the usual concept of convexity defined on $[0, b]$, $b > 0$.

In [9], E. Set, M. Sardari, M. E. Ozdemir and J. Rooin proved the following Hadamard type inequalities for (α, m)-convex functions.

1.6. Theorem. *Let $f : [0, \infty) \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)}{2^\alpha}\,dx. \tag{1.4}$$ □

1.7. Theorem. *Let $f : [0, \infty) \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a) + m\alpha f\left(\frac{b}{m}\right)}{\alpha+1}, \frac{f(b) + m\alpha f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \tag{1.5}$$ □

1.8. Theorem. *Let $f : [0,\infty) \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0,1]^2$. If $0 \le a < b < \infty$ and $f \in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[\frac{f(a)+f(b)+m\alpha f\left(\frac{a}{m}\right)+m\alpha f\left(\frac{b}{m}\right)}{\alpha+1}\right]. \tag{1.6}$$
□

Let us now consider a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$, with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ is said to be *convex on* Δ if the following inequality:

$$f(tx+(1-t)z, ty+(1-t)w) \le tf(x,y)+(1-t)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$. A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$ (see [5, p. 317]).

Also, in [4], Dragomir proved the following similar inequalities of Hadamard's type for a co-ordinated convex mapping on a rectangle in the plane $\mathbb{R}^2$.

1.9. Theorem. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy\\
&\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&\qquad\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\end{aligned} \tag{1.7}$$

The above inequalities are sharp. □

For co-ordinated s-convex functions, another version of this result can be found in [1].

The main purpose of this paper is to establish new Hadamard-type inequalities for functions of 2-variables which are m-convex or (α,m)-convex on the co-ordinates.

2. Inequalities for co-ordinated m-convex functions

Firstly, we can define co-ordinated m-convex functions as follows:

2.1. Definition. Consider the bidimensional interval $\Delta := [0,b]\times[0,d]$ in $[0,\infty)^2$. The mapping $f:\Delta\to\mathbb{R}$ is *m-convex on* Δ if

$$f(tx+(1-t)z, ty+m(1-t)w) \le tf(x,y)+m(1-t)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ with $t\in[0,1]$, $b,d>0$, and for some fixed $m\in[0,1]$.

A function $f:\Delta\to\mathbb{R}$ which is m-convex on Δ is called *co-ordinated m-convex on* Δ if the partial mappings

$$f_y:[0,b]\to\mathbb{R},\ \ f_y(u)=f(u,y)$$

and

$$f_x : [0,d] \to \mathbb{R},\ f_x(v) = f(x,v)$$

are m-convex for all $y \in [0,d]$ and $x \in [0,b]$ with $b, d > 0$, and for some fixed $m \in [0,1]$.

We also need the following Lemma for our main results.

2.2. Lemma. *Every m-convex mapping $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is m-convex on the co-ordinates, where $\Delta = [0,b] \times [0,d]$ and $m \in [0,1]$.*

Proof. Suppose that $f : \Delta = [0,b] \times [0,d] \to \mathbb{R}$ is m-convex on Δ. Consider the function

$$f_x : [0,d] \to \mathbb{R},\ f_x(v) = f(x,v),\ (x \in [0,b]).$$

Then for $t, m \in [0,1]$ and $v_1, v_2 \in [0,d]$, we have

$$\begin{aligned} f_x(tv_1 + m(1-t)v_2) &= f(x, tv_1 + m(1-t)v_2) \\ &= f(tx + (1-t)x, tv_1 + m(1-t)v_2) \\ &\le tf(x,v_1) + m(1-t)f(x,v_2) \\ &= tf_x(v_1) + m(1-t)f_x(v_2). \end{aligned}$$

Therefore, $f_x(v) = f(x,v)$ is m-convex on $[0,d]$. The fact that $f_y : [0,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ is also m-convex on $[0,b]$ for all $y \in [0,d]$ goes likewise, and we shall omit the details. □

2.3. Theorem. *Suppose that $f : \Delta = [0,b] \times [0,d] \to \mathbb{R}$ is an m-convex function on the co-ordinates on Δ. If $0 \le a < b < \infty$ and $0 \le c < d < \infty$ with $m \in (0,1]$, then one has the inequality:*

$$\begin{aligned} &\frac{1}{(d-c)(b-a)} \int_c^d \int_a^b f(x,y)\, dx\, dy \\ &\qquad \le \frac{1}{4(b-a)} \min\{v_1, v_2\} + \frac{1}{4(d-c)} \min\{v_3, v_4\}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} v_1 &= \int_a^b f(x,c)\, dx + m \int_a^b f\left(x, \frac{d}{m}\right) dx \\ v_2 &= \int_a^b f(x,d)\, dx + m \int_a^b f\left(x, \frac{c}{m}\right) dx \\ v_3 &= \int_c^d f(a,y)\, dy + m \int_c^d f\left(\frac{b}{m}, y\right) dy \\ v_4 &= \int_c^d f(b,y)\, dy + m \int_c^d f\left(\frac{a}{m}, y\right) dy. \end{aligned}$$

Proof. Since $f : \Delta \to \mathbb{R}$ is co-ordinated m-convex on Δ it follows that the mapping $g_x : [0,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is m-convex on $[0,d]$ for all $x \in [0,b]$. Then by the inequality (1.1) one has:

$$\frac{1}{d-c} \int_c^d g_x(y)\, dy \le \min\left\{ \frac{g_x(c) + mg_x\left(\frac{d}{m}\right)}{2},\ \frac{g_x(d) + mg_x\left(\frac{c}{m}\right)}{2} \right\},$$

or

$$\frac{1}{d-c} \int_c^d f(x,y)\, dy \le \min\left\{ \frac{f(x,c) + mf\left(x, \frac{d}{m}\right)}{2},\ \frac{f(x,d) + mf\left(x, \frac{c}{m}\right)}{2} \right\},$$

where $0 \le c < d < \infty$ and $m \in (0,1]$.

Dividing both sides by $(b-a)$ and integrating this inequality over $[a,b]$ with respect to x, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\quad\leq \min\left\{\frac{1}{2(b-a)}\int_a^b f(x,c)\,dx + \frac{m}{2(b-a)}\int_a^b f\left(x,\frac{d}{m}\right)dx,\right. \\
&\qquad\qquad \left.\frac{1}{2(b-a)}\int_a^b f(x,d)\,dx + \frac{m}{2(b-a)}\int_a^b f\left(x,\frac{c}{m}\right)dx\right\} \\
&\quad= \frac{1}{2(b-a)}\min\left\{\int_a^b f(x,c)\,dx + m\int_a^b f\left(x,\frac{d}{m}\right)dx,\right. \\
&\qquad\qquad \left.\int_a^b f(x,d)\,dx + m\int_a^b f\left(x,\frac{c}{m}\right)dx\right\}
\end{aligned} \tag{2.2}$$

where $0 \leq a < b < \infty$.

By a similar argument applied to the mapping $g_y : [0,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$ with $0 \leq a < b < \infty$, we get

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dx\,dy \\
&\quad\leq \frac{1}{2(d-c)}\min\left\{\int_c^d f(a,y)\,dy + m\int_c^d f\left(\frac{b}{m},y\right)dy,\right. \\
&\qquad\qquad \left.\int_c^d f(b,y)\,dy + m\int_c^d f\left(\frac{a}{m},y\right)dy\right\}.
\end{aligned} \tag{2.3}$$

Summing the inequalities (2.2) and (2.3), we get the inequality (2.1). □

2.4. Corollary. *With the above assumptions, and provided that the partial mappings*

$$f_y : [0,b] \to \mathbb{R},\ \ f_y(u) = f(u,y)$$

and

$$f_x : [0,d] \to \mathbb{R},\ \ f_x(v) = f(x,v)$$

are differentiable on $(0,b)$ *and* $(0,d)$*, respectively, we have the inequalities*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\quad\leq \frac{1}{4(b-a)}\min\left\{(b-ma)\left[f(b,c)+mf\left(b,\frac{d}{m}\right)\right]\right. \\
&\qquad\qquad -(a-mb)\left[f(a,c)+mf\left(a,\frac{d}{m}\right)\right], \\
&\qquad\quad (b-ma)\left[f(b,d)+mf\left(b,\frac{c}{m}\right)\right] \\
&\qquad\qquad \left.-(a-mb)\left[f(a,d)+mf\left(a,\frac{c}{m}\right)\right]\right\},
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dx\,dy \\
&\quad\le \frac{1}{4(d-c)}\min\Bigg\{(d-mc)\left[f(a,d)+mf\left(\frac{b}{m},d\right)\right] \\
&\qquad\qquad -(c-md)\left[f(a,c)+mf\left(\frac{b}{m},c\right)\right], \\
&\qquad (d-mc)\left[f(b,d)+mf\left(\frac{a}{m},d\right)\right] \\
&\qquad\qquad -(c-md)\left[f(b,c)+mf\left(\frac{a}{m},c\right)\right]\Bigg\}.
\end{aligned} \tag{2.5}$$

Proof. Since the partial mappings

$$f_x:[0,d]\to\mathbb{R},\ \ f_x(v)=f(x,v)$$

are differentiable on $[0,d]$, by the inequality (1.2) we have

$$\begin{aligned}
\frac{1}{(b-a)}\int_a^b f(x,c)\,dx &\le \frac{(b-ma)f(b,c)-(a-mb)f(a,c)}{2(b-a)}, \\
\frac{1}{(b-a)}\int_a^b f\left(x,\frac{d}{m}\right)dx &\le \frac{(b-ma)f(b,\frac{d}{m})-(a-mb)f(a,\frac{d}{m})}{2(b-a)}, \\
\frac{1}{(b-a)}\int_a^b f(x,d)\,dx &\le \frac{(b-ma)f(b,d)-(a-mb)f(a,d)}{2(b-a)},\ \text{and} \\
\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c}{m}\right)dx &\le \frac{(b-ma)f(b,\frac{c}{m})-(a-mb)f(a,\frac{c}{m})}{2(b-a)}.
\end{aligned}$$

Hence, using (2.2), we get the inequality (2.4).

Analogously, Since the partial mappings

$$f_y:[0,b]\to\mathbb{R},\ \ f_y(u)=f(u,y)$$

are differentiable on $[0,b]$, using the inequality (2.3), we get the inequality (2.5). The proof is completed. □

2.5. Remark. Choosing $m=1$ in (2.4) or (2.5), we get the relationship between the third and fourth inequalities in (1.7).

2.6. Theorem. *Suppose that $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ is an m-convex function on the co-ordinates on Δ. If $0\le a<b<\infty$ and $0\le c<d<\infty$, $m\in(0,1]$ with $f_x\in L_1[0,d]$ and $f_y\in L_1[0,b]$, then one has the inequality:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\
&\quad\le \frac{1}{(b-a)(d-c)}\Bigg[\int_a^b\int_c^d \frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}\,dy\,dx \\
&\qquad\qquad +\int_c^d\int_a^b \frac{f(x,y)+mf\left(\frac{x}{m},y\right)}{2}\,dx\,dy\Bigg].
\end{aligned} \tag{2.6}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is co-ordinated m-convex on Δ it follows that the mapping $g_x:[0,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is m-convex on $[0,d]$ for all $x\in[0,b]$. Then by the inequality (1.3) one has:

$$g_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \frac{g_x(y)+mg_x\left(\frac{y}{m}\right)}{2}\,dy,$$

or

$$f\left(x,\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}\,dy$$

for all $x \in [0,b]$. Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\ \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}\,dy\,dx.\end{aligned} \tag{2.7}$$

By a similar argument applied to the mapping $g_y : [0,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we get

$$\begin{aligned}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\ \leq \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b \frac{f(x,y)+mf\left(\frac{x}{m},y\right)}{2}\,dx\,dy.\end{aligned} \tag{2.8}$$

Summing the inequalities (2.7) and (2.8), we get the inequality (2.6). □

2.7. Remark. Choosing $m = 1$ in (2.6), we get the second inequality of (1.7).

3. Inequalities for co-ordinated (α, m)-convex functions

3.1. Definition. Consider the bidimensional interval $\Delta := [0,b] \times [0,d]$ in $[0,\infty)^2$. The mapping $f : \Delta \to \mathbb{R}$ is (α,m)*-convex on* Δ if

$$f(tx+(1-t)z, ty+m(1-t)w) \leq t^\alpha f(x,y) + m(1-t^\alpha) f(z,w) \tag{3.1}$$

holds for all $(x,y), (z,w) \in \Delta$ and $(\alpha,m) \in [0,1]^2$, with $t \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ which is (α,m)-convex on Δ is called *co-ordinated* (α,m)*-convex on* Δ if the partial mappings

$$f_y : [0,b] \to \mathbb{R},\ f_y(u) = f(u,y)$$

and

$$f_x : [0,d] \to \mathbb{R},\ f_x(v) = f(x,v)$$

are (α,m)-convex for all $y \in [0,d]$ and $x \in [0,b]$ with some fixed $(\alpha,m) \in [0,1]^2$.

Note that for $(\alpha,m) = (1,1)$ and $(\alpha,m) = (1,m)$, one obtains the class of co-ordinated convex and of co-ordinated m-convex functions on Δ, respectively.

3.2. Lemma. *Every* (α,m)*-convex mapping* $f : \Delta \to \mathbb{R}$ *is* (α,m)*-convex on the co-ordinates, where* $\Delta = [0,b] \times [0,d]$ *and* $\alpha, m \in [0,1]$.

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is (α,m)-convex on Δ. Consider the function

$$f_x : [0,d] \to \mathbb{R},\ f_x(v) = f(x,v),\ (x \in [0,b]).$$

Then for $t \in [0,1]$, $(\alpha, m) \in [0,1]^2$ and $v_1, v_2 \in [0,d]$, one has

$$\begin{aligned} f_x\left(tv_1 + m(1-t)v_2\right) &= f\left(x, tv_1 + m(1-t)v_2\right) \\ &= f\left(tx + (1-t)x, tv_1 + m(1-t)v_2\right) \\ &\leq t^\alpha f(x, v_1) + m(1-t^\alpha) f(x, v_2) \\ &= t^\alpha f_x(v_1) + m(1-t^\alpha) f_x(v_2). \end{aligned}$$

Therefore, $f_x(v) = f(x,v)$ is (α, m)-convex on $[0,d]$. The fact that $f_y : [0,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ is also (α, m)-convex on $[0,b]$ for all $y \in [0,d]$ goes likewise, and we shall omit the details. □

3.3. Theorem. *Suppose that $f : \Delta = [0,b] \times [0,d] \to \mathbb{R}$ is an (α, m)-convex function on the co-ordinates on Δ, where $(\alpha, m) \in (0,1]^2$. If $0 \leq a < b < \infty$, $0 \leq c < d < \infty$ and $f_x \in L_1[0,d]$, $f_y \in L_1[0,b]$, then the following inequalities hold:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy \\ (3.2)\qquad &\leq \frac{1}{(d-c)(b-a)} \\ &\quad \times \int_c^d \int_a^b \frac{2f(x,y) + m(2^\alpha - 1)\left(f\left(x, \frac{y}{m}\right) + f\left(\frac{x}{m}, y\right)\right)}{2^\alpha} \, dx \, dy, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_c^d \int_a^b f(x,y)\, dx\, dy \\ (3.3)\qquad &\leq \frac{1}{2(\alpha+1)(b-a)} \min\{w_1, w_2\} + \frac{1}{2(\alpha+1)(d-c)} \min\{w_3, w_4\}, \end{aligned}$$

where

$$\begin{aligned} w_1 &= \int_a^b f(x,c)\, dx + \alpha m \int_a^b f\left(x, \frac{d}{m}\right) dx \\ w_2 &= \int_a^b f(x,d)\, dx + \alpha m \int_a^b f\left(x, \frac{c}{m}\right) dx \\ w_3 &= \int_c^d f(a,y)\, dy + \alpha m \int_c^d f\left(\frac{b}{m}, y\right) dy \\ w_4 &= \int_c^d f(b,y)\, dy + \alpha m \int_c^d f\left(\frac{a}{m}, y\right) dy. \end{aligned}$$

Proof. Since $f : \Delta \to \mathbb{R}$ is co-ordinated (α, m)-convex on Δ it follows that the mapping $g_x : [0,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is (α, m)-convex on $[0,d]$ for all $x \in [0,b]$. Then by the inequality (1.4) one has:

$$g_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \frac{g_x(y) + m(2^\alpha - 1) g_x\left(\frac{y}{m}\right)}{2^\alpha} \, dy,$$

that is

$$f\left(x, \frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \frac{f(x,y) + m(2^\alpha - 1) f\left(x, \frac{y}{m}\right)}{2^\alpha} \, dy,$$

where $0 \le c < d < \infty$ and $(\alpha, m) \in (0,1]^2$. Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}(3.4)\qquad &\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx \\ &\le \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d \frac{f(x,y) + m(2^\alpha - 1) f\left(x, \frac{y}{m}\right)}{2^\alpha}\, dy\, dx,\end{aligned}$$

where $0 \le a < b < \infty$.

By a similar argument applied for the mapping $g_y : [0,b] \to [0,\infty)$, $g_y(x) = f(x,y)$ with $0 \le a < b < \infty$, we get

$$\begin{aligned}(3.5)\qquad &\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy \\ &\le \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d \frac{f(x,y) + m(2^\alpha - 1) f\left(\frac{x}{m}, y\right)}{2^\alpha}\, dy\, dx.\end{aligned}$$

Summing the inequalities (3.4) and (3.5), we get the inequality (3.2).

The inequality (3.3) can be obtained in a similar way to the proof of Theorem 2.3 by using (1.5). □

3.4. Remark. If we take $\alpha = 1$, (3.2) and (3.3) reduce to (2.6) and (2.1), respectively.

3.5. Theorem. *Suppose that $f : \Delta = [0,b] \times [0,d] \to \mathbb{R}$ is (α, m)-convex function on the co-ordinates on Δ, where $(\alpha, m) \in (0,1]^2$. If $0 \le a < b < \infty$, $0 \le c < d < \infty$ and $f_x \in L_1[0,d]$, $f_y \in L_1[0,b]$, then the following inequality holds:*

$$\begin{aligned}(3.6)\qquad &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\, dy\, dx \\ &\le \frac{1}{4(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\, dx + \frac{1}{b-a}\int_a^b f(x,d)\, dx\right. \\ &\quad + \frac{m\alpha}{b-a}\int_a^b f\left(x, \frac{c}{m}\right) dx + \frac{m\alpha}{b-a}\int_a^b f\left(x, \frac{d}{m}\right) dx \\ &\quad + \frac{1}{d-c}\int_c^d f(a,y)\, dy + \frac{1}{d-c}\int_c^d f(b,y)\, dy \\ &\quad \left. + \frac{m\alpha}{d-c}\int_c^d f\left(\frac{a}{m}, y\right) dy + \frac{m\alpha}{d-c}\int_c^d f\left(\frac{b}{m}, y\right) dy\right].\end{aligned}$$

Proof. Since $f : \Delta \to \mathbb{R}$ is co-ordinated (α, m)-convex on Δ it follows that the mapping $g_x : [0,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is (α, m)-convex on $[0,d]$ for all $x \in [0,b]$. Then by inequality (1.6) one has:

$$\frac{1}{d-c}\int_c^d g_x(y)\, dy \le \frac{1}{2}\left[\frac{g_x(c) + g_x(d) + m\alpha\left(g_x\left(\frac{c}{m}\right) + g_x\left(\frac{d}{m}\right)\right)}{\alpha+1}\right],$$

that is

$$\frac{1}{d-c}\int_c^d f(x,y)\, dy \le \frac{1}{2}\left[\frac{f(x,c) + f(x,d) + m\alpha\left(f\left(x, \frac{c}{m}\right) + f\left(x, \frac{d}{m}\right)\right)}{\alpha+1}\right],$$

where $0 \leq c < d < \infty$ and $(\alpha, m) \in (0,1]^2$. Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \\
&\leq \frac{1}{2(\alpha+1)} \left[\frac{1}{b-a} \int_a^b f(x,c)\, dx + \frac{1}{b-a} \int_a^b f(x,d)\, dx \right. \\
&\left. + \frac{m\alpha}{b-a} \int_a^b f\left(x, \frac{c}{m}\right) dx + \frac{m\alpha}{b-a} \int_a^b f\left(x, \frac{d}{m}\right) dx \right]
\end{aligned} \tag{3.7}$$

where $0 \leq a < b < \infty$.

By a similar argument applied to the mapping $g_y : [0,b] \to [0,\infty)$, $g_y(x) = f(x,y)$ with $0 \leq a < b < \infty$, we get

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)} \int_c^d \int_a^b f(x,y)\, dx\, dy \\
&\leq \frac{1}{2(\alpha+1)} \left[\frac{1}{d-c} \int_c^d f(a,y)\, dy + \frac{1}{d-c} \int_c^d f(b,y)\, dy \right. \\
&\left. + \frac{m\alpha}{d-c} \int_c^d f\left(\frac{a}{m}, y\right) dy + \frac{m\alpha}{d-c} \int_c^d f\left(\frac{b}{m}, y\right) dy \right].
\end{aligned} \tag{3.8}$$

Summing the inequalities (3.7) and (3.8), we get the inequality (3.6). □

3.6. Corollary. *Choosing $m = 1$ in Theorem 3.5, we get the following inequality*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \\
&\leq \frac{1}{4(\alpha+1)} \left[\frac{1}{b-a} \int_a^b f(x,c)\, dx + \frac{1}{b-a} \int_a^b f(x,d)\, dx \right. \\
&+ \frac{\alpha}{b-a} \int_a^b f(x,c)\, dx + \frac{\alpha}{b-a} \int_a^b f(x,d)\, dx \\
&+ \frac{1}{d-c} \int_c^d f(a,y)\, dy + \frac{1}{d-c} \int_c^d f(b,y)\, dy \\
&\left. + \frac{\alpha}{d-c} \int_c^d f(a,y)\, dy + \frac{\alpha}{d-c} \int_c^d f(b,y)\, dy \right].
\end{aligned}$$

3.7. Remark. Choosing $(\alpha, m) = (1,1)$ in (3.6), we get the third inequality of (1.7).

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