Hacettepe Journal of Mathematics and Statistics
Volume 40 (2) (2011), 231 – 240

INEQUALITIES FOR ONE SIDED APPROXIMATION IN ORLICZ SPACES

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Received 16 : 06 : 2010 : Accepted 10 : 01 : 2011

Abstract

In the present article some inequalities of trigonometric approximation are proved in Orlicz spaces generated by a quasiconvex Young function. Also, the main one-sided approximation problems are investigated.

Keywords: Fractional moduli of smoothness, Direct theorem, Converse theorem, Fractional derivative, One-sided approximation.

2010 AMS Classification: 26 A 33, 41 A 17, 41 A 20, 41 A 25, 41 A 27, 42 A 10.

Communicated by Cihan Orhan

1. Introduction

A function Φ is called a *Young function* if Φ is even, continuous, nonnegative in $\mathbb{R} := (-\infty, +\infty)$, increasing on $\mathbb{R}^+ := (0, \infty)$ and such that

$$\Phi(0) = 0, \lim_{x\to\infty} \Phi(x) = \infty.$$

A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be *quasiconvex* if there exist a convex Young function Φ and a constant $c_1 \geq 1$ such that

$$\Phi(x) \leq \varphi(x) \leq \Phi(c_1 x) \;\; \forall x \geq 0.$$

Set $\mathsf{T} := [0, 2\pi]$ and let φ be a quasiconvex Young function. We denote by $\varphi(L)$ the class of complex valued Lebesgue measurable functions $f : \mathsf{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathsf{T}} \varphi(|f(x)|)\, dx < \infty.$$

The class of functions $f : \mathsf{T} \to \mathbb{C}$ having the property

$$\int_{\mathsf{T}} \varphi(c_2 |f(x)|)\, dx < \infty$$

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for some $c_2 \in \mathbb{R}^+$ is denoted by $L_\varphi(\mathsf{T})$. The set $L_\varphi(\mathsf{T})$ becomes a normed space with the *Orlicz* norm

$$\|f\|_\varphi := \sup\left\{ \int_{\mathsf{T}} |f(x)\, g(x)|\, dx : \int_{\mathsf{T}} \tilde{\varphi}(|g|)\, dx \le 1 \right\},$$

where $\tilde{\varphi}(y) := \sup_{x \ge 0}(xy - \varphi(x))$, $y \ge 0$, is the *complementary function* of φ.

For a quasiconvex function φ we define the index $p(\varphi)$ of φ as

$$\frac{1}{p(\varphi)} := \inf\{p : p > 0,\ \varphi^p \text{ is quasiconvex}\}$$

and the conjugate index of φ as

$$p'(\varphi) := \frac{p(\varphi)}{p(\varphi) - 1}.$$

It can be easily seen that the functions in $L_\varphi(\mathsf{T})$ are summable on T, $L_\varphi(\mathsf{T}) \subset L^1(\mathsf{T})$ and $L_\varphi(\mathsf{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_\varphi(\mathsf{T})$ is called the *Orlicz space*.

A Young function Φ is said to be satisfy the Δ_2 *condition* if there is a constant $c_3 > 0$ such that

$$\Phi(2x) \le c_3 \Phi(x)$$

for all $x \in \mathbb{R}$.

We will denote by $QC_2^\theta(0,1)$ the class of functions g satisfying the condition $\triangle_2$ such that g^θ is quasiconvex for some $\theta \in (0,1)$.

In the present work we consider the trigonometric polynomial approximation problems for functions and their fractional derivatives in the spaces $L_\varphi(\mathsf{T})$, where $\varphi \in QC_2^\theta(0,1)$. We prove a Jackson type direct theorem, and a converse theorem of trigonometric approximation with respect to the fractional order moduli of smoothness in Orlicz spaces. As a particular case, we obtain a constructive description of the Lipschitz class in Orlicz spaces. A direct theorem of one sided trigonometric approximation is also obtained.

Let

$$f(x) \backsim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \text{ and } \tilde{f}(x) \backsim \sum_{k=-\infty}^{\infty} (-i\operatorname{sign} k)\, c_k e^{ikx} \tag{1.1}$$

be the *Fourier* and the *conjugate Fourier series* of $f \in L^1(\mathsf{T})$, respectively. We define

$$S_n(f) := S_n(x,f) := \sum_{k=-n}^{n} c_k e^{ikx},\ n = 0,1,2,\ldots.$$

For a given $f \in L^1(\mathsf{T})$, assuming $c_0 = 0$ in (1.1), we define the α^{th} *fractional* $(\alpha \in \mathbb{R}^+)$ *integral* of f as in [7, v.2, p.134] by

$$I_\alpha(x,f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where $\mathbb{Z}$ is the set of integers, $\mathbb{Z}^* := \{z \in \mathbb{Z} : z \ne 0\}$, and

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define the *fractional derivative* of a function $f \in L^1(\mathsf{T})$, satisfying $c_0 = 0$ in (1.1), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+\alpha-[\alpha]}(x,f),$$

provided the righthand side exists, where $[x]$ denotes the integer part of the real number x.

Setting $h \in \mathsf{T}$, $r \in \mathbb{R}^+$, $\varphi \in QC_2^\theta(0,1)$ and $f \in L_\varphi(\mathsf{T})$, we define

$$\Delta_h^r f(\cdot) := (T_h - I)^r f(\cdot) = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} f(\cdot + (r-k)h),$$

where $\binom{r}{k} := \dfrac{r(r-1)\ldots(r-k+1)}{k!}$ for $k > 1$, $\binom{r}{1} := r$ and $\binom{r}{0} := 1$ are the binomial coefficients, $T_h f(x) := f(x+h)$ is the translation operator and I the identity operator.

Since $\sum\limits_{k=0}^{\infty} \left|\binom{r}{k}\right| < \infty$ we get

$$\left\|\Delta_h^r f\right\|_\varphi \le c\left\|f\right\|_\varphi < \infty \tag{1.2}$$

under the condition $f \in L_\varphi(\mathsf{T})$, where $\varphi \in QC_2^\theta(0,1)$.

Here and in the following we will denote by B a translation invariant Banach Function Space. Also, the notation $\|\cdot\|_B$ stands for the norm of B.

For $r \in \mathbb{R}^+$, we define the *fractional modulus of smoothness of order r for $f \in B$*, as

$$\omega_B^r(f,\delta) := \sup_{|h|\le\delta} \left\|\Delta_h^r f\right\|_B, \ \delta \ge 0.$$

If $\varphi \in QC_2^\theta(0,1)$ and $B = L_\varphi(\mathsf{T})$, we will set $\omega_B^r(f,\cdot) =: \omega_\varphi^r(f,\cdot)$. Hence for $\varphi \in QC_2^\theta(0,1)$ and $f \in L_\varphi(\mathsf{T})$, we have by (1.2) that

$$\omega_\varphi^r(f,\delta) \le c\left\|f\right\|_\varphi,$$

where the constant $c > 0$ dependent only on r and φ.

Let $\mathcal{T}_n$ be the class of trigonometric polynomials of degree not greater than n. We begin with the fractional Nikolski-Civin inequality:

1.1. Theorem. *Suppose that $\alpha \in \mathbb{R}^+$, $T_n \in \mathcal{T}_n$ and $0 < h < 2\pi/n$. Then*

$$\left\|T_n^{(\alpha)}\right\|_B \le \left(\frac{n}{2\sin(nh/2)}\right)^\alpha \left\|\Delta_h^\alpha T_n\right\|_B.$$

In particular, if $h = \pi/n$, then

$$\left\|T_n^{(\alpha)}\right\|_B \le 2^{-\alpha} n^\alpha \left\|\Delta_{\pi/n}^\alpha T_n\right\|_B. \tag{1.3}$$

Proof. Let $T_n(x) = \frac{a_0}{2} + \sum\limits_{\nu\in\mathbb{Z}_n^*} c_\nu e^{i\nu x}$, where $\mathbb{Z}_n^* := \{z \in \mathbb{Z} : z < n,\ z > -n,\ z \ne 0\}$. Then $T_n^{(\alpha)}(x) = \sum\limits_{\nu\in\mathbb{Z}_n^*} (i\nu)^\alpha c_\nu e^{i\nu x}$, and

$$\Delta_h^\alpha T_n\left(x + \frac{\alpha}{2}h\right) = \sum_{\nu\in\mathbb{Z}_n^*} \left(2i\sin\frac{h}{2}\nu\right)^\alpha c_\nu e^{i\nu x}.$$

We set

$$\varphi(t) := \left(2i\sin\frac{h}{2}t\right)^\alpha,\ g(t) := \left(\frac{t}{2\sin\frac{h}{2}t}\right)^\alpha \text{ for } -n \le t \le n \text{ and } g(0) := h^{-\alpha}.$$

Then for $x \in \mathbb{R}$, $h \in (0, 2\pi/n)$, we obtain

$$\Delta_h^\alpha T_n\left(x + \frac{\alpha}{2}h\right) = \sum_{\nu\in\mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu x}$$

and

$$T_n^{(\alpha)}(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) g(\nu) c_\nu e^{i\nu x}.$$

The convergence

$$g(t) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi t/n}$$

is uniform for $t \in [-n, n]$. Since $(-1)^k d_k \geq 0$, we find

$$\begin{aligned} T_n^{(\alpha)}(x) &= \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) \sum_{k=-\infty}^{\infty} d_k e^{\frac{ik\pi\nu}{n}} c_\nu e^{i\nu x} \\ &= \sum_{k=-\infty}^{\infty} d_k \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu\left(x+\frac{k\pi}{n}\right)} \\ &= \sum_{k=-\infty}^{\infty} d_k \Delta_h^\alpha T_n\left(x + \frac{k\pi}{n} + \frac{\alpha}{2}h\right). \end{aligned}$$

Hence we conclude

$$\begin{aligned} \left\|T_n^{(\alpha)}\right\|_B &\leq \left\|\Delta_h^\alpha T_n\right\|_B \sum_{k=-\infty}^{\infty} \left|d_k e^{ik\pi}\right| \\ &= \left\|\Delta_h^\alpha T_n\right\|_B \sum_{k=-\infty}^{\infty} d_k e^{ik\pi} \\ &= \left(\frac{n}{2\sin(nh/2)}\right)^\alpha \left\|\Delta_h^\alpha T_n\right\|_B, \end{aligned}$$

and Theorem 1.1 is proved. □

We denote by B^α, $\alpha > 0$, the linear space of 2π-periodic complex valued functions $f \in B$ such that $f^{(\alpha-1)}$ is absolutely continuous (AC), and $f^{(\alpha)} \in B$. If $\varphi \in QC_2^\theta(0,1)$ and $B = L_\varphi(\mathsf{T})$ we will let $B^\alpha =: W_\varphi^\alpha(\mathsf{T})$.

We set $L_0^\infty := \left\{f \in L^\infty : f \text{ is real valued and bounded on } \mathsf{T}\right\}$. If $f \in L_0^\infty$ we define

$\mathcal{T}_n^-(f) := \{t \in \mathcal{T}_n : t \text{ is real valued } 2\pi \text{ periodic and } t(x) \leq f(x) \text{ for every } x \in \mathbb{R}\}$,

$\mathcal{T}_n^+(f) := \{T \in \mathcal{T}_n : T \text{ is real valued } 2\pi \text{ periodic and } f(x) \leq T(x) \text{ for every } x \in \mathbb{R}\}$,

$E_n^-(f)_\varphi := \inf\limits_{t \in \mathcal{T}_n^-(f)} \left\|f - t\right\|_\varphi$, $E_n^+(f)_\varphi := \inf\limits_{T \in \mathcal{T}_n^+(f)} \left\|T - f\right\|_\varphi$.

The quantities $E_n^-(f)_\varphi$ and $E_n^+(f)_\varphi$ are, respectively, called the *best lower* (*upper*) *one sided approximation errors* for $f \in L_0^\infty$. Similarly, the best *trigonometric approximation error* of $f \in L_\varphi(\mathsf{T})$ is defined as $E_n(f)_\varphi := \inf\limits_{S \in \mathcal{T}_n} \left\|f - S\right\|_\varphi$. We note that $E_n(f)_\varphi \leq E_n^\pm(f)_\varphi$.

If $\varphi \in QC_2^\theta(0,1)$, $f \in L_\varphi(\mathsf{T})$, $g \in L^1(\mathsf{T})$, we introduce the convolution

$$(f * g)(x) = \frac{1}{2\pi} \int_{\mathsf{T}} f(x-u) g(u)\, du.$$

This convolution exists for every $x \in \mathbb{R}$ and is a measurable function. Furthermore

$$\left\|f * g\right\|_\varphi \leq \left\|f\right\|_\varphi \left\|g\right\|_{L^1(\mathsf{T})}.$$

If f is continuous (AC) then $f * g$ is continuous (AC).

1.2. Theorem. *Let $\varphi \in QC_2^{\theta}(0,1)$, $1 \le \beta < \infty$ and $f \in W_{\varphi}^{\beta}(\mathsf{T})$. If $0 \le \alpha \le \beta$ and $n = 1,2,3,\ldots$, then there exists a constant $c > 0$ depending only on α and β such that*

$$E_n\big(f^{(\alpha)}\big)_{\varphi} \le \frac{c}{n^{\beta-\alpha}} E_n\big(f^{(\beta)}\big)_{\varphi} \tag{1.4}$$

holds. If f is real valued, $0 \le \alpha \le \beta - 1$ and $n = 1,2,3,\ldots$, then

$$E_n^{\pm}\big(f^{(\alpha)}\big)_{\varphi} \le \frac{c}{n^{\beta-\alpha}} E_n\big(f^{(\beta)}\big)_{\varphi} \tag{1.5}$$

holds.

Proof. 1° First we prove that $f^{(\alpha)}$ is AC for $0 \le \alpha \le \beta - 1$ and $f^{(\alpha)} \in L_{\varphi}(\mathsf{T})$ for $\beta - 1 \le \alpha \le \beta$. It is well known that the function $\Psi_{\alpha}(u) := \lim_{n\to\infty} \sum_{\nu \in \mathbb{Z}_n^*} \frac{e^{i\nu u}}{(i\nu)^{\alpha}} = \sum_{\nu\in\mathbb{Z}^*} \frac{e^{i\nu u}}{(i\nu)^{\alpha}}$, $\alpha \in \mathbb{R}^+$, is defined for every $u \in \mathbb{R}$ if $1 \le \alpha < \infty$ (for $u \ne 2k\pi$, $k \in \mathbb{Z}$ if $0 < \alpha < 1$) and Ψ_{α} is of class $L^1(\mathsf{T})$. In this case

$$f(x) = \big(f^{(\beta)} * \Psi_{\beta}\big)(x) \text{ for every } x \in \mathbb{R}. \tag{1.6}$$

Furthermore,

$$f^{(\alpha)}(x) = \big(f^{(\beta)} * \Psi_{\beta-\alpha}\big)(x) \tag{1.7}$$

is satisfied for every $x \in \mathbb{R}$ if $0 \le \alpha < \beta - 1$ (for almost every $x \in \mathbb{R}$ if $\beta - 1 < \alpha < \beta$). Now (1.6) implies that if $\beta \ge 1$, then f is absolutely continuous, and (1.7) implies that $f^{(\alpha)}$ is AC for $0 \le \alpha \le \beta - 1$ and $f^{(\alpha)} \in L_{\varphi}(\mathsf{T})$ for $\beta - 1 \le \alpha \le \beta$.

2° If $\alpha = \beta$, then (1.4) is obvious. If $\alpha = 0$, then (1.4) was proved in [3]. Let $0 \le \alpha < \beta$. We choose a $S_{\alpha,n} \in \mathcal{T}_n$ with $\|S_{\alpha,n} - \Psi_{\beta-\alpha}\|_{L^1(\mathsf{T})} = E_n(\Psi_{\beta-\alpha})_{L^1(\mathsf{T})}$. Let $U_{n,\alpha}[f] = f^{(\beta)} * S_{\alpha,n}$, $n = 1,2,3\ldots$. Then

$$f^{(\alpha)}(x) - U_{n,\alpha}[f](x) = \frac{1}{2\pi}\int_{\mathsf{T}} f^{(\beta)}(u)\left\{\Psi_{\beta-\alpha}(x-u) - S_{\alpha,n}(x-u)\right\} du$$

holds a.e. Therefore,

$$\big\|f^{(\alpha)} - U_{n,\alpha}[f]\big\|_{\varphi} \le \|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(\mathsf{T})}\|f^{(\beta)}\|_{\varphi}.$$

Since by [4]

$$\|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(\mathsf{T})} \le cn^{\alpha-\beta}$$

we get (since $U_{n,\alpha}[f] \in \mathcal{T}_n$) that

$$E_n\big(f^{(\alpha)}\big)_{\varphi} \le cn^{\alpha-\beta}\|f^{(\beta)}\|_{\varphi}.$$

Let $Q_n \in \mathcal{T}_n$ be such that

$$\|f^{(\beta)} - Q_n\|_{\varphi} = E_n\big(f^{(\beta)}\big)_{\varphi},\ n = 1,2,3\ldots.$$

We suppose

$$\phi(x) = f(x) - I_{\beta}[Q_n](x),\ x \in \mathbb{R}.$$

Then

$$\phi^{(\beta)}(x) = f^{(\beta)}(x) - Q_n(x),$$

and hence

$$\|\phi^{(\beta)}\|_{\varphi} = \|f^{(\beta)} - Q_n\|_{\varphi} = E_n\big(f^{(\beta)}\big)_{\varphi}.$$

Therefore we find

$$E_n\big(\phi^{(\alpha)}\big)_{\varphi} \le cn^{\alpha-\beta}\|\phi^{(\beta)}\|_{\varphi} \le cn^{\alpha-\beta}E_n\big(f^{(\beta)}\big)_{\varphi}.$$

Since

$$E_n\left(\phi^{(\alpha)}\right)_\varphi = E_n\left(f^{(\alpha)}\right)_\varphi,$$

we conclude that (1.4) holds.

3° Let

$$f_+^{(\beta)}(u) = \frac{1}{2}\left\{\left|f^{(\beta)}(u)\right| + f^{(\beta)}(u)\right\} \text{ and } f_-^{(\beta)}(u) = \frac{1}{2}\left\{\left|f^{(\beta)}(u)\right| - f^{(\beta)}(u)\right\}$$

for $u \in \mathbb{R}$. Then

$$f(x) = \left(f_+^{(\beta)} * \Psi_\beta\right)(x) - \left(f_-^{(\beta)} * \Psi_\beta\right)(x),$$
$$f^{(\alpha)}(x) = \left(f_+^{(\beta)} * \Psi_{\beta-\alpha}\right)(x) - \left(f_-^{(\beta)} * \Psi_{\beta-\alpha}\right)(x)$$

for every $0 < \alpha \leq \beta - 1$. Let $t_{\alpha,n} \in \mathcal{T}_n^-(\Psi_{\beta-\alpha})$, $T_{\alpha,n} \in \mathcal{T}_n^+(\Psi_{\beta-\alpha})$ be such that

$$\left\|f - t_{\alpha,n}\right\|_\varphi = E_n^-(\Psi_{\beta-\alpha})_{L^1(\mathsf{T})} \text{ and } \left\|T_{\alpha,n} - f\right\|_\varphi = E_n^+(\Psi_{\beta-\alpha})_{L^1(\mathsf{T})}$$

for $n = 1, 2, 3 \ldots$. Let also

$$U_{0,n}^+[f] = \left(f_+^{(\beta)} * T_{0,n}\right) - \left(f_-^{(\beta)} * t_{0,n}\right),\ U_{0,n}^-[f] = \left(f_+^{(\beta)} * t_{0,n}\right) - \left(f_-^{(\beta)} * T_{0,n}\right),$$

and for $0 < \alpha < \beta - 1$ we set

$$U_{\alpha,n}^+[f] = \left(f_+^{(\beta)} * T_{\alpha,n}\right) - \left(f_-^{(\beta)} * t_{\alpha,n}\right),\ U_{\alpha,n}^-[f] = \left(f_+^{(\beta)} * t_{\alpha,n}\right) - \left(f_-^{(\beta)} * T_{\alpha,n}\right).$$

Hence,

$$U_{\alpha,n}^+[f](x) - f^{(\alpha)}(x) = \frac{1}{2\pi}\int_{\mathsf{T}} f_+^{(\beta)}(u)\left\{T_{\alpha,n}(x-u) - \Psi_{\beta-\alpha}(x-u)\right\} du$$
$$+ \frac{1}{2\pi}\int_{\mathsf{T}} f_-^{(\beta)}(u)\left\{\Psi_{\beta-\alpha}(x-u) - t_{\alpha,n}(x-u)\right\} du$$

for every $x \in \mathbb{R}$. Then

$$U_{\alpha,n}^+[f](x) \geq f^{(\alpha)}(x)$$

for every $x \in \mathbb{R}$. This implies that

$$U_{\alpha,n}^+[f] \in \mathcal{T}_n^+\left(f^{(\alpha)}\right),\ 0 \leq \alpha \leq \beta - 1.$$

Similarly,

$$U_{\alpha,n}^-[f] \in \mathcal{T}_n^-\left(f^{(\beta)}\right),\ 0 \leq \alpha \leq \beta - 1.$$

We obtain

$$\left\|U_{n,\alpha}^\pm[f] - f^{(\alpha)}\right\|_\varphi \leq cn^{\alpha-\beta}\left\|f^{(\beta)}\right\|_\varphi,$$

and hence

$$\left\|U_{n,\alpha}^\pm[f] - f^{(\alpha)}\right\|_\varphi \leq cn^{\alpha-\beta}E_n\left(f^{(\beta)}\right)_\varphi$$

for $0 \leq \alpha \leq \beta - 1$. Since

$$U_{\alpha,n}^-[\phi](x) \leq \phi^{(\alpha)}(x) \leq U_{\alpha,n}^+[\phi](x)$$

and

$$\phi^{(\alpha)}(x) = f^{(\alpha)}(x) - Q_n^{(\alpha-\beta)}(x)$$

we have

$$U_{\alpha,n}^-[\phi](x) + Q_n^{(\alpha-\beta)}(x) \leq f^{(\alpha)}(x) \leq U_{\alpha,n}^+[\phi](x) + Q_n^{(\alpha-\beta)}(x)$$

for every $x \in \mathbb{R}$. Therefore

$$\begin{aligned} E_n^{\pm}\big(f^{(\alpha)}\big)_\varphi &\le \big\|U_{n,\alpha}^{\pm}[\phi] - Q_n^{(\alpha-\beta)} - f^{(\alpha)}\big\|_\varphi \\ &= \big\|U_{n,\alpha}^{\pm}[\phi] - \phi^{(\alpha)}\big\|_\varphi \\ &\le cn^{\alpha-\beta} E_n\big(f^{(\beta)}\big)_\varphi, \end{aligned}$$

and the required result holds. □

1.3. Theorem. *Let $\varphi \in QC_2^\theta(0,1)$. If $1 \le \beta < \infty$, $f \in W_\varphi^\beta(\mathsf{T})$ and $\beta \ge \alpha \ge 0$, then for $n = 1, 2, 3, \ldots$ there is a constant $c > 0$ dependent only on α, β and φ such that*

$$\big\|f^{(\alpha)}(\,\cdot\,) - S_n^{(\alpha)}(\,\cdot\,, f)\big\|_\varphi \le \frac{c}{n^{\beta-\alpha}} E_n\big(f^{(\beta)}\big)_\varphi \tag{1.8}$$

holds. If f is real valued and there exist polynomials $t_n \in \mathcal{T}_n^-(f)$, $T_n \in \mathcal{T}_n^+(f)$ such that $\big\|f - t_n\big\|_\varphi \le cE_n^-(f)_\varphi$, $\big\|T_n - f\big\|_\varphi \le cE_n^+(f)_\varphi$, then for $0 \le \alpha \le \beta$ and $n = 1, 2, 3, \ldots$,

$$\big\|f^{(\alpha)} - t_n^{(\alpha)}\big\|_\varphi \le \frac{c}{n^{\beta-\alpha}} E_n\big(f^{(\beta)}\big)_\varphi, \ \textit{and} \tag{1.9}$$

$$\big\|T_n^{(\alpha)} - f^{(\alpha)}\big\|_\varphi \le \frac{c}{n^{\beta-\alpha}} E_n\big(f^{(\beta)}\big)_\varphi \tag{1.10}$$

hold.

Proof. If $\alpha = 0$, then the results follows from Theorem 1.2. If $\alpha = \beta$, then it was proved in [2] that

$$\big\|f^{(\alpha)}(\,\cdot\,) - S_n^{(\alpha)}(\,\cdot\,, f)\big\|_\varphi \le cE_n\big(f^{(\alpha)}\big)_\varphi. \tag{1.11}$$

From Theorem 1.2 and last inequality, (1.8) follows.

Now we prove (1.9) and (1.10) for $0 \le \alpha \le \beta$. Let

$$W_n(f) := W_n(x, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x, f), \ n = 0, 1, 2, \ldots.$$

Suppose that $u := u(\,\cdot\,, f) \in \mathcal{T}_n$ satisfies $\big\|f - u\big\|_\varphi = E_n(f)_\varphi$. Since

$$W_n(\,\cdot\,, f^{(\alpha)}) = W_n^{(\alpha)}(\,\cdot\,, f)$$

we have

$$\begin{aligned} \big\|f^{(\alpha)} - t_n^{(\alpha)}\big\|_\varphi &\le \big\|f^{(\alpha)} - W_n(\,\cdot\,, f^{(\alpha)})\big\|_\varphi + \big\|u(\,\cdot\,, W_n(f)) - t_n^{(\alpha)}\big\|_\varphi \\ &\qquad + \big\|W_n^{(\alpha)}(\,\cdot\,, f) - u(\,\cdot\,, W_n(f))\big\|_\varphi \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Since $\big\|W_n(f)\big\|_\varphi \le 4\big\|f\big\|_\varphi$, we get

$$\begin{aligned} I_1 &\le \big\|f^{(\alpha)}(\,\cdot\,) - u(\,\cdot\,, f^{(\alpha)})\big\|_\varphi + \big\|u(\,\cdot\,, f^{(\alpha)}) - W_n(\,\cdot\,, f^{(\alpha)})\big\|_\varphi \\ &= E_n\big(f^{(\alpha)}\big)_\varphi + \big\|W_n(\,\cdot\,, u(f^{(\alpha)}) - f^{(\alpha)})\big\|_\varphi \\ &\le 5E_n\big(f^{(\alpha)}\big)_\varphi. \end{aligned}$$

From Theorem 1.1, we get

$$I_2 \le 2(n-1)^\alpha \big\|u(\,\cdot\,, W_n(f)) - t_n\big\|_\varphi$$

and

$$I_3 \le 2(2n-2)^\alpha \big\|W_n(\,\cdot\,, f) - u(\,\cdot\,, W_n(f))\big\|_\varphi \le 2^{\alpha+1} n^\alpha E_n(W_n(f))_\varphi.$$

Now we have

$$\begin{aligned}\left\|u(\,\cdot\,,W_n(f))-t_n\right\|_{\varphi} &\le \left\|u(\,\cdot\,,W_n(f))-W_n(\,\cdot\,,f)\right\|_{\varphi}+\left\|W_n(\,\cdot\,,f)-f\,(\,\cdot\,)\,\right\|_{\varphi}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\left\|f\,(\,\cdot\,)-t_n\right\|_{\varphi}\\ &\le E_n\left(W_n(f)\right)_{\varphi}+5E_n\left(f\right)_{\varphi}+cE_n^{-}\left(f\right)_{\varphi}.\end{aligned}$$

Since

$$E_n\left(W_n(f)\right)_{\varphi}\le\left\|W_n(f)-u\right\|_{\varphi}=\left\|W_n(f-u)\right\|_{\varphi}\le 4E_n\left(f\right)_{\varphi},$$

we get

$$\begin{aligned}\left\|f^{(\alpha)}-t_n^{(\alpha)}\right\|_{\varphi} &\le 5E_n\big(f^{(\alpha)}\big)_{\varphi}+2n^{\alpha}E_n\left(W_n(f)\right)_{\varphi}+10n^{\alpha}E_n\left(f\right)_{\varphi}\\ &\qquad+2^{\alpha+1}n^{\alpha}E_n\left(W_n(f)\right)_{\varphi}+c2n^{\alpha}E_n^{-}\left(f\right)_{\varphi}\\ &\le 5E_n\left(f^{(\alpha)}\right)_{\varphi}+\left(18+2^{3+\alpha}\right)n^{\alpha}E_n\left(f\right)_{\varphi}+c2n^{\alpha}E_n^{-}\left(f\right)_{\varphi}.\end{aligned}\tag{1.12}$$

Using Theorem 1.2 we get (1.9), and (1.10) can be proved using the same procedure. □

Direct theorem of trigonometric approximation:

1.4. Theorem. *Let $\varphi\in QC_2^{\theta}(0,1)$ and $r\in\mathbb{R}^{+}$. If $f\in L_{\varphi}(\mathsf{T})$, then there is a constant $c>0$, dependent only on r and φ, such that the inequality*

$$E_n\left(f\right)_{\varphi}\le c\omega_{\varphi}^{r}\left(f,\frac{1}{n+1}\right)\tag{1.13}$$

holds for $n=0,1,2,3,\ldots$.

Proof. This is a consequence of [3, Theorem 2] and the property $\omega_{\varphi}^{r}(f,\cdot)\le c\omega_{\varphi}^{s}(f,\cdot)$, $(r\ge s\in\mathbb{R}^{+})$, of the smoothness moduli. □

1.5. Theorem. *If $r,\delta\in\mathbb{R}^{+}$ and $f\in B^{\alpha}$, $\alpha\in\mathbb{R}^{+}$, then there exists a constant $c>0$ depending only on r and B such that*

$$\omega_B^{r}\left(f,\delta\right)\le c\delta^{r}\left\|f^{(r)}\right\|_{B},\ \delta\ge 0\tag{1.14}$$

holds.

Proof. For the function $\chi_r\left(\,\cdot\,,h\right)\in L^{1}\left(\mathsf{T}\right)$ of [6, (20.15), p.376] we define

$$\left(A_h^{r}f\right)(x):=\left(f*\chi_r\left(\,\cdot\,,h\right)\right)(x)=\frac{1}{2\pi}\int_{\mathsf{T}}f\left(x-u\right)\chi_r\left(u,h\right)\,du,\ x\in\mathsf{T},\ h\in\mathbb{R}^{+}.$$

Then using Fubini's theorem we get

$$\left\|A_h^{r}f\right\|_{B}\le\left\|\chi_r\left(\,\cdot\,,h\right)\right\|_{L^{1}(\mathsf{T})}\left\|f\right\|_{B}\le c\left\|f\right\|_{B}.\tag{1.15}$$

Since

$$\left(\Delta_h^{r}f\right)(x)=h^{r}\left(A_h^{r}f\right)^{(r)}(x)=h^{r}A_h^{r}\left(f^{(r)}\right)(x)$$

we have from (1.15) that

$$\sup_{|h|\le\delta}\left\|\Delta_h^{r}f\right\|_{B}=\sup_{|h|\le\delta}h^{r}\left\|A_h^{r}\left(f^{(r)}\right)\right\|_{B}\le c\delta^{r}\left\|f^{(r)}\right\|_{B},$$

from which we obtain (1.14). □

The converse theorem of trigonometric approximation:

1.6. Theorem. *Let $\varphi \in QC_2^{\theta}(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathsf{T})$, then there is a constant $c > 0$, dependent only on r and φ, such that for $n = 0, 1, 2, 3, \ldots$*

$$\omega_\varphi^r\Big(f, \frac{\pi}{n+1}\Big) \leq \frac{c}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{r-1} E_\nu(f)_\varphi$$

holds.

Proof. The proof goes similarly to that of the proof of [2, Theorem 3]. □

From Theorems 1.4 and 1.6 we have the following corollaries:

1.7. Corollary. *Let $\varphi \in QC_2^{\theta}(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathsf{T})$ satisfies*

$$E_n(f)_\varphi = \mathcal{O}\left(n^{-\sigma}\right), \ \sigma > 0, \ n = 1, 2, \ldots,$$

then

$$\omega_\varphi^r(f, \delta) = \begin{cases} \mathcal{O}(\delta^\sigma) & \text{if } r > \sigma, \\ \mathcal{O}(\delta^\sigma \left|\log(1/\delta)\right|) & \text{if } r = \sigma, \\ \mathcal{O}(\delta^r) & \text{if } r < \sigma, \end{cases}$$

holds. □

1.8. Definition. Let $\varphi \in QC_2^{\theta}(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathsf{T})$, then for $0 < \sigma < r$ we set $\mathrm{Lip}\sigma(r, \varphi) := \left\{ f \in L_\varphi(\mathsf{T}) : \omega_\varphi^r(f, \delta) = \mathcal{O}(\delta^\sigma), \ \delta > 0 \right\}$.

The following constructive characterization of the Lipschitz class holds:

1.9. Corollary. *Let $0 < \sigma < r$, $M \in QC_2^{\theta}(0,1)$ and $f \in L_\varphi(\mathsf{T})$. Then the conditions*

(a) $f \in \mathrm{Lip}\sigma(r, \varphi)$,
(b) $E_n(f)_\varphi = \mathcal{O}\left(n^{-\sigma}\right)$, $n = 1, 2, \ldots,$

are equivalent. □

1.10. Theorem. *Let $\varphi \in QC_2^{\theta}(0,1)$ and $f \in L_\varphi(\mathsf{T})$. If $\alpha \in \mathbb{R}^+$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_\varphi < \infty,$$

then there exists a constant $c > 0$ dependent only on α and φ, such that

$$E_n\left(f^{(\alpha)}\right)_\varphi \leq c\left(n^\alpha E_n(f)_\varphi + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_\varphi\right) \tag{1.16}$$

holds.

Proof. Since

$$\left\| f^{(\alpha)} - S_n\big(f^{(\alpha)}\big)\right\|_\varphi \leq \left\| S_{2^{m+2}}\big(f^{(\alpha)}\big) - S_n\big(f^{(\alpha)}\big)\right\|_\varphi + \sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}\left(f^{(\alpha)}\right) - S_{2^k}\big(f^{(\alpha)}\big)\right\|_\varphi$$

we have for $2^m < n < 2^{m+1}$ that

$$\left\| S_{2^{m+2}}\big(f^{(\alpha)}\big) - S_n\big(f^{(\alpha)}\big)\right\|_\varphi \leq c 2^{(m+2)\alpha} E_n(f)_\varphi \leq c n^\alpha E_n(f)_\varphi.$$

On the other hand we find

$$\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}\left(f^{(\alpha)}\right)-S_{2^{k}}\left(f^{(\alpha)}\right)\right\|_{\varphi}$$

$$\leq c\sum_{k=m+2}^{\infty}2^{(k+1)\alpha}E_{2^{k}}\left(f\right)_{\varphi}\leq c\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^{k}}\mu^{\alpha-1}E_{\mu}\left(f\right)_{\varphi}$$

$$=c\sum_{\nu=2^{m+1}+1}^{\infty}\nu^{\alpha-1}E_{\nu}\left(f\right)_{\varphi}\leq c\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{\varphi}.$$

Therefore

$$E_{n}\left(f^{(\alpha)}\right)_{\varphi}\leq c\left(n^{\alpha}E_{n}(f)_{\varphi}+\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{\varphi}\right), \qquad \square$$

As a corollary of Theorems 1.4, 1.6 and 1.10,

1.11. Theorem. *Let $f\in W_{\varphi}^{\alpha}\left(\mathsf{T}\right)$, $r\in(0,\infty)$, and*

$$\sum_{\nu=1}^{\infty}\nu^{\alpha-1}E_{\nu}\left(f\right)_{\varphi}<\infty$$

for some $\alpha>0$. In this case, for $n=0,1,2,\ldots$ there exists a constant $c>0$, dependent only on α, r and φ such that

$$\omega_{\varphi}^{r}\left(f^{(\alpha)},\frac{\pi}{n+1}\right)\leq c\left(\frac{1}{\left(n+1\right)^{r}}\sum_{\nu=0}^{n}\left(\nu+1\right)^{\alpha+r-1}E_{\nu}\left(f\right)_{\varphi}+\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}\left(f\right)_{\varphi}\right)$$

holds. $\square$

As a corollary of Theorem 1.4,

1.12. Theorem. *Let $\varphi\in QC_{2}^{\theta}\left(0,1\right)$, $r\in\mathbb{R}^{+}$ and $1\leq\beta<\infty$. If $f\in W_{\varphi}^{\beta}\left(\mathsf{T}\right)$ is real valued and $0\leq\alpha\leq\beta-1$, then there is a constant $c>0$, dependent only on r and φ, such that the inequality*

$$E_{n}^{\pm}\left(f^{(\alpha)}\right)_{\varphi}\leq\frac{c}{n^{\beta-\alpha}}\omega_{\varphi}^{r}\left(f^{(\beta)},\frac{\pi}{n}\right)$$

holds for $n=1,2,3,\ldots$. $\square$

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