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SOME RELATIONS SATISFIED BY ORTHOGONAL MATRIX POLYNOMIALS

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Abstract

The main purpose of this paper is to obtain some properties of orthogonal matrix polynomials. We derive identities for power series satisfied by Laguerre, Hermite and Gegenbauer matrix polynomials. Furthermore, for these matrix polynomials, we give raising operators.

Keywords: Laguerre matrix polynomials, Hermite matrix polynomials, Gegenbauer matrix polynomials, Power series, Raising operator.

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1. Introduction

"Orthogonal matrix polynomials" is a developing field whose development is attaining significant results from both the theoretical and practical examples. The property of orthogonality [9, 10], Rodrigues formula [3, 5], a second-order Sturm-Liouville differential equation [3], a three-term matrix recurrence [5, 6], a relation between different orthogonal matrix polynomials [17] are theoretical examples for orthogonal matrix polynomials. Beside, practical examples for matrix polynomials can be seen in statistics, group representation theory [12], scattering theory [11], differential equations [14, 15], Fourier series expansions [4], interpolation and quadrature [19, 20], splines [7] and medical imaging [2].

Some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials, see [1, 5, 13, 14, 16]. In [18], these matrix polynomials are orthogonal as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Hermite, Laguerre and Gegenbauer matrix polynomials have been introduced and studied in [13, 14, 15] for matrices in $\mathbb{C}^{r\times r}$.

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Our main aim in this paper is to prove new properties for orthogonal matrix polynomials. We calculate summations and derive raising operators for orthogonal matrix polynomials.

Throughout this paper, for a matrix A in $\mathbb{C}^{r\times r}$, its *spectrum* $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus in [8], it follows that: $f(A)g(A) = g(A)f(A)$, Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$, then $f(A)g(B) = g(B)f(A)$. We say that a matrix A in $\mathbb{C}^{r\times r}$ is *positive stable* if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$. Furthermore the identity matrix and the zero matrix of $\mathbb{C}^{r\times r}$ will be denoted by I and $\mathbf{0}$, respectively. From [16], for any matrix A in $\mathbb{C}^{r\times r}$, one can see

$$(A)_n = A(A+I)(A+2I)\cdots(A+(n-1)I);\; n \geq 1;\; (A)_0 = I.$$

For any matrix A in $\mathbb{C}^{r\times r}$, the authors exploited the following relation due to [16]

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n,\; |x| < 1.$$

2. Some identities for orthogonal matrix polynomials

Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $(-z) \notin \sigma(A)\ \forall z \in \mathbb{Z}^+$, and let λ be a complex parameter with $\Re(\lambda) > 0$. Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ are defined by

$$f(x,t,A) = (1-t)^{-A-I} \exp\left(\frac{-\lambda x t}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n;\; |t| < 1, \tag{2.1}$$

see [14].

2.1. Theorem. *Let n and k be positive integers with $k \geq 2$, then we have the equation*

$$\sum_{m=0}^{n} \left(\frac{[(k-1)(A+I)]_m}{m!} L_{n-m}^{(A,\lambda)}(kx) \right) = \sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A,\lambda)}(x) \cdots L_{n_k}^{(A,\lambda)}(x). \tag{2.2}$$

Proof. Taking partial derivatives of $f(xk,t,Ak) = (1-t)^{-Ak-I} \exp\left(\frac{-\lambda x k t}{1-t}\right)$, and using the generating matrix function for Laguerre matrix polynomials, we obtain

$$\frac{\partial f(xk,t,Ak)}{\partial x} = (-1)^1 (\lambda t k)(1-t)^{-Ak-2I} \exp\left(\frac{-\lambda x k t}{1-t}\right)$$

$$\frac{\partial^2 f(xk,t,Ak)}{\partial x^2} = (-1)^2 (\lambda t k)^2 (1-t)^{-Ak-3I} \exp\left(\frac{-\lambda x k t}{1-t}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial^{(k-1)} f(xk,t,Ak)}{\partial x^{(k-1)}}$$
$$= (-1)^{k-1} (\lambda tk)^{k-1} (1-t)^{-Ak-kI} \exp\left(\frac{-\lambda xkt}{1-t}\right)$$
$$= (-1)^{k-1} (\lambda tk)^{k-1} (1-t)^{-A(k-1)-(k-1)I} (1-t)^{-A-I} \exp\left(\frac{-\lambda xkt}{1-t}\right)$$
$$= (-1)^{k-1} (\lambda tk)^{k-1} (1-t)^{-A(k-1)-(k-1)I} \left[\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(kx)t^n\right]. \tag{2.3}$$

Furthermore, we get

$$(1-t)^{-(A+I)(k-1)} = \sum_{m=0}^{\infty} \frac{[(k-1)(A+I)]_m}{m!} t^m. \tag{2.4}$$

Combining (2.3) and (2.4), and taking $(n-m)$ instead of n, we can write

$$\frac{\partial^{(k-1)} f(xk,t,Ak)}{\partial x^{(k-1)}} = (-1)^{k-1} (\lambda tk)^{k-1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{[(k-1)(A+I)]_m}{m!} L_{n-m}^{(A,\lambda)}(kx)\right) t^n. \tag{2.5}$$

On the other hand, using (2.1), we also have

$$(1-t)^{-Ak-kI} \exp\left(\frac{-\lambda xkt}{1-t}\right)$$
$$= \left(\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n\right)^k \tag{2.6}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A,\lambda)}(x) \cdots L_{n_k}^{(A,\lambda)}(x)\right) t^n.$$

Combining (2.3), (2.5) and (2.6) and comparing the coefficients of t^n, we have the desired relation. □

2.2. Theorem. *For any positive integers n and k with $k \geq 2$, the Laguerre matrix polynomials satisfy the following equation:*

$$\sum_{n_1+n_2+\cdots+n_k=n} \int_0^{\infty} L_{n_1}^{(A,\lambda)}(x) \cdots L_{n_k}^{(A,\lambda)}(x) \exp(-k\lambda x)\, dx = \frac{1}{\lambda k} \frac{[(Ak+(k-1)I)]_n}{n!}. \tag{2.7}$$

Proof. Respectively, multiplying (2.6) by $\exp(-k\lambda x)$, integrating with respect to x over the interval $(0,\infty)$ and using power series of $(1-t)^{-Ak-kI+I}$, we may write

$$\sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+\cdots+n_k=n} \int_0^{\infty} L_{n_1}^{(A,\lambda)}(x) \cdots L_{n_k}^{(A,\lambda)}(x) \exp(-k\lambda x)\, dx\right) t^n$$
$$= \int_0^{\infty} (1-t)^{-Ak-kI} \exp\left(\frac{-\lambda xkt}{1-t}\right) \exp(-k\lambda x)\, dx$$
$$= (1-t)^{-Ak-kI} \int_0^{\infty} \exp\left(-k\lambda x \left[\frac{t}{1-t}+1\right]\right)\, dx$$
$$= \frac{(1-t)^{-Ak-kI+I}}{k\lambda} = \frac{1}{k\lambda} \left(\sum_{n=0}^{\infty} \frac{[(Ak+(k-1)I)]_n}{n!} t^n\right).$$

Comparing the coefficients of t^n, we complete the proof. □

2.3. Theorem. *Let $A_1, \cdots, A_k$ be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $(-z) \notin \sigma(A_i)\ \forall z \in \mathbb{Z}^+$, and let λ_i be a complex parameter with $\Re(\lambda_i) > 0$ for $i = 1, \ldots, k$. For any positive integers n and k with $k \geq 2$, we have*

$$\sum_{m=0}^{n} \left(\frac{\left[(\mathbf{A}-A_i)+(k-1)I\right]_m L_{n-m}^{(A_i,\lambda_1+\lambda_2+\cdots+\lambda_k)}(kx)}{m!} \right) = \sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A_1,\lambda_1)}(kx)\cdots L_{n_k}^{(A_k,\lambda_k)}(kx);\ i=1,2,...,k,$$

where $\mathbf{A} = A_1 + \cdots + A_k$, and the matrices $A_1, \ldots, A_k$, are assumed to be commutative.

Proof. Let

$$f(x,t,\mathbf{A}) = (1-t)^{-\mathbf{A}-I} \exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xt}{1-t} \right),$$

then with the help of partial derivatives with respect to x and the generating matrix function for Laguerre matrix polynomials, we can write

$$\begin{aligned} f(xk,t,\mathbf{A}) &= (1-t)^{-\mathbf{A}-I} \exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xkt}{1-t} \right) \\ \frac{\partial f(xk,t,\mathbf{A})}{\partial x} &= (-1)^1(\lambda_1+\lambda_2+\cdots+\lambda_k)(tk) \\ &\quad \times (1-t)^{-\mathbf{A}-2I} \exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xkt}{1-t} \right) \\ \frac{\partial^2 f(xk,t,\mathbf{A})}{\partial x^2} &= (-1)^2(\lambda_1+\lambda_2+\cdots+\lambda_k)^2 (tk)^2 \\ &\quad \times (1-t)^{-\mathbf{A}-3I} \exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xkt}{1-t} \right) \\ &\ \ \vdots \end{aligned}$$

$$\begin{aligned} \frac{\partial^{(k-1)} f(xk,t,\mathbf{A})}{\partial x^{(k-1)}} &= (-1)^{k-1}(\lambda_1+\lambda_2+\cdots+\lambda_k)^{k-1} (tk)^{k-1} \\ &\quad \times (1-t)^{-(\mathbf{A}-A_i)-(k-1)I} \left[\sum_{n=0}^{\infty} L_n^{(A_i,\lambda_1+\lambda_2+\cdots+\lambda_k)}(kx)t^n \right]. \end{aligned} \tag{2.8}$$

Using the power series of $(1-t)^{-(\mathbf{A}-A_i)-(k-1)I}$, we can write

$$\begin{aligned} &\frac{\partial^{(k-1)} f(xk,t,\mathbf{A})}{\partial x^{(k-1)}} \\ &\quad = (-1)^{k-1}(\lambda_1+\lambda_2+\cdots+\lambda_k)^{k-1} (tk)^{k-1} \\ &\quad\quad \times \left(\sum_{m=0}^{\infty} \frac{\left[(\mathbf{A}-A_i)+(k-1)I\right]_m}{m!} t^m \right) \left[\sum_{n=0}^{\infty} L_n^{(A_i,\lambda_1+\lambda_2+\cdots+\lambda_k)}(kx)t^n \right] \\ &\quad = (-1)^{k-1}(\lambda_1+\lambda_2+\cdots+\lambda_k)^{k-1} (tk)^{k-1} \\ &\quad\quad \times \left(\sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{\left[(\mathbf{A}-A_i)+(k-1)I\right]_m L_{n-m}^{(A_i,\lambda_1+\lambda_2+\cdots+\lambda_k)}(kx)}{m!} \right] t^n \right) \end{aligned} \tag{2.9}$$

for $1 \leq i \leq k$. On the other hand, we get

$$\begin{aligned}(1-t)^{-(A_1+A_2+\cdots+A_k)-kI}&\exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xkt}{1-t}\right)\\ &=\left(\sum_{n_1=0}^{\infty} L_{n_1}^{(A_1,\lambda_1)}(kx)t^{n_1}\right)\left(\sum_{n_2=0}^{\infty} L_{n_2}^{(A_2,\lambda_2)}(kx)t^{n_2}\right)\cdots\left(\sum_{n_k=0}^{\infty} L_{n_k}^{(A_k,\lambda_k)}(kx)t^{n_k}\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A_1,\lambda_1)}(kx)\cdots L_{n_k}^{(A_k,\lambda_k)}(kx)\right)t^n. \qquad (2.10)\end{aligned}$$

Combining (2.8), (2.9) and (2.10) and comparing the coefficients of t^n, we have desired relation. □

2.4. Theorem. *Let $A_1, \ldots, A_k$ be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $(-z) \notin \sigma(A_i)\ \forall z \in \mathbb{Z}^+$, and let λ_i be a complex parameter with $\Re(\lambda_i) > 0$ for $i = 1, \ldots, k$. For the Laguerre matrix polynomials, we have*

$$\begin{aligned}\sum_{n_1+n_2+\cdots+n_k=n}\int_0^{\infty} L_{n_1}^{(A_1,\lambda_1)}(kx)\cdots L_{n_k}^{(A_k,\lambda_k)}(kx)\exp(-(\lambda_1+\lambda_2+\cdots+\lambda_k)kx)\,dx\\ =\frac{1}{k(\lambda_1+\lambda_2+\cdots+\lambda_k)}\frac{[(\mathbf{A}+(k-1)I)]_n}{n!}\ \text{for } k,n\in\mathbb{N} \text{ with } k\geq 2,\end{aligned}$$

where $\mathbf{A} = A_1 + \cdots + A_k$, and the matrices $A_1, \ldots, A_k$, are assumed to be commutative.

Proof. Multiplying (2.10) by $\exp(-(\lambda_1 + \lambda_2 + \cdots + \lambda_k)kx)$, and then integrating with respect to x over the interval $(0, \infty)$, it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n}\int_0^{\infty} L_{n_1}^{(A_1,\lambda_1)}(kx)\cdots L_{n_k}^{(A_k,\lambda_k)}(kx)\right.\\ &\qquad\qquad\left.\times\exp(-(\lambda_1+\lambda_2+\cdots+\lambda_k)kx)\,dx\right)t^n \qquad (2.11)\\ &=\int_0^{\infty}(1-t)^{-\mathbf{A}-kI}\exp\left(\frac{-(\lambda_1+\lambda_2+\cdots+\lambda_k)xkt}{1-t}\right)\\ &\qquad\qquad\times\exp(-(\lambda_1+\lambda_2+\cdots+\lambda_k)kx)\,dx\\ &=(1-t)^{-\mathbf{A}-kI+I}\frac{1}{k(\lambda_1+\lambda_2+\cdots+\lambda_k)}.\end{aligned}$$

By using the power series expansion of $(1-t)^{-\mathbf{A}-kI+I}$ the above is equal to

$$\frac{1}{k(\lambda_1+\lambda_2+\cdots+\lambda_k)}\left(\sum_{n=0}^{\infty}\frac{[(\mathbf{A}+(k-1)I)]_n}{n!}\,t^n\right). \qquad (2.12)$$

Comparing the coefficients of t^n in (2.11) and (2.12) completes the proof. □

The relation presented in the following theorem is also of interest.

2.5. Theorem. *Let $A_1, \ldots, A_k$ be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $(-z) \notin \sigma(A_i)\ \forall z \in \mathbb{Z}^+$, and let λ_i be a complex parameter with $\Re(\lambda_i) > 0$ for $i = 1, \ldots, k$.*

Let n and k be positive integers with $k \geq 2$, then we have

$$\sum_{m=0}^{n} \frac{[(\mathbf{A}-A_i)+(k-1)I]_m \, L_{n-m}^{(A_i,\lambda_i)}\left(\frac{\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k}{\lambda_i}\right)}{m!} = \sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A_1,\lambda_1)}(x_1)\cdots L_{n_k}^{(A_k,\lambda_k)}(x_k);\ i=1,2,\ldots,k,$$

where $\mathbf{A} = A_1+\cdots+A_k$, and the matrices $A_1,\ldots,A_k$, are assumed to be commutative.

Proof. Let $f(x_1,\ldots,x_k,t,\mathbf{A})$ be defined as

$$f(x_1,\ldots,x_k,t,\mathbf{A}) = (1-t)^{-(A_1+A_2+\cdots+A_k)-I} \times \exp\left(\frac{-(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k)t}{1-t}\right).$$

Then, differentiating $f(x_1,\ldots,x_k,t,\mathbf{A})$ with respect to x_i, $(i=1,2,\ldots,k)$, we have

$$\frac{\partial f(x_1,\ldots,x_k,t,\mathbf{A})}{\partial x_i} = (-1)^1(\lambda_i t)(1-t)^{-(A_1+A_2+\cdots+A_k)-2I} \times \exp\left(\frac{-(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k)t}{1-t}\right)$$

$$\vdots \tag{2.13}$$

$$\frac{\partial^{(k-1)} f(x_1,\ldots,x_k,t,\mathbf{A})}{\partial x_i^{(k-1)}} = (-1)^{k-1}(\lambda_i t)^{k-1}(1-t)^{-(\mathbf{A}-A_i)-(k-1)I}(1-t)^{-A_i-I} \times \exp\left(\frac{-(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k)t}{1-t}\right).$$

On the other hand, we can write

$$\begin{aligned}\frac{\partial^{(k-1)} f(x_1,\ldots,x_k,t,\mathbf{A})}{\partial x_i^{(k-1)}} &= (-1)^{k-1}(\lambda_i t)^{k-1}\left(\sum_{m=0}^{\infty}\frac{[(\mathbf{A}-A_i)+(k-1)I]_m}{m!}t^m\right) \times \left[\sum_{n=0}^{\infty} L_n^{(A_i,\lambda_i)}\left(\frac{\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k}{\lambda_i}\right)t^n\right] \\ &= (-1)^{k-1}(\lambda_i t)^{k-1} \times \left(\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{[(\mathbf{A}-A_i)+(k-1)I]_m \, L_{n-m}^{(A_i,\lambda_i)}\left(\frac{\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k}{\lambda_i}\right)}{m!}\right]t^n\right).\end{aligned} \tag{2.14}$$

Also, we get following relation

$$\begin{aligned}&(1-t)^{-(A_1+A_2+\cdots+A_k)-kI}\exp\left(\frac{-(\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k)t}{1-t}\right) \\ &= \left(\sum_{n_1=0}^{\infty} L_{n_1}^{(A_1,\lambda_1)}(x_1)t^{n_1}\right)\left(\sum_{n_2=0}^{\infty} L_{n_2}^{(A_2,\lambda_2)}(x_2)t^{n_2}\right)\cdots\left(\sum_{n_k=0}^{\infty} L_{n_k}^{(A_k,\lambda_k)}(x_k)t^{n_k}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n} L_{n_1}^{(A_1,\lambda_1)}(x_1)\cdots L_{n_k}^{(A_k,\lambda_k)}(x_k)\right)t^n.\end{aligned} \tag{2.15}$$

If we use (2.13), (2.14) and (2.15), we complete the proof. □

2.6. Theorem. *Let $A_1, \dots, A_k$ be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $(-z) \notin \sigma(A_i)\ \forall z \in \mathbb{Z}^+$, and let λ_i be a complex parameter with $\Re(\lambda_i) > 0$ for $i = 1, \dots, k$. Then the Laguerre matrix polynomials satisfy the following equation:*

$$\sum_{n_1+n_2+\cdots+n_k=n} \left\{ \int_{x_1=0}^{\infty} \int_{x_2=0}^{\infty} \cdots \int_{x_k=0}^{\infty} L_{n_1}^{(A_1,\lambda_1)}(x_1) \cdots L_{n_k}^{(A_k,\lambda_k)}(x_k) \times \exp\left(-(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)\right) dx_k \cdots dx_1 \right\}$$
$$= \frac{(\mathbf{A})_n}{n!\lambda_1\lambda_2\cdots\lambda_k}$$

where $\mathbf{A} = A_1 + \cdots + A_k$, and the matrices $A_1, \dots, A_k$, are assumed to be commutative.

Proof. Successively multiplying (2.15) by $\exp\left(-(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)\right)$, integrating over the domain

$$\Omega = \left\{(x_1,x_2,\dots,x_k) : 0 < x_i < \infty,\ i = 1, 2, \dots, k\right\},$$

and then using the power series of $(1-t)^{-\mathbf{A}}$, we may write

$$\int_{x_1=0}^{\infty} \int_{x_2=0}^{\infty} \cdots \int_{x_k=0}^{\infty} \left\{ (1-t)^{-\mathbf{A}-kI} \exp\left(\frac{-(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)t}{1-t} \right) \times \exp\left(-(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)\right) dx_k \cdots dx_1 \right\}$$
$$= \frac{(1-t)^{-\mathbf{A}}}{\lambda_1.\lambda_2 ... \lambda_k}$$
$$= \frac{1}{\lambda_1.\lambda_2\cdots\lambda_k} \left(\sum_{n=0}^{\infty} \frac{(\mathbf{A})_n}{n!} t^n \right).$$

It is now enough to compare the coefficients of t^n. □

Hermite matrix polynomials $H_n(x, A)$ are defined by

$$f(x,t,A) = \exp(\sqrt{2A}xt - t^2 I) = \sum_{n=0}^{\infty} \frac{H_n(x,A)}{n!} t^n, \tag{2.16}$$

where A is a positive stable matrix in $\mathbb{C}^{r\times r}$ (see [13]).

2.7. Theorem. *Hermite matrix polynomials satisfy*

$$\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(\sqrt{2A}kx)^{n-2s}(-k)^s}{(n-2s)!\ s!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x,A)\cdots H_{n_k}(x,A)}{n_1!\cdots n_k!} \quad \textit{for } k \in \mathbb{N}.$$

Proof. For $f(x\sqrt{k}, t\sqrt{k}, A)$, we can find

$$f(x\sqrt{k}, t\sqrt{k}, A) = \exp(\sqrt{2A}kxt - kt^2 I)$$
$$= \exp(\sqrt{2A}kxt)\exp(-kt^2 I).$$

Using the power series and taking $(n-2s)$ instead of n, we have

$$
\begin{aligned}
f(x\sqrt{k}, t\sqrt{k}, A) &= \left(\sum_{n=0}^{\infty} \frac{(\sqrt{2A}kxt)^n}{n!}\right)\left(\sum_{s=0}^{\infty} \frac{(-kt^2)^s}{s!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(\sqrt{2A}kx)^{n-2s}(-k)^s}{(n-2s)!\ s!}\right) t^n. \qquad (2.17)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
\exp(\sqrt{2A}kxt - kt^2 I) &= \left[\exp(\sqrt{2A}xt - t^2 I)\right]^k = \left(\sum_{n=0}^{\infty} \frac{H_n(x,A)}{n!} t^n\right)^k \\
&= \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x,A)\cdots H_{n_k}(x,A)}{n_1!\cdots n_k!}\right) t^n. \qquad (2.18)
\end{aligned}
$$

If we combine (2.17) and (2.18), we complete the proof. □

2.8. Theorem. *For $k \in \mathbb{N}$, we get the following relation*

$$
\begin{aligned}
&\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)\right]^{n-2s}(-k)^s}{(n-2s)!\,s!} \\
&\qquad = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x_1,A_1)\cdots H_{n_k}(x_k,A_k)}{n_1!\cdots n_k!},
\end{aligned}
$$

where $A_1,\ldots,A_k$ are positive stable matrices in $\mathbb{C}^{r\times r}$ which commute with one another.

Proof. Let $g(x_1,\ldots,x_k,t,A_1,\ldots,A_k) = \exp\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)t - t^2 I\right]$. For $g\left(\frac{x_1}{\sqrt{k}},\ldots,\frac{x_k}{\sqrt{k}}, t\sqrt{k}, A_1,\ldots,A_k\right)$, using the power series and taking $(n-2s)$ instead of n, we can write

$$
\begin{aligned}
&g\left(\frac{x_1}{\sqrt{k}},\ldots,\frac{x_k}{\sqrt{k}}, t\sqrt{k}, A_1,\ldots,A_k\right) \\
&\qquad = \exp\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)t - t^2 kI\right] \\
&\qquad = \left(\sum_{n=0}^{\infty} \frac{\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)t\right]^n}{n!}\right)\left(\sum_{s=0}^{\infty}\frac{(-kt^2)^s}{s!}\right) \\
&\qquad = \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)\right]^{n-2s}(-k)^s}{(n-2s)!\,s!}\right) t^n. \qquad (2.19)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&\exp\left[\left(\sqrt{2A_1}x_1 + \cdots + \sqrt{2A_k}x_k\right)t - t^2 kI\right] \\
&\qquad = \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x_1,A_1)\cdots H_{n_k}(x_k,A_k)}{n_1!\cdots n_k!}\right) t^n. \qquad (2.20)
\end{aligned}
$$

Combining (2.19) and (2.20), the proof is complete. □

Gegenbauer matrix polynomials $C_n^A(x)$ are defined by

$$f(x,t,A) = (1-2xt+t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n,$$

where A is a matrix in $\mathbb{C}^{r\times r}$ satisfying $\left(\frac{-z}{2}\right) \notin \sigma(A)\ \forall\, z \in \mathbb{Z}^+ \cup \{0\}$ (see [15]).

2.9. Theorem. *For any positive integer k we derive*

$$\sum_{n_1+n_2+\cdots+n_k=n} C_{n_1}^A(x)\cdots C_{n_k}^A(x) = \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^s (Ak)_{n-s}(2x)^{n-2s}}{s!\,(n-2s)!}.$$

Proof. Using the power series of $(1-2xt+t^2)^{-Ak}$, and making the necessary arrangements, we have

$$\begin{aligned}(1-2xt+t^2)^{-Ak} &= \sum_{n=0}^{\infty} \frac{(Ak)_n}{n!}(2xt-t^2)^n \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(Ak)_n(2x)^{n-s}(-1)^s(t)^{n+s}}{s!\,(n-s)!} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^s(Ak)_{n-s}(2x)^{n-2s}(t)^n}{s!\,(n-2s)!}. \qquad (2.21)\end{aligned}$$

In addition to this, we can write

$$(1-2xt+t^2)^{-Ak} = \sum_{n=0}^{\infty}\left(\sum_{n_1+n_2+\cdots+n_k=n} C_{n_1}^A(x)\cdots C_{n_k}^A(x)\right)t^n. \qquad (2.22)$$

From (2.21) and (2.22), we complete the proof. □

2.10. Theorem.

$$\sum_{n_1=0}^{n}\cdots\sum_{n_{k-1}=0}^{n-n_1-\cdots-n_{k-2}} C_{n-(n_1+n_2+\cdots+n_{k-1})}^{A_1}(x)C_{n_1}^{A_2}(x)\cdots C_{n_{k-1}}^{A_k}(x) = \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^s(A_1+A_2+\cdots+A_k)_{n-s}(2x)^{n-2s}}{s!\,(n-2s)!}$$

where $A_1,\ldots,A_k$, $k\in\mathbb{N}$, are matrices in $\mathbb{C}^{r\times r}$ satisfying $\left(\frac{-z}{2}\right)\notin\sigma(A_i)\ \forall\, z\in\mathbb{Z}^+\cup\{0\}$, which commute with one another.

Proof. Using the power series of $(1-2xt+t^2)^{-(A_1+A_2+\cdots+A_k)}$, and then taking $(n-s)$ instead of n, we obtain

$$\begin{aligned}&(1-2xt+t^2)^{-(A_1+A_2+\cdots+A_k)} \\ &\quad= \sum_{n=0}^{\infty}\frac{(A_1+A_2+...+A_k)_n}{n!}(2xt-t^2)^n \\ &\quad= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(A_1+A_2+\cdots+A_k)_n(2x)^{n-s}(-1)^s t^{n+s}}{s!\,(n-s)!} \\ &\quad= \sum_{n=0}^{\infty}\sum_{s=0}^{\left[\frac{n}{2}\right]}\frac{(A_1+A_2+\cdots+A_k)_{n-s}(2x)^{n-2s}(-1)^s t^n}{s!\,(n-2s)!}. \qquad (2.23)\end{aligned}$$

On the other hand, we get

$$\begin{aligned}(1-2xt+t^2)^{-(A_1+A_2+\cdots+A_k)} \\ &= \left(\sum_{n_1=0}^{\infty} C_{n_1}^{A_1}(x)t^{n_1}\right)\cdots\left(\sum_{n_k=0}^{\infty} C_{n_k}^{A_k}(x)t^{n_k}\right) \\ &= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n}\cdots\sum_{n_{k-1}=0}^{n-n_1-\cdots-n_{k-2}} C_{n-(n_1+n_2+\cdots+n_{k-1})}^{A_1}(x)C_{n_1}^{A_2}(x)\cdots C_{n_{k-1}}^{A_k}(x)t^n. \end{aligned} \tag{2.24}$$

It is enough now to use (2.23) and (2.24). □

3. Raising operators for matrix polynomials

3.1. Lemma. [14] *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $\Re(z) > -1$ for every $z \in \sigma(A)$, and let λ be a complex parameter with $\Re(\lambda) > 0$. Then it follows that for any fixed matrix polynomial $P(t)$,*

$$\lim_{t\to 0^+} e^{-\lambda t}t^{A+I}P(t) = \mathbf{0} \text{ and } \lim_{t\to\infty} e^{-\lambda t}t^{A+I}P(t) = \mathbf{0},$$

where $t^{A+I} = \exp((A+I)\ln t)$ for $t > 0$. □

3.2. Theorem. *The raising operator for Laguerre matrix polynomials is*

$$\frac{d}{dx}\left[L_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}\right] = (n+1)x^{A-I}e^{-\lambda x}L_{n+1}^{(A-I,\lambda)}(x),$$

where A is positive stable matrix in $\mathbb{C}^{r\times r}$ and $\Re(\lambda) > 0$.

Proof. To prove the theorem, we start by taking the derivative of $L_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}$ with respect to x

$$\begin{aligned}\frac{d}{dx}\left[L_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}\right] \\ &= \frac{d}{dx}\left[L_n^{(A,\lambda)}(x)\right]x^Ae^{-\lambda x} + L_n^{(A,\lambda)}(x)\left[Ax^{A-I}e^{-\lambda x} - \lambda x^Ae^{-\lambda x}\right] \\ &= x^{A-I}e^{-\lambda x}\left[x\frac{d}{dx}\left(L_n^{(A,\lambda)}(x)\right) + (A-\lambda x)L_n^{(A,\lambda)}(x)\right] \\ &= x^{A-I}e^{-\lambda x}\Theta_{n+1}(x),\end{aligned}$$

where $\Theta_{n+1}(x) := x\frac{d}{dx}\left(L_n^{(A,\lambda)}(x)\right) + (A-\lambda x)L_n^{(A,\lambda)}(x)$. In addition to this, using Lemma 3.1 we can write

$$\begin{aligned}\int_0^\infty x^{A-I}e^{-\lambda x}\Theta_{n+1}(x)x^k\,dx &= \int_0^\infty \frac{d}{dx}\left[L_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}\right]x^k\,dx,\ k=0,1,\ldots,n \\ &= x^kL_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}\Big|_0^\infty - \int_0^\infty kL_n^{(A,\lambda)}(x)x^Ae^{-\lambda x}x^{k-1}\,dx \\ &= \mathbf{0}.\end{aligned}$$

Therefore we should have

$$\Theta_{n+1}(x) = CL_{n+1}^{(A-I,\lambda)}(x),$$

because the family of polynomials are orthogonal with respect to the weight function $x^{A-I}e^{-\lambda x}$ over the interval $(0,\infty)$, which is unique up to a constant. Comparing the coefficients of x^{n+1}, we have $C = n+1$. Thus, the theorem is proved. □

Let us consider the operator $R = \frac{d}{dx}\left[x^{A}e^{-\lambda x}\right]$. Taking $A+nI$ instead of A in this operator and using $L_0^{(A+nI,\lambda)}(x) = I$, we have

$$\frac{d}{dx}\left[x^{A+nI}e^{-\lambda x}I\right] = x^{A+(n-1)I}e^{-\lambda x}L_1^{(A+(n-1)I,\lambda)}(x)$$
$$\frac{d^2}{dx^2}\left[x^{A+nI}e^{-\lambda x}I\right] = 2x^{A+(n-2)I}e^{-\lambda x}L_2^{(A+(n-2)I,\lambda)}(x)$$
$$\vdots$$
$$\frac{d^n}{dx^n}\left[x^{A+nI}e^{-\lambda x}I\right] = n!x^{A}e^{-\lambda x}L_n^{(A,\lambda)}(x).$$

Thus we obtain the Rodrigues formula for Laguerre matrix polynomials,

$$L_n^{(A,\lambda)}(x) = \frac{1}{n!}x^{-A}e^{\lambda x}\frac{d^n}{dx^n}\left[x^{A+nI}e^{-\lambda x}\right],$$

which is given in [14].

3.3. Lemma. [13] *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition $\Re(z) > 0$ for every $z \in \sigma(A)$. Then it follows that for any fixed matrix polynomial $P(x)$,*

$$\lim_{x\to\pm\infty} P(x)\ e^{-\frac{A\ x^2}{2}} = \mathbf{0}. \qquad \square$$

3.4. Theorem. *The raising operator for Hermite matrix polynomials is*

$$\frac{d}{dx}\left[H_n(x,A)e^{-\frac{A}{2}x^2}\right] = -\sqrt{\tfrac{A}{2}}e^{-\frac{A}{2}x^2}H_{n+1}(x,A),$$

where A is positive stable matrix in $\mathbb{C}^{r\times r}$.

Proof. To prove the theorem, we start by taking the derivative of $H_n(x,A)e^{-\frac{A}{2}x^2}$ with respect to x,

$$\begin{aligned}\frac{d}{dx}\left[H_n(x,A)e^{-\frac{A}{2}x^2}\right] &= e^{-\frac{A}{2}x^2}\frac{d}{dx}\left[H_n(x,A)\right] + H_n(x,A)\left[-Axe^{-\frac{A}{2}x^2}\right]\\ &= e^{-\frac{A}{2}x^2}\left[\frac{d}{dx}\left(H_n(x,A)\right) - xAH_n(x,A)\right]\\ &= e^{-\frac{A}{2}x^2}\Theta_{n+1}(x),\end{aligned}$$

where $\Theta_{n+1}(x) := \frac{d}{dx}\left(H_n(x,A)\right) - xAH_n(x,A)$. On the other hand, using Lemma 3.2, we get

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-\frac{A}{2}x^2}\Theta_{n+1}(x)H_k(x,A)\,dx\\ &= \int_{-\infty}^{\infty}\frac{d}{dx}\left[H_n(x,A)e^{-\frac{A}{2}x^2}\right]H_k(x,A)\,dx,\ k=0,1,\dots,n\\ &= \left[H_k(x,A)H_n(x,A)e^{-\frac{A}{2}x^2}\right]\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}H_n(x,A)e^{-\frac{A}{2}x^2}\frac{d}{dx}\left[H_k(x,A)\right]dx\\ &= -k\sqrt{2A}\int_{-\infty}^{\infty}H_n(x,A)e^{-\frac{A}{2}x^2}H_{k-1}(x,A)\,dx = \mathbf{0}.\end{aligned}$$

Therefore, we should have $\Theta_{n+1}(x) = CH_{n+1}(x, A)$ because the family of polynomials is orthogonal with respect to the weight function $e^{-\frac{A}{2}x^2}$ over the interval $(-\infty, \infty)$, which is unique up to a constant. Comparing the coefficients of x^{n+1} we have $C = -\sqrt{\frac{A}{2}}$. Thus, the theorem is proved. $\square$

Let us consider the operator $R = \frac{d}{dx}\left[e^{-\frac{A}{2}x^2}\right]$. Applying this operator to $H_0(x, A) = I$, we have

$$\begin{aligned}
\frac{d}{dx}\left[e^{-\frac{A}{2}x^2}I\right] &= -\sqrt{\tfrac{A}{2}}e^{-\frac{A}{2}x^2}H_1(x, A)\\
\frac{d^2}{dx^2}\left[e^{-\frac{A}{2}x^2}I\right] &= -\sqrt{\tfrac{A}{2}}\frac{d}{dx}\left[e^{-\frac{A}{2}x^2}H_1(x, A)\right] = \tfrac{A}{2}e^{-\frac{A}{2}x^2}H_2(x, A)\\
&\vdots\\
\frac{d^n}{dx^n}\left[e^{-\frac{A}{2}x^2}I\right] &= (-1)^n\left(\sqrt{\tfrac{A}{2}}\right)^n e^{-\frac{A}{2}x^2}H_n(x, A).
\end{aligned}$$

Thus, by means of the raising operator the Rodrigues formula for Hermite matrix polynomials is

$$H_n(x, A) = (-1)^n\left(\sqrt{\tfrac{A}{2}}\right)^{-n} e^{\frac{A}{2}x^2}\frac{d^n}{dx^n}\left[e^{-\frac{A}{2}x^2}\right],$$

as given in [13].

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