CONSTRUCTION OF A COMPLEX JACOBI MATRIX FROM TWO-SPECTRA

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Abstract

In this paper we study the inverse spectral problem for two-spectra of finite order complex Jacobi matrices (tri-diagonal matrices). The problem is to reconstruct the matrix using two sets of eigenvalues, one for the original Jacobi matrix and one for the matrix obtained by deleting the first column and the first row of the Jacobi matrix. An explicit procedure of reconstruction of the matrix from the two-spectra is given.

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1. Introduction

An $N \times N$ complex Jacobi matrix is a matrix of the form

$$(1.1) \quad J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix},$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$(1.2) \quad a_n, \ b_n \in \mathbb{C}, \ a_n \neq 0.$$

The general inverse spectral problem is to reconstruct the matrix given some of its spectral characteristics (spectral data).

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In the real case

(1.3) $a_n, \ b_n \in \mathbb{R}, \ a_n \neq 0,$

many versions of the inverse spectral problem for J have been investigated in the litera-
ture, see [1] and references given therein.

In [9, 10] Hochstadt investigated the following version of the inverse problem for two-
spectra. Let J_1 be the $(N-1) \times (N-1)$ matrix obtained from J by deleting its first row
and first column. Let $\{\lambda_k\}$ and $\{\mu_k\}$ be the eigenvalues of J and J_1, respectively. The
eigenvalues of a Jacobi matrix with entries satisfying (1.3) are real and distinct, and the
eigenvalues of J and J_1 interlace:

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \ldots < \mu_{N-1} < \lambda_N.$$

The author showed in [9, 10] that if the $\{\lambda_k\}$ are the eigenvalues of some Jacobi matrix
of the form (1.1) with the entries

$$a_n, \ b_n \in \mathbb{R}, \ a_n > 0,$$

and if $\{\mu_k\}$ are the eigenvalues of the corresponding J_1, then there is precisely one such
matrix with these $\{\lambda_k, \mu_k\}$, and gave a constructive method for calculating the entries
of J in terms of the given eigenvalues. The same problem was subsequently considered
in [5, 8, 4], where different proofs of the uniqueness and existence of J were offered (for
the case of infinite Jacobi matrices see [2, 6, 3]).

In the present paper, we consider the above formulated problem about two-spectra for
complex Jacobi matrices of the form (1.1) with entries satisfying (1.2). In the complex
case in general the matrix J is no longer selfadjoint and therefore its eigenvalues may be
complex and multiple. Recently, the author introduced in [7] the concept of spectral data
for complex Jacobi matrices of finite order and solved the inverse problem of recovering
the matrix from its spectral data. Here we show that the inverse problem for two-spectra
of complex Jacobi matrices can be solved by reducing it to the inverse problem for spectral
data.

2. Spectral data and the inverse problem

In this auxiliary section we follow the author's paper [7]. Let J be a complex Jacobi
matrix of the form (1.1) with entries satisfying (1.2). Further, let $R(\lambda) = (J - \lambda I)^{-1}$ be
the resolvent of the matrix J (by I we denote the identity matrix of needed dimension)
and let e_0 be the N-dimensional column vector with the components $1, 0, \ldots, 0$. The
rational function

(2.1) $w(\lambda) = -\left((R(\lambda)e_0, e_0\right) = \left((\lambda I - J)^{-1}e_0, e_0\right)$

we call the *resolvent function* of the matrix J, where $(\cdot, \cdot)$ denotes the standard inner
product in $\mathbb{C}^N$.

Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J, and by $m_1, \ldots, m_p$
their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J - \lambda I)$,
so that $1 \leq p \leq N$ and $m_1 + \ldots + m_p = N$. We can decompose the rational function
$w(\lambda)$ into partial fractions to get

(2.2) $w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \dfrac{\beta_{kj}}{(\lambda - \lambda_k)^j},$

where β_{kj} are some complex numbers uniquely determined by the matrix J.

2.1. Definition. The collection of quantities

(2.3) $\{\lambda_k,\ \beta_{kj}\ (j = 1, \ldots, m_k;\ k = 1, \ldots, p)\},$

we call the spectral data of the matrix J. For each $k \in \{1, \ldots, p\}$ the (finite) sequence

$$\{\beta_{k1}, \ldots, \beta_{km_k}\}$$

we call the normalizing chain (of the matrix J) associated with the eigenvalue λ_k.

Determining the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

Let us indicate a convenient way to compute the spectral data of complex Jacobi matrices. For this purpose we should describe the resolvent function $w(\lambda)$ of the Jacobi matrix. Given a Jacobi matrix J of the form (1.1) with the entries (1.2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the second order linear difference equation

(2.4) $a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \{0, 1, \ldots, N-1\},\ a_{-1} = a_{N-1} = 1,$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0.$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of equation (2.4) satisfying the initial conditions

(2.5) $P_{-1}(\lambda) = 0,\quad P_0(\lambda) = 1;$

(2.6) $Q_{-1}(\lambda) = -1,\ Q_0(\lambda) = 0.$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n called a polynomial of the first kind, and $Q_n(\lambda)$ is a polynomial of degree $n-1$ known as a polynomial of second kind. The equality

(2.7) $\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda)$

holds, so that the eigenvalues of the matrix J coincide with the roots of the polynomial $P_N(\lambda)$.

In [7] it is shown that the entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \leq n \leq m \leq N-1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \leq m \leq n \leq N-1, \end{cases}$$

where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}.$$

Therefore according to (2.1) and using initial conditions (2.5), (2.6), we get

(2.8) $w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}.$

Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct roots of the polynomial $P_N(\lambda)$ (which coincide by (2.7) with the eigenvalues of the matrix J), and by $m_1, \ldots, m_p$ their multiplicities, respectively:

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p},$$

where c is a constant. We have $1 \leq p \leq N$ and $m_1 + \cdots + m_p = N$. Therefore by (2.8) the decomposition (2.2) can be obtained by rewriting the rational function $Q_N(\lambda)/P_N(\lambda)$ as the sum of partial fractions.

We can also get another convenient representation for the resolvent function as follows. If we delete the first row and the first column of the matrix J given in (1.1), then we get the new matrix

$$J_1 = \begin{bmatrix} b_0^{(1)} & a_0^{(1)} & 0 & \cdots & 0 & 0 & 0 \\ a_0^{(1)} & b_1^{(1)} & a_1^{(1)} & \cdots & 0 & 0 & 0 \\ 0 & a_1^{(1)} & b_2^{(1)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{N-4}^{(1)} & a_{N-4}^{(1)} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-4}^{(1)} & b_{N-3}^{(1)} & a_{N-3}^{(1)} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-3}^{(1)} & b_{N-2}^{(1)} \end{bmatrix},$$

where

$$a_n^{(1)} = a_{n+1}, \ n \in \{0, 1, \ldots, N-3\},$$
$$b_n^{(1)} = b_{n+1}, \ n \in \{0, 1, \ldots, N-2\}.$$

The matrix J_1 is called the *first truncated matrix* (with respect to the matrix J).

2.2. Theorem. *The equality*

$$w(\lambda) = -\frac{\det(J_1 - \lambda I)}{\det(J - \lambda I)}$$

holds.

Proof. Let us denote the polynomials of the first and the second kinds, corresponding to the matrix J_1, by $P_n^{(1)}(\lambda)$ and $Q_n^{(1)}(\lambda)$, respectively. It is easily seen that

(2.9)
$$P_n^{(1)}(\lambda) = a_0 Q_{n+1}(\lambda), \ n \in \{0, 1, \ldots, N-1\},$$
$$Q_n^{(1)}(\lambda) = \frac{1}{a_0}\{(\lambda - b_0)Q_{n+1}(\lambda) - P_{n+1}(\lambda)\}, \ n \in \{0, 1, \ldots, N-1\}.$$

Indeed, both sides of each of these equalities are solutions of the same difference equation

$$a_{n-1}^{(1)} y_{n-1} + b_n^{(1)} y_n + a_n^{(1)} y_{n+1} = \lambda y_n, \ n \in \{0, 1, \ldots, N-2\}, \ a_{N-2}^{(1)} = 1,$$

and the sides coincide for $n = -1$ and $n = 0$. Therefore the equalities hold by the uniqueness theorem for solutions.

Consequently, taking into account (2.7) for the matrix J_1 instead of J and using (2.9), we find

(2.10)
$$\det(J_1 - \lambda I) = (-1)^{N-1} a_0^{(1)} a_1^{(1)} \cdots a_{N-3}^{(1)} P_{N-1}^{(1)}(\lambda)$$
$$= (-1)^{N-1} a_1 \cdots a_{N-2} a_0 Q_N(\lambda).$$

Comparing this with (2.7), we get

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = -\frac{\det(J^{(1)} - \lambda I)}{\det(J - \lambda I)}$$

so that the statement of the theorem follows by (2.8). □

The inverse spectral problem is stated as follows:

(i) To see if it is possible to reconstruct the matrix J given its spectral data (2.3). If it is possible, to describe the reconstruction procedure.

(ii) To find the necessary and sufficient conditions for a given collection (2.3) to be the spectral data for some matrix J of the form (1.1) with entries belonging to the class (1.2).

This problem was recently solved by the author in [7] and the following results were established.

Given the collection (2.3) define the numbers

$$(2.11) \quad s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \ l = 0, 1, 2, \ldots,$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j-1 > l$. Using the numbers s_l let us introduce the determinants

$$(2.12) \quad D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \ n = 0, 1, \ldots, N.$$

2.3. Theorem. *Let an arbitrary collection (2.3) of complex numbers be given, where $\lambda_1, \ldots, \lambda_p$, $(1 \leq p \leq N)$ are distinct, $1 \leq m_k \leq N$, and $m_1 + \ldots + m_p = N$. In order for this collection to be the spectral data for some Jacobi matrix J of the form (1.1) with entries belonging to the class (1.2), it is necessary and sufficient that the following two conditions be satisfied:*

(i) $\sum_{k=1}^{p} \beta_{k1} = 1$;

(ii) $D_n \neq 0$ *for* $n \in \{1, 2, \ldots, N-1\}$, *and* $D_N = 0$, *where* D_n *is the determinant defined by (2.12), (2.11).* □

Under the conditions of Theorem 2.3, the entries a_n and b_n of the matrix J for which the collection (2.3) is spectral data, are recovered by the formulas

$$(2.13) \quad a_n = \frac{\pm\sqrt{D_{n-1}D_{n+1}}}{D_n}, \ n \in \{0, 1, \ldots, N-2\}, \ D_{-1} = 1,$$

$$(2.14) \quad b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \ n \in \{0, 1, \ldots, N-1\}, \ \Delta_{-1} = 0, \ \Delta_0 = s_1,$$

where D_n is defined by (2.12) and (2.11), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

2.4. Remark. It follows from the above solution of the inverse problem that the matrix (1.1) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (2.13) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$. Namely, let $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ be a given finite sequence, where for $n \in \{1, 2, \ldots, N-1\}$ each σ_n is $+$ or $-$. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (2.13) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ of signs $+$ and $-$. Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

3. Inverse problem for two-spectra

Let J be an $N \times N$ Jacobi matrix of the form (1.1) with entries satisfying (1.2). Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \ldots, m_p$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J - \lambda I)$, so

that $1 \leq p \leq N$ and $m_1 + \cdots + m_p = N$. Further, let J_1 be the $(N-1) \times (N-1)$ matrix obtained from J by deleting its first row and first column. Denote by $\mu_1, \ldots, \mu_q$ all the distinct eigenvalues of the matrix J_1, and by $n_1, \ldots, n_q$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J_1 - \lambda I)$, so that $1 \leq q \leq N-1$ and $n_1 + \cdots + n_q = N-1$.

The collections

(3.1) $\{\lambda_k, \ m_k \ (k=1, \ldots, p)\}$ and $\{\mu_k, \ n_k \ (k=1, \ldots, q)\}$

form the spectra of the matrices J and J_1, respectively. We call these collections the *two-spectra* of the matrix J.

The inverse problem for two-spectra consists in the reconstruction of the matrix J by its two-spectra.

We will reduce the inverse problem for two-spectra to the inverse problem for the spectral data solved above in Section 2.

First let us study some necessary properties of the two-spectra of the Jacobi matrix J.

Having the matrix J consider the difference equation (2.4) and let $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ be the solutions of this equation satisfying the initial conditions (2.5) and (2.6), respectively. Then by (2.7) and (2.10), we have

(3.2) $\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda),$

(3.3) $\det(J_1 - \lambda I) = (-1)^{N-1} a_0 a_1 \cdots a_{N-2} Q_N(\lambda),$

so that the eigenvalues and their multiplicities of the matrix J coincide with the roots and their multiplicities of the polynomial $P_N(\lambda)$ and the eigenvalues and their multiplicities of the matrix J_1 coincide with the roots and their multiplicities of the polynomial $Q_N(\lambda)$. The equation

(3.4) $P_{N-1}(\lambda)Q_N(\lambda) - P_N(\lambda)Q_{N-1}(\lambda) = 1$

holds (see [7, Lemma 4]).

3.1. Lemma. *The matrices J and J_1 have no common eigenvalues, that is, $\lambda_k \neq \mu_j$ for all possible values of k and j.*

Proof. Suppose that λ is an eigenvalue of the matrices J and J_1. Then by (3.2) and (3.3) we have $P_N(\lambda) = Q_N(\lambda) = 0$. But this is impossible by (3.4). $\qquad\square$

In virtue of (2.2) and Theorem 2.2, we have

$$\sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j} = \frac{\prod\limits_{i=1}^{q} (\lambda - \mu_i)^{n_i}}{\prod\limits_{l=1}^{p} (\lambda - \lambda_l)^{m_l}}.$$

Hence,

$$\beta_{kj} = \frac{1}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{\prod\limits_{i=1}^{q} (\lambda - \mu_i)^{n_i}}{\prod\limits_{l=1}^{p} (\lambda - \lambda_l)^{m_l}} \right].$$

Therefore

$$(3.5) \qquad \beta_{kj} = \frac{1}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \frac{\prod\limits_{i=1}^{q} (\lambda - \mu_i)^{n_i}}{\prod\limits_{\substack{l=1 \\ l \neq k}}^{p} (\lambda - \lambda_l)^{m_l}}$$

$$(j = 1, \ldots, m_k; \ k = 1, \ldots, p).$$

Formula (3.5) shows that the normalizing numbers β_{kj} of the matrix J are determined uniquely by the two-spectra $\{\lambda_k, \ m_k \ (k = 1, \ldots, p)\}$ and $\{\mu_k, \ n_k \ (k = 1, \ldots, q)\}$ of this matrix. Since the inverse problem for spectral data is solved uniquely up to the signs of the off-diagonal elements of the recovered matrix (see Remark 2.4 made above in Section 2), we get the following result.

3.2. Theorem. *The two spectra in (3.1) determine the matrix J uniquely up to signs of the off-diagonal elements of J.* $\qquad \square$

The procedure of reconstruction of the matrix J from its two-spectra consists in the following. If we are given the two-spectra

$$\{\lambda_k, \ m_k \ (k = 1, \ldots, p)\} \text{ and } \{\mu_k, \ n_k \ (k = 1, \ldots, q)\},$$

we find the quantities β_{kj} from (3.5) and then solve the inverse problem with respect to the spectral data

$$\{\lambda_k, \ \beta_{kj} \ (j = 1, \ldots, m_k; \ k = 1, \ldots, p)\}$$

to recover the matrix J by using formulas (2.13) and (2.14).

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