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THEORY, METHODS AND APPLICATIONS OF INITIAL TIME DIFFERENCE BOUNDEDNESS CRITERIA AND LAGRANGE STABILITY IN TERMS OF TWO MEASURES FOR NONLINEAR SYSTEMS

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Abstract

This paper investigates the initial time difference equi-boundedness criteria in terms of two measures, initial time difference boundedness and Lagrange Stability in terms of two measures. These are unified with Lyapunov-like functions to establish a variational comparison result. We support our new results with analytic examples and numerical applications.

Keywords: Initial time difference, Perturbed differential systems, Boundedness and Lagrange stability, In terms of two measures, Variational comparison result.

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1. Introduction

The problems of modern society are both complex and multidisciplinary [1, 2, 3, 4]. It is now recognized that the concept of Lyapunov function [1, 4, 7, 9] can be employed to investigate various qualitative and quantitative properties of dynamic systems. Lyapunov functions serve as a vehicle to transform a given complicated differential system into a relatively simpler system, and as a result it is enough to study the properties of solutions of the simpler system. The application of Lyapunov's second method in boundedness theory [2, 3, 6, 10, 11] has the advantage of not requiring knowledge of the solutions. In order to unify a variety of known concepts of stability and boundedness, it is beneficial to employ two different measures [3, 6, 8, 10, 11] and obtain criteria in terms of these

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measures. In this paper we apply these effective and fruitful techniques with Lyapunov-like functions [3, 4, 7, 9] to obtain boundedness and Lagrange stability criteria [2, 3, 4] for nonlinear differential systems with an initial time difference [5, 7, 8, 9, 10, 11]. We give boundedness and Lagrange stability criteria for a perturbed differential system with respect to an unperturbed differential system that differs both in initial time and initial position in terms of two measures [8, 10, 11] by applying an initial time difference variational comparison result [1, 8, 10].

In Section 2, we give the definitions necessary for our analysis of a new notion of initial time difference boundedness and Lagrange stability in terms of two measures. In Section 3 we have investigated that initial time difference (h_0, h)-boundedness and Lagrange Stability related to Comparison Equations. In Section 4, we present a variational comparison result of boundedness and Lagrange Stability in terms of two measures for vector Lyapunov-like functions. In Section 5 we give two examples that apply the main results of Section 4. Finally, in Section 6, we have investigated and focused on numerical methods that how could be applied to obtain an approximation to the solution of perturbed system with respect to unperturbed system in terms of boundedness and Lagrange stability with initial time difference. In Figures 1, 2 and 3 we illustrate initial time difference boundedness and Lagrange stability of perturbed systems with respect to the unperturbed systems by giving the graphs of the approximate solutions of perturbed and unperturbed systems provided by Runge-Kutta, Improved Euler and Euler methods for $h = 0.2$, respectively. Moreover, in Figures 4, 5 and 6 we give the same graphs where the step size is decreased to $h = 0.01$.

2. Preliminaries

Consider the differential systems

(2.1) $x' = f(t, x), x(t_0) = x_0$ for $t \geq t_0,\ t_0 \in \mathbb{R}_+,$

(2.2) $x' = f(t, x), x(\tau_0) = y_0$ for $t \geq \tau_0 \geq t_0,$

and the perturbed systems

(2.3) $y' = F(t, y), y(\tau_0) = y_0$ for $t \geq \tau_0,$

(2.4) $\omega' = H(t, \omega), \omega(\tau_0) = y_0 - x_0$ for $t \geq \tau_0,$

where $f, F, H \in C\left[\mathbb{R}_+ \times \mathbb{R}^n,\ \mathbb{R}^n\right]$ satisfy a local Lipschitz condition on the set $\mathbb{R}_+ \times S(\rho)$ and $S(\rho)$ is defined by

$$S(\rho) = \{x \in \mathbb{R}^n : \| x \| \leq \rho < \infty\}.$$

The above assumptions imply the existence and uniqueness of solutions on (t_0, x_0) and (τ_0, y_0).

We introduce definitions for a variety of classes of functions that we use in Sections 3 and 4, and for generalized (Dini-like) derivatives and initial time difference boundedness and Lagrange stability in terms of two measures. All inequalities between vectors are componentwise.

2.1. Definition. A function $\phi(r)$ is said to belong to the class $\mathcal{K}$ if $\phi \in C[(0, \rho),\ \mathbb{R}_+], \phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r.

2.2. Definition. A function $a(t, u)$ is said to belong to the class $\mathcal{CK}$ if $a \in C[\mathbb{R}_+^2,\ \mathbb{R}_+]$, $a(t, u) \in \mathcal{K}$ for each $t \in \mathbb{R}_+$.

2.3. Definition. A function $h(t, x)$ is said to belong to the class Γ if $h \in C[\mathbb{R}_+ \times \mathbb{R}^n,\ \mathbb{R}_+]$, $\inf_{(t,x)} h(t, x) = 0$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

2.4. Definition. A function $h(t,x)$ is said to belong to the class Γ_0 if $h \in \Gamma$ and $\sup_{t\in\mathbb{R}_+} h(t,x)$ exists for $x \in \mathbb{R}^n$.

2.5. Definition. A real-valued function $V(t,x)$ defined on $\mathbb{R}_+ \times S(\rho)$ with $V(t,0) = 0$ for $t > 0$ is said to be *positive definite* if there exists a function $\phi(r) \in \mathcal{K}$ such that the relation

$$\phi(\|x\|) \leq V(t,x), \ (V(t,x) \leq -\phi(\|x\|)) \tag{2.5}$$

for $(t,x) \in \mathbb{R}_+ \times S(\rho)$. Also, V is called *negative definite* if $-V$ is positive definite. A function $V(t,x)$ is called *decrescent* if there exists a function $\psi \in \mathcal{K}$ such that $V(t,x) \leq \psi(\|x\|)$ for $(t,x) \in \mathbb{R}_+ \times S(\rho)$.

2.6. Definition. For a real-valued function $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, we define the *generalized derivatives* (Dini-like derivatives) as follows

$$D_*^+ V(t,s,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(s+h, y(t,s+h,x+hf(s,x))) - V(s,y(t,s,x))], \tag{2.6}$$

$$D_{*-} V(t,s,x) = \lim_{h\to 0^-} \inf \frac{1}{h}[V(s+h, y(t,s+h,x+hf(s,x))) - V(s,y(t,s,x))]. \tag{2.7}$$

2.7. Definition. The solution $y(t,\tau_0,y_0)$ of the system (2.3) is said to be *initial time difference* (h_0,h)*-equibounded with respect to the solution* $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.1) for $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$, if and only if given any $\alpha > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists $\beta = \beta(\alpha,\tau_0) > 0$ such that $h_0(\tau_0, y_0 - x_0) < \alpha$ implies $h(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) < \beta$ for $t \geq \tau_0$.

If $\beta = \beta(\alpha,\tau_0) > 0$ is independent of τ_0, then we have (h_0,h)*-uniform equi-boundedness.*

Observe that if $\beta = \beta(\alpha,\tau_0) > 0$ is such that $\beta(\cdot,\tau_0) \in \mathcal{K}$, then (h_0,h)-equiboundedness with initial time difference implies (h_0,h)-stability with initial time difference. Indeed, given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon,\tau_0) > 0$, which is continuous in τ_0 for each ϵ, such that $\beta(\alpha,\tau_0) < \epsilon$ whenever $\alpha < \delta$.

2.8. Definition. The solution $y(t,\tau_0,y_0)$ of the system (2.3) is said to be *initial time difference* (h_0,h)*-equi-attractive in the large with respect to the solution* $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.1) for $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$, if and only if given any $\epsilon, \alpha > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists a positive number $T = T(\tau_0,\epsilon,\alpha)$ such that $h_0(\tau_0, y_0 - x_0) < \alpha$ implies $h(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) < \epsilon$ for $t \geq \tau_0 + T(\tau_0,\epsilon,\alpha)$.

If $T = T(\tau_0,\epsilon,\alpha) > 0$ is independent of τ_0, then we have (h_0,h)*-uniform equi-attractiveness in the large.*

2.9. Definition. If the solution $y(t,\tau_0,y_0)$ of system (2.3) satisfies Definitions 2.7 and 2.8, then it is said to be *initial time difference* (h_0,h)*-Lagrange stable* with respect to the solution $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the system (2.1) for $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$.

If $\beta = \beta(\alpha,\tau_0) > 0$ in Definition 2.7 and $T = T(\tau_0,\epsilon,\alpha) > 0$ in Definition 2.8 are independent of τ_0, then we have (h_0,h)*-uniform Lagrange Stability.*

3. Initial time difference $(\mathbf{h}_0, h)$-boundedness and Lagrange stability related to comparison equations

In earlier work [3, 4, 5, 6], comparisons between the classical notion of boundedness and Lagrange stability and ITD boundedness and Lagrange stability in terms of two measures did not allow the use of the behavior of the null solution in ITD boundedness and Lagrange stability in terms of two measures [2, 3, 4]. The main result presented in this section resolves those difficulties with a new approach that allows the use of boundedness and Lagrange stability in terms of two measures of the null solution of the comparison system to predict the boundedness properties and Lagrange stability in terms of two measures of the solution of (2.3) with respect to $\widetilde{x}(t) = x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.1).

In order to obtain this result we need the following two Lemmas from [9].

3.1. Lemma. (please see [9]) *Let* $f, F \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, *and*

$$G(t,r) = \max_{\tilde{x}, y \in \bar{B}(x_0;r)} \left\| F(t,y) - \tilde{f}(t,\tilde{x}) \right\|, \tag{3.1}$$

where $G(t,r) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *and* $\bar{B}$ *is the closed ball with center at* x_0 *and radius* r. *Assume that* $r^*(t, \tau_0, \|y_0 - x_0\|)$ *is the maximal solution of* $u' = G(t,u), u(\tau_0) = \|y_0 - x_0\|$ *on* $(\tau_0, \|y_0 - x_0\|)$. *Here,* $\tilde{x}(t, \tau_0, x_0) = x(t-\eta, t_0, x_0)$, *where* $x(t, t_0, x_0)$ *is any solution of the system (2.1) for* $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$, $\eta = \tau_0 - t_0 \in \mathbb{R}_+$, *and* $y(t, \tau_0, y_0)$ *is the solution of (2.3). Then, for* $t \geq \tau_0$,

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| \leq r^*(t, \tau_0, \|y_0 - x_0\|).$$

holds. □

3.2. Lemma. (please see [9]) *Let* $V(t,\tilde{x}) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *and* $V(t,\tilde{x})$ *be locally Lipschitzian in* $\tilde{x}$. *Assume that the function*

$$D_*^+ V(t, y-\tilde{x}) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, y-\tilde{x}+h(F(t,y)-\tilde{f}(t,\tilde{x}))) - V(t, y-\tilde{x})] \tag{3.2}$$

satisfies $D_*^+ V(t, y-\tilde{x}) \leq G(t, V(t, y-\tilde{x}))$ *with* $(t,\tilde{x}), (t,y) \in \mathbb{R}_+ \times \mathbb{R}^n$, *where* $G(t,u) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$. *Let* $r(t) = r(t, \tau_0, u_0)$ *be the maximal solution of the comparison equation*

$$u' = G(t,u), u(\tau_0) = u_0 \geq 0 \text{ for } t \geq \tau_0. \tag{3.3}$$

If $\tilde{x}(t) = x(t-\eta, t_0, x_0)$, *where* $x(t, t_0, x_0)$ *is any solution of the system (2.1) for* $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$, $\eta = \tau_0 - t_0 \in \mathbb{R}_+$, *and* $y(t) = y(t, \tau_0, y_0)$ *is any solution of (2.3) for* $t \geq \tau_0$ *such that* $V(\tau_0, y_0 - x_0) \leq u_0$, *then* $V(t, y(t) - x(t)) \leq r(t)$ *for* $t \geq \tau_0$. □

3.3. Theorem. *Assume that*

(i) $h_0, h \in \Gamma$, $h(t,w) \leq \varphi(h_0(t,w))$, *for each* $(t,w) \in \mathbb{R}_+ \times \mathbb{R}^n$ *and* $\varphi \in \mathcal{K}$;

(ii) $V(t,w) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *and* $V(t,w)$ *is locally Lipschitzian in* w, V *is h-positive definite and* h_0*-decresent;*

(iii) $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+]$, $G(t,0) \equiv 0, (t,w) \in S(h,\rho)$, *and*

$$D^+ V(t,w) \leq G(t, V(t,w)) \text{ for } t \geq \tau_0. \tag{3.4}$$

Then the boundedness properties of the comparison equation imply the corresponding initial time difference (h_0, h)*-equi-boundedness properties of the system* $y(t, \tau_0, y_0)$ *(2.3) with respect to the solution* $x(t-\eta, t_0, x_0)$, *where* $x(t, t_0, x_0)$ *is any solution of the system (2.1) for* $t \geq \tau_0 \geq t_0$, *and* $\eta = \tau_0 - t_0 \in \mathbb{R}_+$.

Proof. Since V is $h-$positive definite, there exists a $\lambda \in (0, \rho]$ and $b \in \mathcal{K}$ such that

$$b(h(t,w)) \leq V(t,w), (t,w) \in S(h,\lambda), \tag{3.5}$$

V is h_0-decresent and h_0 is uniformly finer than h, there exists a $\lambda_0 = \varphi^{-1}(\lambda) > 0$ and a function $a \in \mathcal{K}$ such that for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$,

$$h(\tau_0, y_0 - x_0) < \lambda \text{ and } V(\tau_0, y_0 - x_0) \leq a(h_0(\tau_0, y_0 - x_0)), \tag{3.6}$$

and it then follows from (3.4) and (3.5) that we have

$$b(h(\tau_0, y_0 - x_0)) \leq V(\tau_0, y_0 - x_0) \leq a(h_0(\tau_0, y_0 - x_0)) \tag{3.7}$$

for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$.

Let $0 < \alpha \leq \lambda_0$. Set $\alpha_1 = a(\alpha)$, and suppose that the solution of the comparison system (3.3) is equi-bounded. Then, given $a(\alpha) > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists a function $\beta_1 = \beta_1(\tau_0, \alpha)$ such that

$$u_0 < \alpha_1 \text{ implies that } u(t, \tau_0, u_0) < \beta_1 \text{ for all } t \geq \tau_0, \tag{3.8}$$

where $u(t, \tau_0, u_0)$ is any solution of the comparison equation (3.3).

Choose $\beta = \beta(\tau_0, \alpha)$ such that $\beta_1 \leq b(\beta)$, and let $h_0(\tau_0, y_0 - x_0) < \alpha$. Then (3.7) shows that $h(\tau_0, y_0 - x_0) < \beta$ since $\alpha_1 \leq \beta_1$. We claim that, for $t \geq \tau_0$,

$$h(t, y(t) - \tilde{x}(t)) < \beta \text{ whenever } h_0(\tau_0, y(\tau_0) - \tilde{x}(\tau_0)) < \alpha,$$

where $y(t) - \tilde{x}(t)$ is any solution of (2.4) with $h_0(\tau_0, y(\tau_0) - \tilde{x}(\tau_0)) < \alpha$. If this is not true, then there exists a $t_1 > \tau_0$ and a solution $y(t) - \tilde{x}(t)$ of (2.4) such that

$$h(t_1, y(t_1) - \tilde{x}(t_1)) = \beta \text{ and } h(t, y(t) - \tilde{x}(t)) < \beta, \tag{3.9}$$

for $\tau_0 \leq t \leq t_1$, in view of the fact that $h(\tau_0, y(\tau_0) - \tilde{x}(\tau_0)) < \beta$ whenever $h_0(\tau_0, y_0 - x_0) < \alpha$. Set $u_0 = V(\tau_0, y_0 - x_0)$. Then, by Lemma 3.2, we have

$$V(t, y(t) - \tilde{x}(t)) \leq r(t, \tau_0, u_0), \tau_0 \leq t \leq t_1, \tag{3.10}$$

where $r(t) = r(t, \tau_0, u_0)$ is the maximal solution of (3.3) for $t \geq \tau_0$. Now the relations (3.6), (3.8), (3.9) and (3.10) yield

$$b(\beta) \leq V(t_1, y(t_1) - \widetilde{x}(t_1)) \leq r(t_1, \tau_0, u_0) < \beta_1 \leq b(\beta).$$

This is a contradiction, which establishes that for each $\alpha > 0$ there exist $\beta = \beta(\tau_0, \alpha) > 0$ such that $h(t, y(t) - \tilde{x}(t)) < \beta$ provided that $h_0(\tau_0, y_0 - x_0) \leq \alpha$ for all $t \geq \tau_0$. Hence $y(t, \tau_0, y_0)$ is (h_0, h)-equi-bounded with respect to $x(t - \eta, t_0, x_0)$ for $t \geq \tau_0$. □

3.4. Theorem. *Assume that the assumptions of Theorem 3.3 hold. Then the uniform equi-boundedness properties of the comparison equation imply the corresponding initial time difference (h_0, h)-uniform equi-boundedness properties of the system $y(t, \tau_0, y_0)$ of (2.3) with respect to the solution $x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.1) for $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$.*

Proof. The theorem can be proved using arguments similar to that used in the proof of the Theorem 3.3. There exists a $\beta_1 = \beta_1(\alpha)$, independent of τ_0, by considering the uniform equi-boundedness properties of the comparison equation (3.3). We can choose $\beta = \beta(\alpha)$ to be independent of τ_0, then considering Definition 2.7 the proof is complete. □

3.5. Theorem. *Let the assumptions of Theorem 3.3 hold. Then the equi-attractiveness in the large properties of the comparison equation imply the corresponding initial time difference (h_0, h)-equi-attractiveness in the large properties of the system $y(t, \tau_0, y_0)$ in (2.3) with respect to the solution $x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.1) for $t \geq \tau_0 \geq t_0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$.*

Proof. By using the assumption (ii) in Theorem 3.3, the proof is very much similar to the proof of Theorem 3.3. Since V is h-positive definite, there exists a $\lambda \in (0, \rho]$ and $b \in \mathcal{K}$ such that

$$b(h(t, w)) \leq V(t, w), (t, w) \in S(h, \lambda). \tag{3.11}$$

On the other hand, since V is h_0-decresent and h_0 is uniformly finer than h, there exists a $\lambda_0 = \varphi^{-1}(\lambda) > 0$ and a function $a \in \mathcal{K}$ such that for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$,

$$h(\tau_0, y_0 - x_0) < \lambda \text{ and } V(\tau_0, y_0 - x_0) \leq a(h_0(\tau_0, y_0 - x_0)), \tag{3.12}$$

and hence

$$(3.13)\quad b\left(h\left(\tau_0, y_0 - x_0\right)\right) \le V\left(\tau_0, y_0 - x_0\right) \le a\left(h_0\left(\tau_0, y_0 - x_0\right)\right)$$

for $(\tau_0, y_0 - x_0) \in S(h_0, \lambda_0)$.

Let $0 < \alpha \le \lambda_0$. Set $\alpha_1 = a(\alpha)$, and suppose that the solution of the comparison system (3.3) is equi-attractive in the large. Then, given $a(\alpha), b(\epsilon) > 0$ and $\tau_0 \in \mathbb{R}_+$, there exists a number $T = T(\tau_0, \epsilon, \alpha) > 0$ such that

$$(3.14)\quad u_0 < \alpha_1 \text{ implies that } u(t, \tau_0, u_0) < b(\epsilon) \text{ for all } t \ge \tau_0 + T,$$

where $u(t, \tau_0, u_0)$ is any solution of comparison equation (3.3). Set $u_0 = V(\tau_0, y_0 - x_0)$. Then, by Lemma 3.2 we have

$$(3.15)\quad V(t, y(t) - \tilde{x}(t)) \le r(t, \tau_0, u_0), t \ge \tau_0,$$

where $r(t) = r(t, \tau_0, u_0)$ is the maximal solution of (3.3) for $t \ge \tau_0$.

Let $h_0(\tau_0, y_0 - x_0) < \alpha$. Then (3.13) shows that

$$(3.16)\quad h\left(\tau_0, y_0 - x_0\right) < \epsilon$$

since $\alpha_1 < b(\epsilon)$. We claim that $h(t, y(t) - \tilde{x}(t)) < \epsilon$ for all $t \ge \tau_0 + T$.

Suppose now that there exists a sequence $\{t_k\}$, $t_k \ge \tau_0 + T$ and $t_k \to \infty$ as $k \to \infty$, such that

$$(3.17)\quad h(t_k, y(t_k) - \tilde{x}(t_k)) \ge \epsilon,$$

where $y(t) - \tilde{x}(t)$ is any solution of (2.4) such that $h_0(\tau_0, y_0 - x_0) < \epsilon$. This leads to a contradiction from (3.11), (3.14), (3.15) and (3.17). Hence,

$$b(\epsilon) \le V(t_k, y(t_k) - \tilde{x}(t_k)) \le r(t_k, \tau_0, u_0) < b(\epsilon),$$

which establishes that $h_0(\tau_0, y_0 - x_0) < \alpha$ implies $h(t, y(t) - \tilde{x}(t)) < \epsilon$ for $t \ge \tau_0 + T(\tau_0, \epsilon, \alpha)$. □

3.6. Theorem. *Let the assumptions of the Theorem 3.3 hold. Then the uniform equi-attractiveness in the large properties of the comparison equation imply the corresponding initial time difference (h_0, h)-uniform equi-attractiveness in the large properties of the system $y(t, \tau_0, y_0)$ of (2.3) with respect to the solution $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.1) for $t \ge \tau_0 \ge t_0$, $t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$.*

Proof. Theorem 3.6 can be proved using arguments similar to those used in the proof of Theorem 3.5. There exists a $T = T(\epsilon, \alpha)$, independent of τ_0, by considering the uniform equi-attractiveness in the large properties of the comparison equation (3.3). Then, considering Definition 2.8 the proof is complete. □

3.7. Theorem. *Let the assumptions of Theorem 3.3 hold, namely*

(i) $h_0, h \in \Gamma$, $h(t, w) \le \varphi(h_0(t, w))$, *for each* $(t, w) \in \mathbb{R}_+ \times \mathbb{R}^n$ *and* $\varphi \in \mathcal{K}$;

(ii) $V(t, w) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ *and* $V(t, w)$ *is locally Lipschitzian in* w, V *is* h*-positive definite and* h_0*-decresent;*

(iii) $G \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+]$, $G(t, 0) \equiv 0, (t, w) \in S(h, \rho)$, *and*

$$D^+V(t, w) \le G(t, V(t, w)) \text{ for } t \ge \tau_0.$$

Then the equi-Lagrange stability of the comparison system assures the initial time difference (h_0, h)-equi-Lagrange stability of the system $y(t, \tau_0, y_0)$ of (2.3) with respect to the solution $x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.1) *for $t \ge \tau_0 \ge t_0$, $t_0 \in \mathbb{R}_+$, and $\eta = \tau_0 - t_0 \in \mathbb{R}_+$.*

Proof. By using Theorem 3.3 and Theorem 3.5 we have respectively the initial time difference (h_0, h)-equi-boundedness and initial time difference (h_0, h)-equi-attractiveness in the large of the system $y(t, \tau_0, y_0)$ of (2.3) with respect to the solution $x(t-\eta, t_0, x_0)$. Therefore, we have obtained initial time difference (h_0, h)-equi-Lagrange stability. This completes the proof. □

3.8. Theorem. *Let the assumptions of the Theorem 3.3 hold. Then the uniform equi-Lagrange stability properties of the comparison equation imply the corresponding is initial time difference* (h_0, h)*-uniformly equi-Lagrange stability properties of the system* $y(t, \tau_0, y_0)$ *of (2.3) with respect to the solution* $x(t-\eta, t_0, x_0)$*, where* $x(t, t_0, x_0)$ *is any solution of the system (2.1) for* $t \geq \tau_0 \geq t_0$*,* $t_0 \in \mathbb{R}_+$ *and* $\eta = \tau_0 - t_0 \in \mathbb{R}_+$*.*

Proof. By using Theorems 3.4 and 3.6, we have the initial time difference (h_0, h)-uniform equi-boundedness and initial time difference (h_0, h)-uniform equi-attractiveness in the large of the system $y(t, \tau_0, y_0)$ of (2.3) with respect to the solution $x(t-\eta, t_0, x_0)$, respectively. Therefore, we have obtained initial time difference (h_0, h)-uniform equi-Lagrange stability. This completes the proof. □

4. Boundedness and Lagrange stability in terms of two measures with initial time difference

The main result in this section is a new initial time difference comparison result in terms of Lyapunov-like functions [3, 4, 7, 9]. This is presented in Theorems 4.1, 4.2 and 4.3, that are used in the next section to obtain an initial time difference boundedness and Lagrange stability result in terms of two measures.

4.1. Theorem. *Assume that*

(i) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^N_+]$*,* $V(t,z)$ *and* $\|\omega(t,s,z)\|$ *are locally Lipschitz in z for each* (t,s)*, where* $\omega(t) = \omega(t, \tau_0, y_0 - x_0)$ *is the solution of (2.4) and for* $t \geq s \geq \tau_0$ *and* $\widetilde{x}(t) = x(t-\eta, t_0, x_0)$*, where* $x(t, t_0, x_0)$ *is the solution of the system (2.1),* $y(t) = y(t, \tau_0, y_0)$ *is the solution of (2.3), and* $z(t) = y(t) - \widetilde{x}(t)$*;*

(ii) *For*

$$D_{*-}V(t,s,z) = \lim_{h \to 0^-} \inf \frac{1}{h}\left[T\left(t,s,h,\omega,F-f\right)\right],$$

where $T = V(s+h, \omega(t, s+h, z+h(F(s,y) - \widetilde{f}(s,\widetilde{x})))) - V(s, \omega(t,s,z))$ *we have*

$$(4.1) \qquad D_{*-}V(t,s,z) \leq g\left(t,s,V\left(s,\omega\left(t,s,z\right)\right)\right);$$

(iii) $g \in C\left[\mathbb{R}^2_+ \times \mathbb{R}^N_+, \mathbb{R}^N\right]$*,* $g(t,s,u)$ *is quasimonotone nondecreasing in u for each* (t,s) *and* $r(t,s,\tau_0,y_0)$ *is the maximum solution of*

$$(4.2) \qquad \frac{du(s)}{ds} = g\left(t,s,u\left(s\right)\right), u(\tau_0) = u_0 \geq 0$$

existing for $\tau_0 \leq s \leq t < \infty$*.*

Then

$$(4.3) \qquad V\left(t, z(t,\tau_0,y_0-x_0)\right) \leq r_0\left(t,\tau_0, V\left(\tau_0, \omega(t,\tau_0,y_0-x_0)\right)\right),$$

whenever $V(\tau_0, \omega(t,\tau_0,y_0-x_0)) = u_0$*, where* $r_0(t,\tau_0,u_0) = r(t,t,\tau_0,u_0)$*.*

Proof. Let us set $m(t,s) = V(s, \omega(t,s,z(s)))$. Then by using the assumptions (i) and (ii), we have the differential inequality

$$D_{*-}m(t,s) \leq g\left(t,s,m\left(t,s\right)\right) \text{ for } \tau_0 \leq s \leq t.$$

This yields, by a comparison result in [2, Volume 1, Theorem 1.7.1],

$$m(t,s) \leq r\left(t,s,\tau_0,V\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right)\right) \text{ for } \tau_0 \leq s \leq t.$$

If we choose $s = t$, then we get the desired estimate in (4.3) to complete the proof. □

4.2. Theorem. *Under the assumptions of Theorem 4.1 for $N = 1$ and $g(t,s,u) \equiv 0$ we have*

$$V\left(t,z(t,\tau_0,y_0-x_0)\right) \leq V\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right), \ t \geq \tau_0. \tag{4.4}$$

Proof. Integrating both sides of the inequality $D_{*-}m(t,s) \leq g\left(t,s,m\left(t,s\right)\right)$ for $\tau_0 \leq s \leq t$, obtained from the proof of Theorem 4.1, and using $g(t,s,u) \equiv 0$ we obtain

$$\int_{\tau_0}^{t} D_{*-}m(t,s)ds = V\left(t,w(t,t,z(t))\right) - V\left(\tau_0,\omega(t,\tau_0,z(\tau_0))\right) \leq 0.$$

Therefore, we have

$$V\left(t,z(t,\tau_0,y_0-x_0)\right) \leq V\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right) \text{ for } t \geq \tau_0.$$ □

4.3. Theorem. *We assume*

(i) $D_{*-}V(t,s,z) \leq -c\left(h_1\left(s,\omega(t,s,z)\right)\right), \tau_0 \leq s \leq t < \infty$, *where* $c \in \mathcal{K} = \{\phi \in C[\mathbb{R}_+,\mathbb{R}_+]$ *such that* $\phi(0) = 0$ *and* $\phi(s)$ *is increasing with* $s\}$, *and* $h_1 \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$.

Then for $t \geq \tau_0$,

$$V\left(t,z(t,\tau_0,y_0-x_0)\right) \leq V\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right) - \int_{\tau_0}^{t} c\left(h_1\left(s,\omega(t,s,z(s))\right)\right) ds \tag{4.5}$$

for $t \geq s \geq \tau_0$.

Proof. Setting

$$W\left(s,\omega(t,s,z(s))\right) \equiv V\left(s,\omega(t,s,z(s))\right) + \int_{\tau_0}^{s} c\left(h_1\left(\sigma,\omega(t,\sigma,z(\sigma))\right)\right) d\sigma,$$

taking the Dini derivative of both sides and using assumption (i), we have

$$\begin{aligned} D_{*-}W\left(t,s,z(s)\right) &= D_{*-}V\left(t,s,z(s)\right) + c\left(h_1\left(s,\omega(t,s,z(s))\right)\right) \\ &\qquad - c\left(h_1\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right)\right) \\ &\leq D_{*-}V\left(t,s,z(s)\right) + c\left(h_1\left(s,\omega(t,s,z(s))\right)\right) \\ &\leq -c\left(h_1\left(s,\omega(t,s,z(s))\right)\right) + c\left(h_1\left(s,\omega(t,s,z(s))\right)\right) \\ &= 0. \end{aligned}$$

Therefore, we obtain

$$D_{*-}W\left(t,s,z(s)\right) \leq 0.$$

Then, from Theorem 4.2 we have

$$W(t,z(t)) \leq W(\tau_0,\omega(t))$$

for $t \geq \tau_0$, which implies, by the definition of W, that

$$V\left(t,z(t,\tau_0,y_0-x_0)\right) + \int_{\tau_0}^{t} c\left(h_1\left(\sigma,\omega(t,\sigma,z(\sigma))\right)\right) d\sigma \leq V\left(\tau_0,\omega(t,\tau_0,y_0-x_0)\right)$$

for $t \geq \tau_0$, which is (4.5). That is complete the proof. □

Our main result uses Theorems 4.1, 4.2 and 4.3 to establish a boundedness and Lagrange stability criteria with initial time difference in terms of two measures. We show that if the solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h_0)-bounded and Lagrange stable with respect to the solution $x(t-\eta,t_0,x_0)$, then the solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h)-bounded and Lagrange stable with respect to the solution $x(t-\eta,t_0,x_0)$.

4.4. Theorem. *Assume that*

(i) $V\in C[\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}_+]$, *$V(t,z)$ and $\|\omega(t,s,z)\|$ are locally Lipschitz in z for each (t,s), where $\omega(t)=\omega(t,\tau_0,y_0-x_0)$ is the solution of (2.4) and $z(t,\tau_0,y_0-x_0)=y(t)-\widetilde{x}(t)$ for $t\geq\tau_0$;*

(ii)
$$D_{*-}V(t,s,z)=\lim_{h\to0^-}\inf\frac{1}{h}\left[T(t,s,h,\omega,F-f)\right]\leq -c\left(h_1\left(s,\omega(t,s,z(s))\right)\right)$$
in $S(h,\rho)=\{(t,z):h(t,z)<\rho$ for some $h\in\Gamma$ and $\rho>0\}$;

(iii) *$b(h(t,z))+\int_{\tau_0}^{t}c\left(h_1\left(s,\omega(t,s,z(s))\right)\right)ds\leq V(t,z)$ in $S(h,\rho)$ and $V(t,z)\leq a_1(t,h_1(t,z))+a_0(t,h_0(t,z))$ in $S(h_1,\rho)\cap S(h_2,\rho)$, where $b\in\mathcal{K}$ and $a_1,a_0\in\mathcal{CK}$;*

(iv) *h_0 is finer than h_1, that is, there exists a function $\phi\in\mathcal{K}$ such that $h_1(t,z)\leq\phi(h_0(t,z))$ whenever $h_0(t,z)\leq\rho_0$, for some ρ_0 with $\phi(\rho_0)\leq\rho$;*

(v) *The solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h_0)-bounded with respect to the solution $x(t-\eta,t_0,x_0)$.*

Then the solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h)-bounded with respect to the solution $x(t-\eta,t_0,x_0)$.

Proof. Given $0<\alpha<\rho$ and the existence of ρ_0 with $\phi(\rho_0)\leq\rho$, let us choose $\rho_0>\eta(\tau_0,\alpha)>0$, then

$$a_0\left(t,h_0\left(t,z\left(t\right)\right)\right)<\frac{b\left(\alpha\right)}{2}\text{ whenever }h_0\left(t,z\left(t\right)\right)<\eta,\ t\geq\tau_0.$$

By considering hypothesis (v), there exists a $\beta_1=\beta_1(\tau_0,\eta)$ such that

$$h_0\left(t,z\left(t\right)\right)<\eta\text{ provided }h_0\left(\tau_0,y_0-x_0\right)<\beta_1.$$

Thus, for $t\geq\tau_0$,

$$a_0\left(t,h_0\left(t,z\left(t\right)\right)\right)<\frac{b\left(\alpha\right)}{2}\text{ whenever }h_0\left(\tau_0,y_0-x_0\right)<\beta_1. \tag{4.6}$$

Similarly, we may choose $\rho_0>\sigma(\tau_0,\alpha)>0$ such that

$$h_1\left(t,z\left(t\right)\right)<\sigma\text{ implies }a_1\left(t,h_1\left(t,z(t)\right)\right)<\frac{b\left(\alpha\right)}{2}$$

for $t\geq\tau_0$. Since there exists a $\beta_2=\beta_2(\tau_0,\sigma)$ such that $h_0(t,z(t))<\phi^{-1}(\sigma)$ whenever $h_0(\tau_0,y_0-x_0)<\beta_2$ for h_0 finer than h_1. We have $h_1(t,z(t))\leq\phi(h_0(t,z(t)))<\sigma$. Then, for $t\geq\tau_0$,

$$a_1\left(t,h_1\left(t,z\left(t\right)\right)\right)<\frac{b\left(\alpha\right)}{2}\text{ provided }h_0\left(\tau_0,y_0-x_0\right)<\beta_2. \tag{4.7}$$

We want to show that the solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h)-bounded with respect to the solution $x(t-\eta,t_0,x_0)$, that is

$$h\left(t,z\left(t\right)\right)<\alpha\text{ whenever }h_0\left(\tau_0,y_0-x_0\right)<\beta\text{ for }t\geq\tau_0, \tag{4.8}$$

where $\beta = \min\{\beta_1, \beta_2\}$. And, in view of Theorem 4.3, (4.6), (4.7) and the hypotheses (ii) and (iii),

$$\begin{aligned} b\left(h\left(t, z\left(t\right)\right)\right) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t, s, z(s))\right)\right) ds \\ &\le V\left(t, z\left(t\right)\right) \\ &\le V\left(\tau_0, \omega(t, \tau_0, y_0 - x_0)\right) - \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t, s, z(s))\right)\right) ds \\ &\le V\left(\tau_0, \omega(t, \tau_0, y_0 - x_0)\right) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t, s, z(s))\right)\right) ds \\ &\le a_1\left(\tau_0, h_1\left(\tau_0, \omega(t_1, \tau_0, y_0 - x_0)\right)\right) + a_0\left(\tau_0, h_0\left(\tau_0, \omega(t_1, \tau_0, y_0 - x_0)\right)\right) \\ &\qquad + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t, s, z(s))\right)\right) ds \\ &< b\left(\alpha\right) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t, s, z(s))\right)\right) ds, \end{aligned}$$

implying $h\left(t, z\left(t\right)\right) < \alpha$ whenever $h_0\left(\tau_0, y_0 - x_0\right) < \beta$ for $t \ge \tau_0$. Thus, the solution $y(t, \tau_0, y_0)$ of the system (2.3) is initial time difference (h_0, h)-bounded with respect to the solution $x\left(t - \eta, t_0, x_0\right)$. If (4.8) is not true, then there exist solutions $\widetilde{x}(t) = x\left(t - \eta, t_0, x_0\right)$, where $x\left(t, t_0, x_0\right)$ is the solution of (2.1) and $y(t) = y(t, \tau_0, y_0)$ of (2.3), and $t_1 > \tau_0$ such that

$$h_0\left(\tau_0, y_0 - x_0\right) < \beta, h\left(t_1, z\left(t_1\right)\right) = \alpha \text{ and } h\left(t, z\left(t\right)\right) \le \alpha$$

for all $t \in [\tau_0, t_1]$ where $z(t) = y(t) - \widetilde{x}\left(t\right)$ for $t \ge \tau_0$. Applying Theorems 4.2 4.3 yields

$$V\left(t, z(t)\right) \le V(\tau_0, \omega(t, \tau_0, y_0 - x_0)) - \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_1, s, z(s))\right)\right) ds,\ t \in \left[\tau_0, t_1\right].$$

At $t = t_1$,

$$\begin{aligned} b\left(\alpha\right) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_1, s, z(s))\right)\right) ds \\ &\le V\left(t_1, z(t_1)\right) \\ &\le V(\tau_0, \omega(t_1, \tau_0, y_0 - x_0)) - \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_1, s, z(s))\right)\right) ds \\ &\le a_1\left(\tau_0, h_1\left(\tau_0, \omega(t_1, \tau_0, y_0 - x_0)\right)\right) + a_0\left(\tau_0, h_0\left(\tau_0, \omega(t_1, \tau_0, y_0 - x_0)\right)\right) \\ &\qquad + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_1, s, z(s))\right)\right) ds \\ &< b\left(\alpha\right) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_1, s, z(s))\right)\right) ds \end{aligned}$$

by assumption (iii), (4.6) and (4.7). This contradiction proves that the solution $y(t, \tau_0, y_0)$ of the system (2.3) on (τ_0, y_0) is initial time difference (h_0, h)-bounded with respect to the solution $x\left(t - \eta, t_0, x_0\right)$. □

4.5. Theorem. *Assume that*

(i) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t, z)$ *and* $\|\omega\left(t, s, z\right)\|$ *are locally Lipschitz in* z *for each* (t, s), *where* $\omega\left(t\right) = \omega(t, \tau_0, y_0 - x_0)$ *is the solution of (2.4) and* $z(t, \tau_0, y_0 - x_0) = y(t) - \widetilde{x}(t)$ *for* $t \ge \tau_0$;

(ii) $$D_{*-}V(t,s,z) = \lim_{h\to 0^-}\inf \frac{1}{h}\left[T\left(t,s,h,\omega,F-f\right)\right],$$
$$D_{*-}V(t,s,z) \le -c\left(h_1\left(s,\omega(t,s,z(s))\right)\right)$$

in $S\left(h,\rho\right)=\{(t,z): h\left(t,z\right)<\rho$ *for some* $h\in\Gamma$ *and* $\rho>0\}$*;*

(iii) $b\left(h\left(t,z\right)\right)+\int_{\tau_0}^{t} c\left(h_1\left(s,\omega(t,s,z(s))\right)\right)ds \le V\left(t,z\right)$ *in* $S\left(h,\rho\right)$ *and* $V\left(t,z\right)\le a_1\left(t,h_1\left(t,z\right)\right)+a_0\left(t,h_0\left(t,z\right)\right)$ *in* $S\left(h_1,\rho\right)\cap S(h_2,\rho)$*, where* $b\in\mathcal{K}$ *and* $a_1,a_0\in\mathcal{CK}$*;*

(iv) h_0 *is finer than* h_1*, that is, there exists a function* $\phi\in\mathcal{K}$ *such that* $h_1\left(t,z\right)\le\phi\left(h_0\left(t,z\right)\right)$ *whenever* $h_0\left(t,z\right)\le\rho_0$*, for some* ρ_0 *with* $\phi\left(\rho_0\right)\le\rho$*;*

(v) *The solution* $y(t,\tau_0,y_0)$ *of the system (2.3) is initial time difference* (h_0,h_0)*-equi-attractive in the large with respect to the solution* $x\left(t-\eta,t_0,x_0\right)$*.*

Then the solution $y(t,\tau_0,y_0)$ *of the system (2.3) is initial time difference* (h_0,h)*-equi-attractive in the large with respect to the solution* $x\left(t-\eta,t_0,x_0\right)$*.*

Proof. Given $0<\epsilon<\rho$ and the existence of a ρ_0 with $\phi\left(\rho_0\right)\le\rho$, let us choose $\rho_0>\eta\left(\tau_0,\epsilon\right)>0$. Then

$$a_0\left(t,h_0\left(t,z\left(t\right)\right)\right)<\frac{b\left(\epsilon\right)}{2} \text{ whenever } h_0\left(t,z\left(t\right)\right)<\eta,\ t\ge\tau_0.$$

By hypothesis (v), there exists a β_1 and a positive number $T_1=T_1\left(\tau_0,\eta,\beta_1\right)$ such that

$$h_0\left(t,z\left(t\right)\right)<\eta \text{ provided } h_0\left(\tau_0,y_0-x_0\right)<\beta_1,\ t\ge\tau_0+T_1.$$

Thus, for $t\ge\tau_0+T_1$,

$$a_0\left(t,h_0\left(t,z\left(t\right)\right)\right)<\frac{b\left(\epsilon\right)}{2} \text{ whenever } h_0\left(\tau_0,y_0-x_0\right)<\beta_1. \tag{4.9}$$

Similarly, we may choose $\rho_0>\sigma\left(\tau_0,\epsilon\right)>0$ such that

$$h_1\left(t,z\left(t\right)\right)<\sigma \text{ implies } a_1\left(t,h_1\left(t,z(t)\right)\right)<\frac{b\left(\epsilon\right)}{2}$$

for $t\ge\tau_0$. Since there exists a β_2 and a positive number $T_2=T_2\left(\tau_0,\sigma,\beta_2\right)$ such that $h_0\left(t,z\left(t\right)\right)<\phi^{-1}\left(\sigma\right)=\sigma^*$ whenever $h_0\left(\tau_0,y_0-x_0\right)<\beta_2$ for h_0 finer than h_1. We have $h_1\left(t,z(t)\right)\le\phi(h_0\left(t,z\left(t\right)\right))<\sigma$. Then, for $t\ge\tau_0+T_2$,

$$a_1\left(t,h_1\left(t,z\left(t\right)\right)\right)<\frac{b\left(\epsilon\right)}{2} \text{ provided } h_0\left(\tau_0,y_0-x_0\right)<\beta_2. \tag{4.10}$$

We want to show that the solution $y(t,\tau_0,y_0)$ of the system (2.3) is initial time difference (h_0,h)-equi-attractive in the large with respect to the solution $x\left(t-\eta,t_0,x_0\right)$, that is

$$h\left(t,z\left(t\right)\right)<\epsilon \text{ whenever } h_0\left(\tau_0,y_0-x_0\right)<\beta \text{ for } t\ge\tau_0+T^\star, \tag{4.11}$$

where $\beta=\min\left\{\beta_1,\beta_2\right\}$ and $T^\star=\min\left\{T_1,T_2\right\}$.

Suppose now that there exists a sequence $\{t_k\}$, $t_k\ge\tau_0+T^\star$ and $t_k\to\infty$ as $k\to\infty$ such that

$$h(t_k,z(t_k))\ge\epsilon, \tag{4.12}$$

where $z(t) = y(t) - \widetilde{x}(t)$ is any solution of (2.4) for $t \geq \tau_0$ such that $h_0(\tau_0, y_0 - x_0) < \beta$. Now from the assumption (iii) in Theorem 4.3, (4.9), (4.10) and (4.12), we have

$$
\begin{aligned}
b(\epsilon) &+ \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_k, s, z(s))\right)\right) ds \\
&\leq V(t_k, z(t_k)) \\
&\leq V\left(\tau_0, \omega(t, \tau_0, y_0 - x_0)\right) - \int_{\tau_0}^{t_k} c\left(h_1\left(s, \omega(t_k, s, z(s))\right)\right) ds \\
&\leq a_1\left(\tau_0, h_1\left(\tau_0, \omega(t_k, \tau_0, y_0 - x_0)\right)\right) + a_0\left(\tau_0, h_0\left(\tau_0, \omega(t_k, \tau_0, y_0 - x_0)\right)\right) \\
&\qquad + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_k, s, z(s))\right)\right) ds \\
&< b(\epsilon) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t_k, s, z(s))\right)\right) ds,
\end{aligned}
$$

which is a contradiction that establishes $h_0(\tau_0, y_0 - x_0) < \beta$ implies $h(t, y(t) - \tilde{x}(t)) < \epsilon$ for $t \geq \tau_0 + T^*(\tau_0, \epsilon, \beta)$. □

4.6. Theorem. *Assume that*

(i) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, *$V(t,z)$ and $\|\omega(t,s,z)\|$ are locally Lipschitz in z for each (t,s), where $\omega(t) = \omega(t, \tau_0, y_0 - x_0)$ is the solution of (2.4) and $z(t, \tau_0, y_0 - x_0) = y(t) - \widetilde{x}(t)$ for $t \geq \tau_0$;*

(ii)
$$
\begin{aligned}
D_{*-}V(t,s,z) &= \lim_{h \to 0^-} \inf \frac{1}{h}\left[T(t,s,h,\omega,F-f)\right], \\
D_{*-}V(t,s,z) &\leq -c\left(h_1\left(s, \omega(t,s,z(s))\right)\right)
\end{aligned}
$$
in $S(h,\rho) = \{(t,z) : h(t,z) < \rho \text{ for some } h \in \Gamma \text{ and } \rho > 0\}$;

(iii) *$b(h(t,z)) + \int_{\tau_0}^{t} c\left(h_1\left(s, \omega(t,s,z(s))\right)\right) ds \leq V(t,z)$ in $S(h,\rho)$ and $V(t,z) \leq a_1(t, h_1(t,z)) + a_0(t, h_0(t,z))$ in $S(h_1,\rho) \cap S(h_2,\rho)$, where $b \in \mathcal{K}$ and $a_1, a_0 \in \mathcal{CK}$;*

(iv) *h_0 is finer than h_1, that is, there exists a function $\phi \in \mathcal{K}$ such that $h_1(t,z) \leq \phi(h_0(t,z))$ whenever $h_0(t,z) \leq \rho_0$, for some ρ_0 with $\phi(\rho_0) \leq \rho$;*

(v) *The solution $y(t, \tau_0, y_0)$ of the system (2.3) is initial time difference (h_0, h_0)-equi-Lagrange stable with respect to the solution $x(t - \eta, t_0, x_0)$.*

Then the solution $y(t, \tau_0, y_0)$ of the system (2.3) is initial time difference (h_0, h)-equi-Lagrange stable with respect to the solution $x(t - \eta, t_0, x_0)$.

Proof. By (v) the solution $y(t, \tau_0, y_0)$ of the system (2.3) is initial time difference (h_0, h_0)-equi-bounded and initial time difference (h_0, h_0)-equi-attractive in the large with respect to the solution $x(t-\eta, t_0, x_0)$, so by Theorem 4.4 and Theorem 4.5, respectively, it is initial time difference (h_0, h)-equi-bounded and initial time difference (h_0, h)-equi-attractive in the large with respect to the solution $x(t-\eta, t_0, x_0)$. Thus, the solution $y(t, \tau_0, y_0)$ is initial time difference (h_0, h)-equi-Lagrange stable with respect to the solution $x(t - \eta, t_0, x_0)$, as required. □

5. Examples and applications

In this section, we give two examples to illustrate how the main results of Section 4 might be applied.

5.1. Example. Let us consider the nonlinear vector differential system

$$x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_1 - y_1^2 x_1 \\ -x_2 \exp(-3t) + x_2 (2\cosh(t) - 1) - x_2 y_2^2 \end{bmatrix} \text{ for } t \geq t_0,$$

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = x_0, \text{ where } \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_1(t_0, t_0, x_0) \\ x_2(t_0, t_0, x_0) \end{bmatrix}$$

and its perturbed system

$$\begin{aligned} y' &= \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \\ &= \begin{bmatrix} -y_1 - y_1^2 y_1 + (-1 + \sin(t) - \exp(-4t))(y_1 - x_1) \\ -y_2 (1 - 2\cosh(t)) - y_2 \left(\exp(-4t) + y_2^2\right) + \left(y_2^4 \exp(-3t) + t^2\right)(x_2 - y_2) \end{bmatrix} \\ &\qquad \text{for } t \geq \tau_0, \end{aligned}$$

$$\begin{bmatrix} y_1(\tau_0) \\ y_2(\tau_0) \end{bmatrix} = \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} = y_0 \text{ where } \begin{bmatrix} y_1(\tau_0) \\ y_2(\tau_0) \end{bmatrix} = \begin{bmatrix} y_1(\tau_0, \tau_0, y_0) \\ y_2(\tau_0, \tau_0, y_0) \end{bmatrix},$$

where the perturbation term $R(t, x, y)$ is

$$R(t, x, y) = \begin{bmatrix} (-1 + \sin(t) - \exp(-4t))(y_1 - x_1) \\ (y_2^4 \exp(-3t) + t^2)(x_2 - y_2) \end{bmatrix} \text{ for } t \geq \tau_0.$$

Let us choose the Lyapunov function $V(t, y - \tilde{x}) = (1 + 2\exp(t)) \parallel y - \tilde{x} \parallel_2^2$, where $\parallel y - \tilde{x} \parallel_2$ is the norm defined by

$$\parallel y - \tilde{x} \parallel_2^2 = (y - \tilde{x}) \cdot (y - \tilde{x}) = |y_1 - \tilde{x}_1|^2 + |y_2 - \tilde{x}_2|^2.$$

Let $h_0(t, z) = \parallel z \parallel_1$, $h_1(t, z) = \parallel z \parallel_2$ and $h(t, z) = \parallel z \parallel_2$, and $\phi(h_0(t, z)) = h_0(t, z)$. Let $a_1(t, h_1(t, z)) = (2\exp(2t)) \parallel y - \tilde{x} \parallel_2^2$, and let $b(h(t, z))$, $a_0(t, h_0(t, z))$ be defined by

$$\begin{aligned} b(h(t, z)) &= b(\parallel y - \tilde{x} \parallel_2) = \parallel y - \tilde{x} \parallel_2^2 = |y_1 - \tilde{x}_1|^2 + |y_2 - \tilde{x}_2|^2, \\ a_0(t, h_0(t, z)) &= a_0(t, \parallel y - \tilde{x} \parallel_1) = (1 + 2\exp(t)) \parallel y - \tilde{x} \parallel_1^2 \\ &= (1 + 2\exp(t)) \left(|y_1 - \tilde{x}_1| + |y_2 - \tilde{x}_2|\right)^2, \end{aligned}$$

so that we have

$$\begin{aligned} b(\parallel y - \tilde{x} \parallel_2) &\leq V(t, y - \tilde{x}) \leq a(\parallel y - \tilde{x} \parallel_2) \\ &\leq a_0(t, \parallel y - \tilde{x} \parallel_1) + a_1(t, \parallel y - \tilde{x} \parallel_2), \\ h_1(t, z) &\leq \phi(h_0(t, z)). \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{\tau_0}^{t} c\left(h_1(s, \omega(t, s, z(s)))\right) ds &\leq V(t, z) - b(h(t, z)) \\ &\leq \left(1 + 2e^t\right) \parallel y - \tilde{x} \parallel_2^2 - \parallel y - \tilde{x} \parallel_2^2. \end{aligned}$$

In this example, we choose $H(t, \omega(t)) = 0$. Then $D_{*-}V = D_{-}V = D^{+}V = V'$. Thus, V is positive definite and decrescent. The Dini-like derivative of $V(t, y - \tilde{x})$ is

$$\begin{aligned} D_{*-}V(t, y - \tilde{x}) &= 2(1 + 2\exp(t)) \left[(y_1 - \tilde{x}_1)\left(y_1' - \tilde{x}_1'\right) + (y_2 - \tilde{x}_2)\left(y_2' - \tilde{x}_2'\right)\right] \\ &\quad + 2\exp(t)\left(|y_1 - \tilde{x}_1|^2 + |y_2 - \tilde{x}_2|^2\right) \\ &\leq 6\exp(t)\left[(y_1 - \tilde{x}_1)\left(y_1' - \tilde{x}_1'\right) + (y_2 - \tilde{x}_2)\left(y_2' - \tilde{x}_2'\right)\right] \\ &\quad + 2\exp(t)\left(|\, y_1 - \tilde{x}_1 \,|^2 + |\, y_2 - \tilde{x}_2 \,|^2\right) \end{aligned}$$

$$\leq -6\exp(t)\left[|\, y_1-\tilde{x}_1\,|^2+|\, y_2-\tilde{x}_2\,|^2\,(-1+2\cosh(t))\right] \\ +2\exp(t)\left(|y_1-\tilde{x}_1|^2+|y_2-\tilde{x}_2|^2\right) \\ \leq -6\exp(t)\left(\|\, y-\tilde{x}\,\|_2^2\right)+2\exp(t)\left(\|\, y-\tilde{x}\,\|_2^2\right) \\ = -4\exp(t)\left(\|\, y-\tilde{x}\,\|_2^2\right)=-c\left(h_1\left(s,\omega(t,s,z(s))\right)\right),$$

and

$$D_{*-}V(t,y-\tilde{x})=2\left(1+2\exp(t)\right)\left[(y_1-\tilde{x}_1)\left(y_1'-\tilde{x}_1'\right)+(y_2-\tilde{x}_2)\left(y_2'-\tilde{x}_2'\right)\right] \\ +2\exp(t)\left(|y_1-\tilde{x}_1|^2+|y_2-\tilde{x}_2|^2\right) \\ \leq -2\left(1+2\exp(t)\right)\left[(y_1-\tilde{x}_1)^2+(y_2-\tilde{x}_2)^2\,(-1+2\cosh(t))\right] \\ +\left(1+2\exp(t)\right)\left(|y_1-\tilde{x}_1|^2+|y_2-\tilde{x}_2|^2\right) \\ \leq -2\left(1+2\exp(t)\right)\|\, y-\tilde{x}\,\|_2^2+\left(1+2\exp(t)\right)\|\, y-\tilde{x}\,\|_2^2 \\ = -V(t,y-\tilde{x})\leq 0.$$

And we have

$$D_*^+V(t,y-\tilde{x})\leq -V(t,y-\tilde{x}).$$

We use Theorem 3.7 to establish assumption (v). We apply Theorem 3.7 with the comparison equation (3.3), that is the differential equation

$$u'=-u,u(\tau_0)=\|\, y_0-\tilde{x}_0\,\| \quad \text{for } t\geq\tau_0.$$

Then the solution $y(t,\tau_0,y_0)$ of system (2.3) is initial time difference (h_0,h_0)-Lagrange stable with respect to the solution $x(t-\eta,t_0,x_0)$, and the assumption (v) is satisfied in Theorem 4.6. Applying Theorem 4.6 we have that the solution $y(t,\tau_0,y_0)$ of system (2.3) is initial time difference (h_0,h)-Lagrange stable with respect to the solution $x(t-\eta,t_0,x_0)$.

In the following example we investigate the Boundedness and Lagrange stability of a nonlinear vector differential system as in the previous example. Then we shall support this idea with some numerical computation and graphics.

5.2. Example. Let us consider the nonlinear vector differential system

$$x'=\begin{bmatrix}x_1'\\x_2'\end{bmatrix}=\begin{bmatrix}-x_2+\left(1-x_1^2-x_2^2\right)x_1\exp(-t)\\x_1+\left(1-x_1^2-x_2^2\right)x_2\sin^2(t)\end{bmatrix}\ \text{for } t\geq t_0,$$
$$\begin{bmatrix}x_1(t_0)\\x_2(t_0)\end{bmatrix}=\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix}=x_0,\ \text{where}\ \begin{bmatrix}x_1(t_0)\\x_2(t_0)\end{bmatrix}=\begin{bmatrix}x_1\left(t_0,t_0,x_0\right)\\x_2(t_0,t_0,x_0)\end{bmatrix},$$

and its perturbed system

$$y'=\begin{bmatrix}y_1'\\y_2'\end{bmatrix} \\ =\begin{bmatrix}-y_2-(-y_1^2-y_2^2+2x_1y_1+2x_2y_2)\exp(-t)\,(x_1-y_1)+(1-x_1^2-x_2^2)y_1\exp(-t)\\y_1-(-y_1^2-y_2^2+2x_1y_1+2x_2y_2)\,(x_2-y_2)\sin^2(t)+(1-x_2^2-x_1^2)y_2\sin^2(t)\end{bmatrix} \\ \text{for } t\geq\tau_0,$$

$$\begin{bmatrix}y_1(\tau_0)\\y_2(\tau_0)\end{bmatrix}=\begin{bmatrix}y_{01}\\y_{02}\end{bmatrix}=y_0,\ \text{where}\ \begin{bmatrix}y_1(\tau_0)\\y_2(\tau_0)\end{bmatrix}=\begin{bmatrix}y_1\left(\tau_0,\tau_0,y_0\right)\\y_2\left(\tau_0,\tau_0,y_0\right)\end{bmatrix},$$

where the perturbation term $R(t,x,y)$ is

$$R(t,x,y)=\begin{bmatrix}-\exp(-t)\left(x_1-y_1\right)\left(-y_1^2-y_2^2+2x_1y_1+2x_2y_2\right)\\-\left(x_2-y_2\right)\sin^2(t)\left(-y_1^2-y_2^2+2x_1y_1+2x_2y_2\right)\end{bmatrix}$$

for $t \geq \tau_0$. Let us choose the Lyapunov function $V(t, y-\tilde{x}) = (-1+ \| y-\tilde{x} \|_2^2)^2$, where $\| y-\tilde{x} \|_2$ is the norm defined by

$$\| y-\tilde{x} \|_2^2 = (y-\tilde{x}) \cdot (y-\tilde{x}) = |y_1-\tilde{x}_1|^2 + |y_2-\tilde{x}_2|^2.$$

Let $h_0(t,z) = \| z \|_1$, $h_1(t,z) = |-1+ \| y-\tilde{x} \|_2|$, $h(t,z) = \big(-1+ \| y-\tilde{x} \|_2^2 \big)$ and $\phi(h_0(t,z)) = h_0^2(t,z)$. Let $a_1(t, h_1(t,z)) = \exp(t)(-1+ \| y-\tilde{x} \|_2)$, and let b and a_0 be defined by

$$\begin{aligned} b(h(t,z)) &= b\big(-1+ \| y-\tilde{x} \|_2 \big) \\ &= (-1+ \| y-\tilde{x} \|_2^2) = -1 + |y_1-\tilde{x}_1|^2 + |y_2-\tilde{x}_2|^2, \\ a_0\big(t, h_0(t,z)\big) &= a_0\big(t, \| y-\tilde{x} \|_1 \big) = \big(\exp(t) + \| y-\tilde{x} \|_1^2 \big)^2, \end{aligned}$$

so that we have

$$\begin{aligned} &b(\| y-\tilde{x} \|_2) \leq V(t, y-\tilde{x}) \leq a(\| y-\tilde{x} \|_2) \leq a_0(t, \| y-\tilde{x} \|_1) + a_1(t, \| y-\tilde{x} \|_2), \\ &\quad h_1(t,z) \leq \phi(h_0(t,z)). \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{\tau_0}^{t} c(h_1(s, \omega(t,s,z(s))))\, ds &\leq V(t,z) - b(h(t,z)) \\ &\leq \big[(-1+ \| y-\tilde{x} \|_2^2)^2 - \big(-1+ \| y-\tilde{x} \|_2^2 \big) \big] \end{aligned}$$

In this example, we choose $H(t, \omega(t)) = 0$, then $D_{*-}V = D_{-}V = D^{+}V = V'$. Thus, V is positive definite and decrescent. The Dini-like derivative of $V(t, y-\tilde{x})$ is

$$\begin{aligned} &D_{*-}V(t, y-\tilde{x}) \\ &\quad = 2\left[-1+|y_1-x_1|^2+|y_2-x_2|^2\right] 2\left[(y_1-x_1)\left(y_1'-x_1'\right)+(y_2-x_2)\left(y_2'-x_2'\right)\right] \\ &\quad = 2\left[-1+|y_1-x_1|^2+|y_2-x_2|^2\right] 2(x_1-y_1) \\ &\qquad \times \left\{-(x_2-y_2)+(1-(x_1-y_1)^2-(\tilde{x}_2-y_2)^2)(x_1-y_1)\exp(-t)\right\} \\ &\qquad\quad + (y_2-x_2)\left\{(x_1-y_1)+(1-(x_1-y_1)^2-(x_2-y_2)^2)(x_2-y_2)\sin^2(t)\right\} \\ &\quad = -4\left[-1+ \| y-\tilde{x} \|_2^2\right]^2 \left[(x_1-y_1)^2\exp(-t)+(x_2-y_2)^2\sin^2(t)\right] \\ &\quad \leq -4\alpha V(t, y-\tilde{x}) \leq 0, \end{aligned}$$

$\alpha = \max\limits_{t \geq t_0 \geq 0} \left[(x_1-y_1)^2 \exp(-t) + (x_2-y_2)^2 \sin^2(t)\right] > 0$. From this inequality,

$$\begin{aligned} &D_{*-}V(t, y-\tilde{x}) \\ &\quad = -4\left[-1+|y_1-x_1|^2+|y_2-x_2|^2\right]^2 \left[\big(x_1-y_1\big)^2\exp(-t)+\big(x_2-y_2\big)^2\sin^2(t)\right] \\ &\quad \leq -4\left[-1+|y_1-x_1|^2+|y_2-x_2|^2\right]^2 \left[\exp(-t)+\sin^2(t)\right] \\ &\quad = -c(h_1(s, \omega(t,s,z(s)))). \end{aligned}$$

Choosing the function $c(h_1(s, \omega(t,s,z(s))))$ to be $4\left[\exp(-t)+\sin^2(t)\right] h_1^2(t,z)$, where $h_1(t,z) = -1 + \|z\|^2$, we have

$$D_*^{+} V(t, y-\tilde{x}) \leq -4\alpha V(t, y-\tilde{x}) \text{ for } \alpha > 0.$$

Applying Theorem 3.7 with the comparison equation (3.3), $u' = -4\alpha u, u(\tau_0) = \| y_0 - \tilde{x}_0 \|$ for $t \geq \tau_0$, we see that the solution $y(t, \tau_0, y_0)$ of system (2.3) is initial time difference (h_0, h_0)-Lagrange stable with respect to the solution $x(t-\eta, t_0, x_0)$. Hence, by Theorem 4.6 we have that $y(t, \tau_0, y_0)$ is initial time difference (h_0, h)-Lagrange stable with respect to the solution $x(t-\eta, t_0, x_0)$.

6. Numerical solution of a perturbed system with respect to an unperturbed system with ITD

In this section we focus on numerical techniques that could be applied to obtain an approximation to the solution of a perturbed system with respect to an unperturbed system in terms of boundedness and Lagrange stability [2, 3, 4] with initial time difference [8, 9, 10, 11]. Although one of these numerical techniques, namely Euler's formula, is attractive for its simplicity, it is seldom used in serious calculations. On the other hand, the improved Euler method gives significantly greater accuracy than Euler's method. However, one of the most popular as well as the most accurate numerical procedure used in obtaining approximate solutions to perturbed systems with respect to unperturbed systems in terms of boundedness and Lagrange stability with initial time difference is the fourth-order Runge-Kutta method. The main reason for resorting to computers is the non-availability of explicit solutions for a large class of nonlinear coupled differential systems started with different initial time and initial positions. It is impossible to use the Mathematical software directly to obtain the approximate solutions of coupled perturbed and unperturbed systems and their norms, respectively.

6.1. Example. Let us consider the fourth-order Runge-Kutta method with $h = 0.2$ used to obtain a four-decimal-place approximation to the solutions of the nonlinear initial value problem of the vector differential system of Example 5.2 given by

$$x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -x_2 + (1 - x_1^2 - x_2^2)x_1 \exp(-t) \\ x_1 + (1 - x_1^2 - x_2^2)x_2 \sin^2(t) \end{bmatrix} \text{ for } t \geq 0.2,$$

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = x_0, \text{ where } \begin{bmatrix} x_1(0.2) \\ x_2(0.2) \end{bmatrix} = \begin{bmatrix} x_1(0.2, 0.2, 0.9801) \\ x_2(0.2, 0.2, 0.1987) \end{bmatrix} = \begin{bmatrix} 0.9801 \\ 0.1987 \end{bmatrix},$$

and its nonlinear perturbed system

$$y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$
$$= \begin{bmatrix} -y_2 - (-y_1^2 - y_2^2 + 2x_1y_1 + 2x_2y_2) \exp(-t)\,(x_1 - y_1) + (1 - x_1^2 - x_2^2)y_1 \exp(-t) \\ y_1 - (-y_1^2 - y_2^2 + 2x_1y_1 + 2x_2y_2)\,(x_2 - y_2) \sin^2(t) + (1 - x_2^2 - x_1^2)y_2 \sin^2(t) \end{bmatrix}$$
$$\text{for } t \geq 0.4,$$

$$\begin{bmatrix} y_1(\tau_0) \\ y_2(\tau_0) \end{bmatrix} = \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} = y_0, \text{ where } \begin{bmatrix} y_1(0.4) \\ y_2(0.4) \end{bmatrix} = \begin{bmatrix} y_1(0.4, 0.4, 1.8421) \\ y_2(0.4, 0.4, 0.7788) \end{bmatrix} = \begin{bmatrix} 1.8421 \\ 0.7788 \end{bmatrix},$$

and the nonlinear perturbation term $R(t, x, y)$ is

$$R(t, x, y) = \begin{bmatrix} -\exp(-t)\big(x_1 - y_1\big)\big(-y_1^2 - y_2^2 + 2x_1y_1 + 2x_2y_2\big) \\ -(x_2 - y_2)\sin^2(t)\big(-y_1^2 - y_2^2 + 2x_1y_1 + 2x_2y_2\big) \end{bmatrix}$$

for $t \geq 0.4$. We compare the results obtained from using the fourth-order Runge-Kutta, second-order Runge-Kutta (improved Euler) and Euler methods over the interval $[0.2, 6]$ with step sizes $h = 0.2$ and $h = 0.01$. We have used Mathematical Software to obtain graphs of the approximate solutions on the interval $[0.2, 6]$.

Table 1. Unperturbed and Perturbed Systems using the Runge-Kutta Method with Step Size h=0.2.

Time	Unperturbed Exact Sol.		Unperturbed Runge-Kutta Sol.		Unperturbed Abs. Error		Unperturbed Rel. Error		Exact sol. ITD
tn	$X1$	$X2$	$X1$	$X2$	$X1$	$X2$	$X1$	$X2$	Norm$(Y-X)$
0.20	0.9801	0.1987	0.9801	0.1987	0.0000	0.0000	0.0000	0.0002	1.4422
0.40	0.9211	0.3894	0.9211	0.3894	0.0000	0.0000	0.0000	0.0000	0.0797
0.60	0.8253	0.5646	0.8253	0.5646	0.0000	0.0000	0.0000	0.0001	0.0797
0.80	0.6967	0.7174	0.6967	0.7174	0.0000	0.0000	0.0000	0.0001	0.0797
1.00	0.5403	0.8415	0.5403	0.8415	0.0000	0.0000	0.0000	0.0000	0.0797
1.20	0.3624	0.9320	0.3624	0.9320	0.0000	0.0000	0.0001	0.0000	0.0797
1.40	0.1700	0.9854	0.1700	0.9854	0.0000	0.0000	0.0002	0.0001	0.0797
1.60	-0.0292	0.9996	-0.0292	0.9995	0.0000	0.0001	0.0000	0.0001	0.0797
1.80	-0.2272	0.9738	-0.2272	0.9738	0.0000	0.0000	0.0000	0.0000	0.0797
2.00	-0.4161	0.9093	-0.4161	0.9093	0.0000	0.0000	0.0001	0.0000	0.0797
2.20	-0.5885	0.8085	-0.5885	0.8085	0.0000	0.0000	0.0000	0.0000	0.0797
2.40	-0.7374	0.6755	-0.7374	0.6755	0.0000	0.0000	0.0000	0.0001	0.0797
2.60	-0.8569	0.5155	-0.8569	0.5155	0.0000	0.0000	0.0000	0.0000	0.0797
2.80	-0.9422	0.3350	-0.9422	0.3350	0.0000	0.0000	0.0000	0.0000	0.0797
3.00	-0.9900	0.1411	-0.9900	0.1412	0.0000	0.0001	0.0000	0.0006	0.0797
3.20	-0.9983	-0.0584	-0.9983	-0.0583	0.0000	0.0001	0.0000	0.0013	0.0797
3.40	-0.9668	-0.2555	-0.9668	-0.2555	0.0000	0.0000	0.0000	0.0002	0.0797
3.60	-0.8968	-0.4425	-0.8968	-0.4425	0.0000	0.0000	0.0000	0.0000	0.0797
3.80	-0.7910	-0.6119	-0.7910	-0.6118	0.0000	0.0001	0.0000	0.0001	0.0797
4.00	-0.6536	-0.7568	-0.6537	-0.7568	0.0001	0.0000	0.0001	0.0000	0.0797
4.20	-0.4903	-0.8716	-0.4903	-0.8715	0.0000	0.0001	0.0001	0.0001	0.0797
4.40	-0.3073	-0.9516	-0.3074	-0.9516	0.0001	0.0000	0.0002	0.0000	0.0797
4.60	-0.1122	-0.9937	-0.1122	-0.9936	0.0000	0.0001	0.0004	0.0001	0.0797
4.80	0.0875	-0.9962	0.0875	-0.9961	0.0000	0.0001	0.0000	0.0001	0.0797
5.00	0.2837	-0.9589	0.2836	-0.9589	0.0001	0.0000	0.0002	0.0000	0.0797
5.20	0.4685	-0.8835	0.4685	-0.8835	0.0000	0.0000	0.0000	0.0001	0.0797
5.40	0.6347	-0.7728	0.6347	-0.7728	0.0000	0.0000	0.0000	0.0000	0.0797
5.60	0.7756	-0.6313	0.7755	-0.6313	0.0001	0.0000	0.0001	0.0001	0.0797
5.80	0.8855	-0.4646	0.8855	-0.4647	0.0000	0.0001	0.0000	0.0002	0.0797

Table 1. Continued

Time	Perturbed Exact Sol.		Perturbed Runge-Kutta Sol.		Perturbed Abs. Error		Perturbed Rel. Error		Exact sol. ITD
tn	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Norm$(Y-X)$
0.40	1.8421	0.7788	1.8421	0.7788	0.0000	0.0000	0.0000	0.0000	1.4421
0.60	1.6507	1.1293	1.6484	1.1282	0.0023	0.0011	0.0014	0.0010	0.0748
0.80	1.3934	1.4347	1.3903	1.4315	0.0031	0.0032	0.0022	0.0022	0.0707
1.00	1.0806	1.6829	1.0777	1.6770	0.0029	0.0059	0.0027	0.0035	0.0660
1.20	0.7247	1.8641	0.7229	1.8553	0.0018	0.0088	0.0025	0.0047	0.0611
1.40	0.3399	1.9709	0.3399	1.9599	0.0000	0.0110	0.0001	0.0056	0.0571
1.60	-0.0584	1.9991	-0.0563	1.9871	0.0021	0.0120	0.0359	0.0060	0.0546
1.80	-0.4544	1.9477	-0.4500	1.9357	0.0044	0.0120	0.0097	0.0062	0.0535
2.00	-0.8323	1.8186	-0.8257	1.8075	0.0066	0.0111	0.0079	0.0061	0.0533
2.20	-1.1770	1.6170	-1.1684	1.6076	0.0086	0.0094	0.0073	0.0058	0.0536
2.40	-1.4748	1.3509	-1.4646	1.3436	0.0102	0.0073	0.0069	0.0054	0.0539
2.60	-1.7138	1.0310	-1.7023	1.0260	0.0115	0.0050	0.0067	0.0049	0.0539
2.80	-1.8844	0.6700	-1.8722	0.6675	0.0122	0.0025	0.0065	0.0037	0.0539
3.00	-1.9800	0.2822	-1.9675	0.2822	0.0125	0.0000	0.0063	0.0001	0.0540
3.20	-1.9966	-0.1167	-1.9844	-0.1143	0.0122	0.0024	0.0061	0.0210	0.0541
3.40	-1.9336	-0.5111	-1.9221	-0.5062	0.0115	0.0049	0.0059	0.0096	0.0540
3.60	-1.7935	-0.8850	-1.7832	-0.8778	0.0103	0.0072	0.0058	0.0082	0.0538
3.80	-1.5819	-1.2237	-1.5733	-1.2142	0.0086	0.0095	0.0055	0.0078	0.0532
4.00	-1.3073	-1.5136	-1.3006	-1.5021	0.0067	0.0115	0.0051	0.0076	0.0523
4.20	-0.9805	-1.7432	-0.9762	-1.7300	0.0043	0.0132	0.0044	0.0075	0.0511
4.40	-0.6147	-1.9032	-0.6128	-1.8889	0.0019	0.0143	0.0030	0.0075	0.0501
4.60	-0.2243	-1.9874	-0.2251	-1.9727	0.0008	0.0147	0.0035	0.0074	0.0494
4.80	0.1750	-1.9923	0.1715	-1.9781	0.0035	0.0142	0.0200	0.0071	0.0497
5.00	0.5673	-1.9178	0.5614	-1.9048	0.0059	0.0130	0.0104	0.0068	0.0503
5.20	0.9370	-1.7669	0.9288	-1.7556	0.0082	0.0113	0.0088	0.0064	0.0510
5.40	1.2694	-1.5455	1.2593	-1.5365	0.0101	0.0090	0.0079	0.0058	0.0518
5.60	1.5511	-1.2625	1.5395	-1.2560	0.0116	0.0065	0.0075	0.0052	0.0521
5.80	1.7710	-0.9292	1.7584	-0.9253	0.0126	0.0039	0.0071	0.0042	0.0525
6.00	1.9203	-0.5588	1.9072	-0.5576	0.0131	0.0012	0.0068	0.0022	0.0525

Table 1 for the Runge-Kutta method with $h = 0.2$ shows exact solutions, approximate solutions of the perturbed and unperturbed systems, absolute error, relative error and $(h_0 - h)$-approximate and exact norms for the initial time difference boundedness and Lagrange stability of the perturbed system with respect to the unperturbed system.

Table 2. Unperturbed and Perturbed Systems using the Improved Euler Method with Step Size h=0.2

Time	Unperturbed Exact Sol.		Unperturbed Impr. Euler		Unperturbed Abs. Error		Unperturbed Rel. Error		Exact sol. ITD
tn	X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2	Norm$(Y-X)$
0.20	0.9801	0.1987	0.9801	0.1987	0.0000	0.0000	0.0000	0.0002	1.4422
0.40	0.9211	0.3894	0.9182	0.3905	0.0029	0.0011	0.0031	0.0028	0.0797
0.60	0.8253	0.5646	0.8203	0.5657	0.0050	0.0011	0.0061	0.0019	0.0797
0.80	0.6967	0.7174	0.6900	0.7174	0.0067	0.0000	0.0096	0.0001	0.0797
1.00	0.5403	0.8415	0.5323	0.8395	0.0080	0.0020	0.0148	0.0023	0.0797
1.20	0.3624	0.9320	0.3535	0.9274	0.0089	0.0046	0.0244	0.0050	0.0797
1.40	0.1700	0.9854	0.1608	0.9779	0.0092	0.0075	0.0539	0.0077	0.0797
1.60	-0.0292	0.9996	-0.0383	0.9894	0.0091	0.0102	0.3117	0.0102	0.0797
1.80	-0.2272	0.9738	-0.2357	0.9613	0.0085	0.0125	0.0374	0.0129	0.0797
2.00	-0.4161	0.9093	-0.4235	0.8948	0.0074	0.0145	0.0177	0.0159	0.0797
2.20	-0.5885	0.8085	-0.5943	0.7924	0.0058	0.0161	0.0099	0.0199	0.0797
2.40	-0.7374	0.6755	-0.7410	0.6579	0.0036	0.0176	0.0049	0.0260	0.0797
2.60	-0.8569	0.5155	-0.8579	0.4968	0.0010	0.0187	0.0012	0.0363	0.0797
2.80	-0.9422	0.3350	-0.9401	0.3154	0.0021	0.0196	0.0023	0.0585	0.0797
3.00	-0.9900	0.1411	-0.9844	0.1211	0.0056	0.0200	0.0056	0.1419	0.0797
3.20	-0.9983	-0.0584	-0.9889	-0.0782	0.0094	0.0198	0.0094	0.3396	0.0797
3.40	-0.9668	-0.2555	-0.9534	-0.2744	0.0134	0.0189	0.0139	0.0738	0.0797
3.60	-0.8968	-0.4425	-0.8795	-0.4594	0.0173	0.0169	0.0192	0.0381	0.0797
3.80	-0.7910	-0.6119	-0.7700	-0.6257	0.0210	0.0138	0.0265	0.0226	0.0797
4.00	-0.6536	-0.7568	-0.6293	-0.7664	0.0243	0.0096	0.0372	0.0127	0.0797
4.20	-0.4903	-0.8716	-0.4633	-0.8760	0.0270	0.0044	0.0550	0.0051	0.0797
4.40	-0.3073	-0.9516	-0.2786	-0.9500	0.0287	0.0016	0.0935	0.0017	0.0797
4.60	-0.1122	-0.9937	-0.0827	-0.9859	0.0295	0.0078	0.2626	0.0078	0.0797
4.80	0.0875	-0.9962	0.1165	-0.9821	0.0290	0.0141	0.3314	0.0141	0.0797
5.00	0.2837	-0.9589	0.3110	-0.9391	0.0273	0.0198	0.0964	0.0207	0.0797
5.20	0.4685	-0.8835	0.4930	-0.8582	0.0245	0.0253	0.0523	0.0286	0.0797
5.40	0.6347	-0.7728	0.6551	-0.7428	0.0204	0.0300	0.0322	0.0388	0.0797
5.60	0.7756	-0.6313	0.7907	-0.5972	0.0151	0.0341	0.0195	0.0540	0.0797
5.80	0.8855	-0.4646	0.8944	-0.4274	0.0089	0.0372	0.0100	0.0801	0.0797

Table 2. Continued

Time	Perturbed Exact Sol.		Perturbed Impr. Euler		Perturbed Abs. Error		Perturbed Rel. Error		Exact sol. ITD
tn	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Norm$(Y-X)$
0.40	1.8421	0.7788	1.8421	0.7788	0.0000	0.0000	0.0000	0.0000	1.4421
0.60	1.6507	1.1293	1.6428	1.1278	0.0079	0.0015	0.0048	0.0013	0.0687
0.80	1.3934	1.4347	1.3811	1.4274	0.0123	0.0073	0.0088	0.0051	0.0570
1.00	1.0806	1.6829	1.0667	1.6667	0.0139	0.0162	0.0129	0.0097	0.0431
1.20	0.7247	1.8641	0.7117	1.8379	0.0130	0.0262	0.0180	0.0140	0.0290
1.40	0.3399	1.9709	0.3297	1.9362	0.0102	0.0347	0.0301	0.0176	0.0182
1.60	-0.0584	1.9991	-0.0645	1.9589	0.0061	0.0402	0.1045	0.0201	0.0131
1.80	-0.4544	1.9477	-0.4558	1.9052	0.0014	0.0425	0.0031	0.0218	0.0130
2.00	-0.8323	1.8186	-0.8286	1.7764	0.0037	0.0422	0.0044	0.0232	0.0159
2.20	-1.1770	1.6170	-1.1681	1.5768	0.0089	0.0402	0.0076	0.0249	0.0196
2.40	-1.4748	1.3509	-1.4605	1.3134	0.0143	0.0375	0.0097	0.0278	0.0217
2.60	-1.7138	1.0310	-1.6942	0.9963	0.0196	0.0347	0.0114	0.0337	0.0231
2.80	-1.8844	0.6700	-1.8597	0.6383	0.0247	0.0317	0.0131	0.0473	0.0236
3.00	-1.9800	0.2822	-1.9501	0.2538	0.0299	0.0284	0.0151	0.1008	0.0239
3.20	-1.9966	-0.1167	-1.9618	-0.1414	0.0348	0.0247	0.0174	0.2112	0.0242
3.40	-1.9336	-0.5111	-1.8943	-0.5307	0.0393	0.0196	0.0203	0.0384	0.0245
3.60	-1.7935	-0.8850	-1.7503	-0.8979	0.0432	0.0129	0.0241	0.0145	0.0238
3.80	-1.5819	-1.2237	-1.5356	-1.2278	0.0463	0.0041	0.0293	0.0033	0.0209
4.00	-1.3073	-1.5136	-1.2591	-1.5070	0.0482	0.0066	0.0369	0.0044	0.0159
4.20	-0.9805	-1.7432	-0.9319	-1.7246	0.0486	0.0186	0.0496	0.0106	0.0097
4.40	-0.6147	-1.9032	-0.5674	-1.8727	0.0473	0.0305	0.0769	0.0160	0.0042
4.60	-0.2243	-1.9874	-0.1801	-1.9461	0.0442	0.0413	0.1971	0.0208	0.0019
4.80	0.1750	-1.9923	0.2144	-1.9424	0.0394	0.0499	0.2252	0.0251	0.0032
5.00	0.5673	-1.9178	0.6003	-1.8617	0.0330	0.0561	0.0581	0.0293	0.0078
5.20	0.9370	-1.7669	0.9621	-1.7065	0.0251	0.0604	0.0268	0.0342	0.0128
5.40	1.2694	-1.5455	1.2852	-1.4823	0.0158	0.0632	0.0125	0.0409	0.0171
5.60	1.5511	-1.2625	1.5566	-1.1976	0.0055	0.0649	0.0035	0.0514	0.0195
5.80	1.7710	-0.9292	1.7653	-0.8636	0.0057	0.0656	0.0032	0.0706	0.0208
6.00	1.9203	-0.5588	1.9028	-0.4939	0.0175	0.0649	0.0091	0.1162	0.0213

Table 2 for the Improved Euler method with $h = 0.2$ shows exact solutions, approximate solutions of the perturbed and unperturbed systems, absolute error, relative error and $(h_0 - h)$-approximate and exact norms for the initial time difference boundedness and Lagrange stability of the perturbed system with respect to the unperturbed system.

Table 3. Unperturbed and Perturbed Systems using the Euler Method with Step Size h=0.2.

Time	Unperturbed Exact Sol.		Unperturbed Impr. Euler		Unperturbed Abs. Error		Unperturbed Rel. Error		Exact sol. ITD
tn	$X1$	$X2$	$X1$	$X2$	$X1$	$X2$	$X1$	$X2$	Norm$(Y-X)$
0.20	0.9801	0.1987	0.9801	0.1987	0.0000	0.0000	0.0000	0.0002	1.4422
0.40	0.9211	0.3894	0.9403	0.3947	0.0192	0.0053	0.0209	0.0136	0.0797
0.60	0.8253	0.5646	0.8564	0.5823	0.0311	0.0177	0.0376	0.0313	0.0797
0.80	0.6967	0.7174	0.7331	0.7509	0.0364	0.0335	0.0522	0.0468	0.0797
1.00	0.5403	0.8415	0.5763	0.8897	0.0360	0.0482	0.0666	0.0573	0.0797
1.20	0.3624	0.9320	0.3931	0.9893	0.0307	0.0573	0.0848	0.0614	0.0797
1.40	0.1700	0.9854	0.1921	1.0450	0.0221	0.0596	0.1302	0.0604	0.0797
1.60	-0.0292	0.9996	-0.0182	1.0573	0.0110	0.0577	0.3767	0.0578	0.0797
1.80	-0.2272	0.9738	-0.2295	1.0287	0.0023	0.0549	0.0101	0.0563	0.0797
2.00	-0.4161	0.9093	-0.4344	0.9611	0.0183	0.0518	0.0439	0.0570	0.0797
2.20	-0.5885	0.8085	-0.6253	0.8564	0.0368	0.0479	0.0625	0.0593	0.0797
2.40	-0.7374	0.6755	-0.7949	0.7174	0.0575	0.0419	0.0780	0.0621	0.0797
2.60	-0.8569	0.5155	-0.9362	0.5488	0.0793	0.0333	0.0926	0.0646	0.0797
2.80	-0.9422	0.3350	-1.0435	0.3564	0.1013	0.0214	0.1075	0.0639	0.0797
3.00	-0.9900	0.1411	-1.1121	0.1459	0.1221	0.0048	0.1233	0.0339	0.0797
3.20	-0.9983	-0.0584	-1.1384	-0.0766	0.1401	0.0182	0.1403	0.3122	0.0797
3.40	-0.9668	-0.2555	-1.1203	-0.3043	0.1535	0.0488	0.1588	0.1908	0.0797
3.60	-0.8968	-0.4425	-1.0568	-0.5270	0.1600	0.0845	0.1785	0.1909	0.0797
3.80	-0.7910	-0.6119	-0.9491	-0.7302	0.1581	0.1183	0.1999	0.1934	0.0797
4.00	-0.6536	-0.7568	-0.8013	-0.8963	0.1477	0.1395	0.2259	0.1843	0.0797
4.20	-0.4903	-0.8716	-0.6207	-1.0108	0.1304	0.1392	0.2661	0.1597	0.0797
4.40	-0.3073	-0.9516	-0.4178	-1.0724	0.1105	0.1208	0.3594	0.1269	0.0797
4.60	-0.1122	-0.9937	-0.2030	-1.0929	0.0908	0.0992	0.8100	0.0998	0.0797
4.80	0.0875	-0.9962	0.0157	-1.0827	0.0718	0.0865	0.8206	0.0869	0.0797
5.00	0.2837	-0.9589	0.2322	-1.0425	0.0515	0.0836	0.1814	0.0872	0.0797
5.20	0.4685	-0.8835	0.4407	-0.9691	0.0278	0.0856	0.0594	0.0969	0.0797
5.40	0.6347	-0.7728	0.6344	-0.8608	0.0003	0.0880	0.0005	0.1139	0.0797
5.60	0.7756	-0.6313	0.8065	-0.7191	0.0309	0.0878	0.0399	0.1391	0.0797
5.80	0.8855	-0.4646	0.9502	-0.5482	0.0647	0.0836	0.0730	0.1799	0.0797

Table 3. Continued

Time	Perturbed Exact Sol.		Perturbed Impr. Euler		Perturbed Abs. Error		Perturbed Rel. Error		Exact sol. ITD
tn	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Y_1	Y_2	Norm$(Y-X)$
0.40	1.8421	0.7788	1.8421	0.7788	0.0000	0.0000	0.0000	0.0000	1.4421
0.60	1.6507	1.1293	1.6864	1.1473	0.0357	0.0180	0.0216	0.0160	0.1231
0.80	1.3934	1.4347	1.4495	1.4816	0.0561	0.0469	0.0403	0.0327	0.1605
1.00	1.0806	1.6829	1.1435	1.7602	0.0629	0.0773	0.0582	0.0459	0.1871
1.20	0.7247	1.8641	0.7829	1.9636	0.0582	0.0995	0.0803	0.0534	0.1959
1.40	0.3399	1.9709	0.3847	2.0802	0.0448	0.1093	0.1317	0.0555	0.1901
1.60	-0.0584	1.9991	-0.0336	2.1092	0.0248	0.1101	0.4246	0.0550	0.1835
1.80	-0.4544	1.9477	-0.4553	2.0551	0.0009	0.1074	0.0020	0.0551	0.1867
2.00	-0.8323	1.8186	-0.8647	1.9220	0.0324	0.1034	0.0389	0.0569	0.2015
2.20	-1.1770	1.6170	-1.2465	1.7140	0.0695	0.0970	0.0590	0.0600	0.2264
2.40	-1.4748	1.3509	-1.5859	1.4372	0.1111	0.0863	0.0753	0.0639	0.2601
2.60	-1.7138	1.0310	-1.8692	1.1010	0.1554	0.0700	0.0907	0.0679	0.3013
2.80	-1.8844	0.6700	-2.0844	0.7168	0.2000	0.0468	0.1061	0.0699	0.3466
3.00	-1.9800	0.2822	-2.2224	0.2965	0.2424	0.0143	0.1224	0.0505	0.3934
3.20	-1.9966	-0.1167	-2.2760	-0.1483	0.2794	0.0316	0.1399	0.2703	0.4412
3.40	-1.9336	-0.5111	-2.2407	-0.6035	0.3071	0.0924	0.1588	0.1808	0.4927
3.60	-1.7935	-0.8850	-2.1149	-1.0489	0.3214	0.1639	0.1792	0.1851	0.5437
3.80	-1.5819	-1.2237	-1.9005	-1.4557	0.3186	0.2320	0.2014	0.1896	0.5743
4.00	-1.3073	-1.5136	-1.6057	-1.7887	0.2984	0.2751	0.2283	0.1817	0.5515
4.20	-0.9805	-1.7432	-1.2454	-2.0187	0.2649	0.2755	0.2701	0.1581	0.4570
4.40	-0.6147	-1.9032	-0.8401	-2.1431	0.2254	0.2399	0.3668	0.1260	0.3302
4.60	-0.2243	-1.9874	-0.4108	-2.1851	0.1865	0.1977	0.8314	0.0995	0.2382
4.80	0.1750	-1.9923	0.0264	-2.1655	0.1486	0.1732	0.8491	0.0869	0.2031
5.00	0.5673	-1.9178	0.4595	-2.0861	0.1078	0.1683	0.1901	0.0877	0.2038
5.20	0.9370	-1.7669	0.8766	-1.9402	0.0604	0.1733	0.0645	0.0981	0.2211
5.40	1.2694	-1.5455	1.2645	-1.7245	0.0049	0.1790	0.0038	0.1158	0.2493
5.60	1.5511	-1.2625	1.6093	-1.4421	0.0582	0.1796	0.0375	0.1422	0.2883
5.80	1.7710	-0.9292	1.8975	-1.1010	0.1265	0.1718	0.0714	0.1849	0.3361
6.00	1.9203	-0.5588	2.1175	-0.7118	0.1972	0.1530	0.1027	0.2737	0.3894

Table 3 for the Euler method with $h = 0.2$ shows exact solutions, approximate solutions of the perturbed and unperturbed systems, absolute error, relative error and $(h_0 - h)$-approximate and exact norm for the initial time difference boundedness and Lagrange stability the of perturbed system with respect to the unperturbed system.

Figure 1. Approximate Solutions provided by the Runge-Kutta Method with h = 0.2

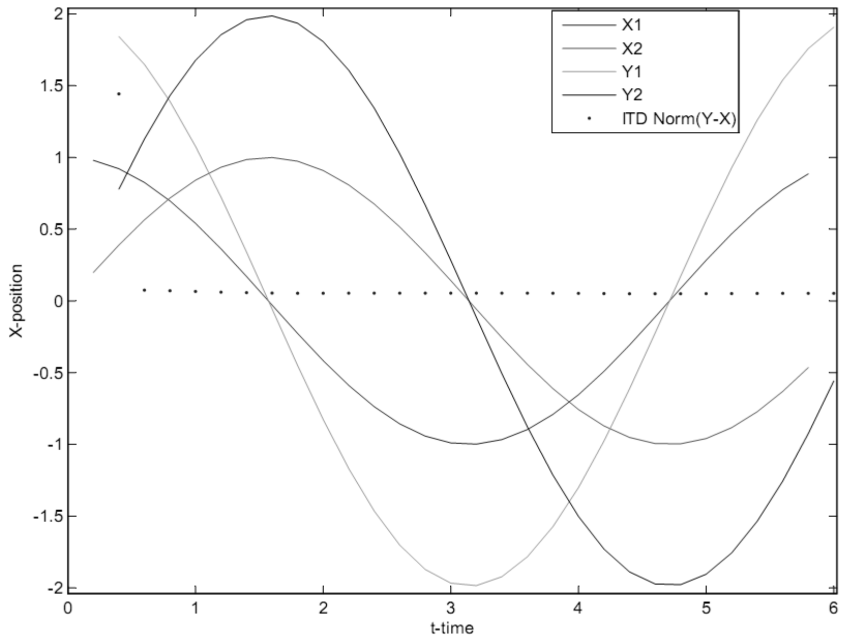


Figure 2. Approximate Solutions provided by the Improved Euler Method with h = 0.2

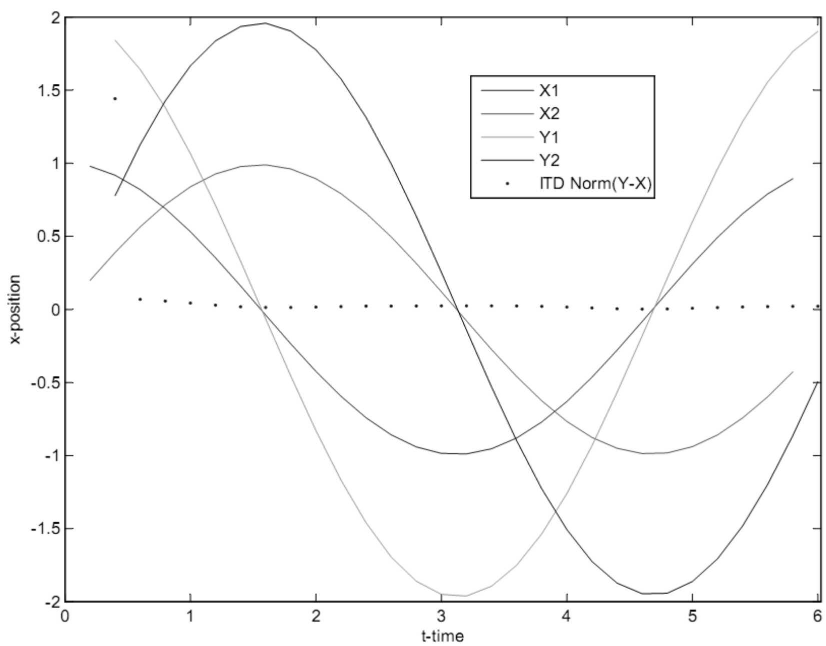


Figure 3. Approximate Solutions provided by the Euler Method with h = 0.2

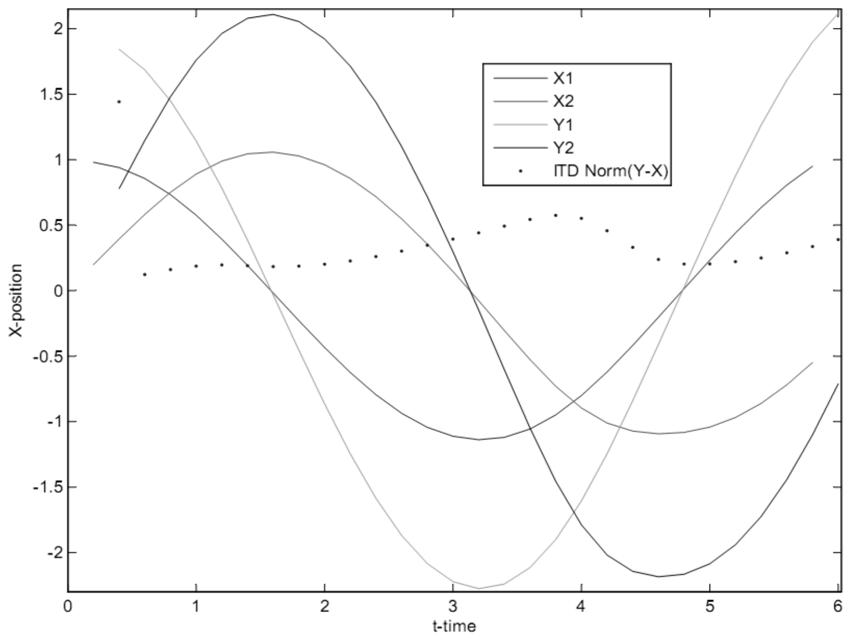


Figure 4. Approximate Solutions provided by the Runge-Kutta Method with h = 0.01

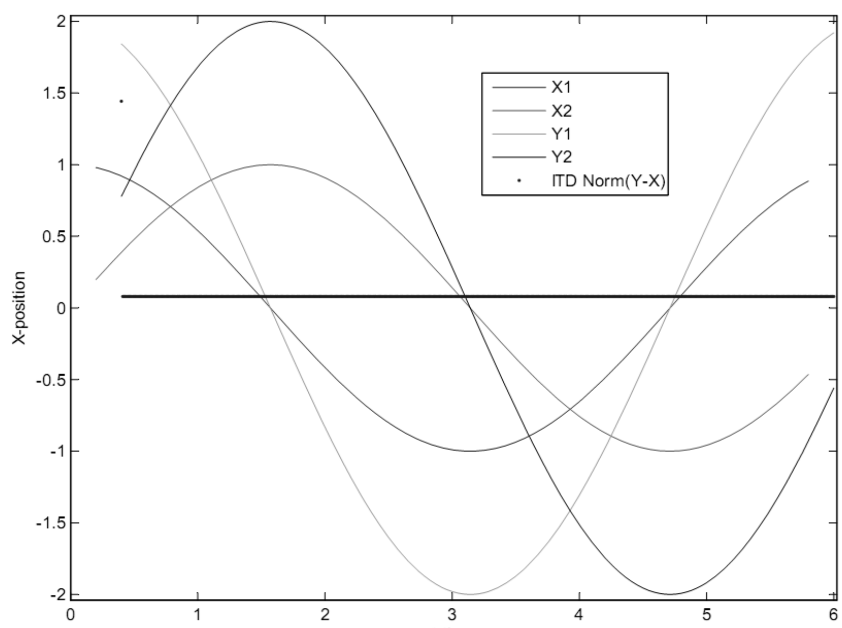


Figure 5. Approximate Solutions provided by the Improved Euler Method with h = 0.01

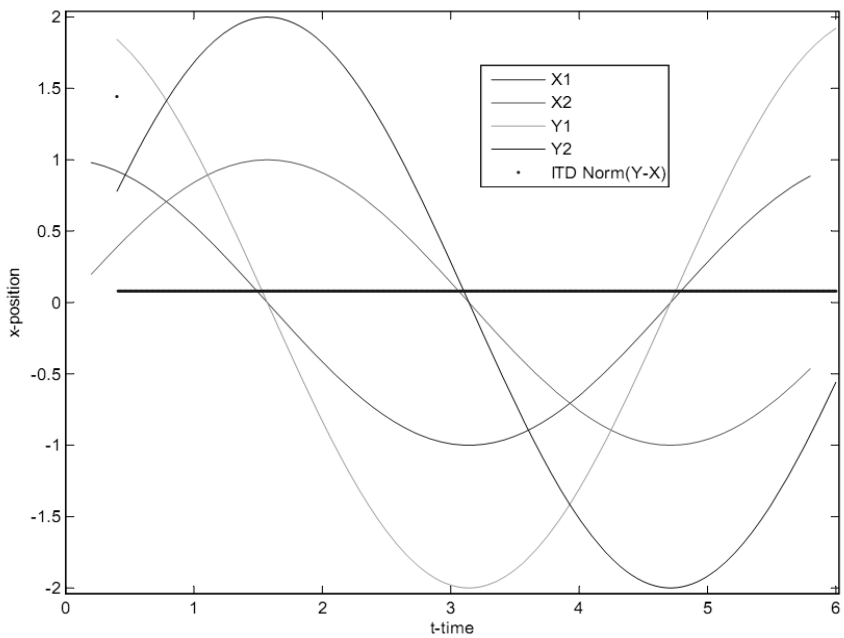
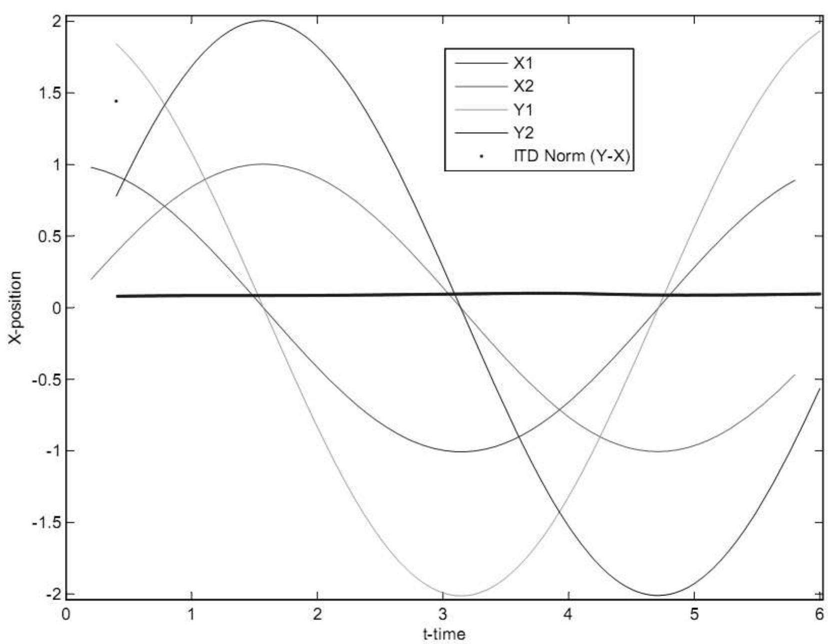


Figure 6. Approximate Solutions provided by the Euler Method with h = 0.01



In Figures 1–3, the graphs of the approximate solutions of the perturbed and unperturbed system, and initial time difference boundedness and Lagrange stability of the perturbed system with respect to the unperturbed system are given for $h = 0.2$ using the Runge-Kutta, Improved Euler and Euler Methods, respectively.

In Figures 4–6, the corresponding graphs are given for $h = 01$.

Inspection of Tables 1–3, which give the results for the three methods when $h = 0.2$, shows why the fourth-order Runge-Kutta method is so popular and fruitful and has more accuracy than the others. However, from the point of view of the number of calculations the Euler and improved Euler methods are faster than Runge-Kutta in real time applications. If four-decimal-place accuracy is all that we desire, there is no need to use a smaller step size.

At the expense of doubling the number of calculations, some important improvements in accuracy; absolute error, relative error and initial time difference $(h_0 - h)$-boundedness and Lagrange stability of a perturbed system with respect to the unperturbed system are obtained by decreasing the step size to $h = 0.01$. It is apparent from Tables 4–6 and Figures 4–6 that the approximations improve as the step size decreases.

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