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MONOTONE ITERATIVE TECHNIQUE WITH INITIAL TIME DIFFERENCE FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we employe the monotone iterative technique for fractional differential equations of Riemann-Liouville type by choosing upper and lower solutions that start at different initial times.

Keywords: Initial time difference, Monotone iterative technique, R-L fractional differential equations, Existence result, Comparison result.

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1. Introduction

The concept of derivative of an arbitrary order or fractional order arose 300 years ago with L'Hospital's query to Leibnitz, and his reply to L'Hospital, in which the meaning of the derivative of order one-half (1/2) was discussed. Since that time the fractional calculus has drawn the attention of many famous mathematicians. By the end of the 19th century, due to the works of Liouville, Grünwald, Letnikov and Riemann, the theory of the calculus of arbitrary order was more or less developed, mainly as a pure theoretical field of mathematics useful only for mathematicians.

It has been shown recently that fractional differential equations provide an excellent model for real world problems in a variety of disciplines. This is the main advantage of fractional derivatives in comparison with conventional integer order models. There has been a growing interest in this new area to study, the concept of fractional differential equations and fractional dynamic systems [2, 5, 7, 10, 11].

The monotone iterative technique [4], coupled with the method of upper and lower solutions, offers monotone sequences that converge uniformly and monotonically to the

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extremal solutions of the given nonlinear problem. Since each member of such a sequence is the solution of a certain linear fractional order differential equations which can be explicitly computed, the advantage and the importance of this technique needs no special emphasis. Moreover, this method can successfully be employed to generate two sided pointwise bounds on solutions of initial value problems of fractional order differential equations, from which qualitative and quantitative behavior can be investigated [1, 5, 6, 8, 9, 12]. Furthermore, the monotone flows that appear in this technique are shown to converge quadratically to the unique solution of the given problem under certain restrictions when we utilize the method of quasilinearization, which is a part of these constructive methods [3, 13, 14, 15].

Consider the nonlinear fractional differential equation

$$D^q x(t) = f(t,x), \; x(t)(t-t_0)^{1-q}|_{t=t_0} = x^0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ and D^q is the Riemann-Liouville (R-L) fractional derivative of order q, $0 < q < 1$.

The corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds, \tag{1.2}$$

where $x^0(t) = \frac{x^0 (t-t_0)^{q-1}}{\Gamma(q)}$, that is, every solution of (1.2) is a solution of (1.1) and *vice versa*.

2. Preliminaries

We give some basic definitions and theorems for the development of our main result. One can see [5] for detailed proofs of these theorems.

2.1. Definition. A function $v \in C_p([t_0,T],\mathbb{R})$, $1-q=p$, $0<q<1$, is said to be a *lower solution* of the initial value problem (IVP) (1.1) if

$$D^q v(t) \le f(t, v(t)), \; v^0 \le x^0, \tag{2.1}$$

where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and

$$C_p([t_0,T],\mathbb{R}) = \{u \in C([t_0,T],\mathbb{R}) \mid u(t)\cdot(t-t_0)^p \in C([t_0,T],\mathbb{R})\}.$$

It is an *upper solution* if the reverse inequalities hold.

The following theorem is a comparison result relative to strict fractional differential inequalities.

2.2. Theorem. *Let $v, w \in C_p([t_0,T],\mathbb{R})$ be locally Holder continuous for an exponent $0<\lambda<1$ and $\lambda > q$, $p = 1-q$, $f \in C([t_0,T]\times\mathbb{R},\mathbb{R})$, where $C_p([t_0,T],\mathbb{R}) = \{u \in C([t_0,T],\mathbb{R}) \mid u(t)\cdot(t-t_0)^p \in C([t_0,T],\mathbb{R})\}$ and*

$$\begin{aligned} &\text{(i)} \quad D^q v(t) \le f(t, v(t)), \\ &\text{(ii)} \quad D^q w(t) \ge f(t, w(t)), \end{aligned} \tag{2.2}$$

$t_0 < t \le T$, one of the inequalities (i) *or* (ii) *being strict. Then*

$$v^0 < w^0, \tag{2.3}$$

where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$ implies $v(t) < w(t)$, $t_0 \le t \le T$. □

The next result is also a comparison result for nonstrict fractional differential inequalities, which requires the usual Lipschitz condition.

2.3. Theorem. *Assume that the conditions of Theorem 2.2 hold with nonstrict inequalities (i) and (ii). Suppose further that the standard Lipschitz condition is satisfied such that*

$$f(t,x) - f(t \cdot y) \leq L(x-y), \; x \geq y \text{ and } L > 0.$$

Then $v^0 \leq w^0$ implies $v(t) \leq w(t)$, $t_0 \leq t \leq T$. □

2.4. Corollary. *The function $f(t,u) = \sigma(t)u$, where $\sigma(t) \leq L$, is admissible in Theorem 2.3 to yield $u(t) \leq 0$ on $t_0 \leq t \leq T$.* □

We wish to give the following Lemma about the theory of fractional differential inequalities.

2.5. Lemma. *Let $m \in C_p\big([t_0,T], \mathbb{R}\big)$ be locally Holder continuous with exponent $\lambda > q$, and suppose that for any $t_1 \in [t_0,T]$ we have*

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \text{ for } t_0 \leq t \leq t_1. \tag{2.4}$$

Then it follows that,

$$D^q m(t_1) \geq 0. \tag{2.5}$$

□

If we know the existence of upper and lower solutions w, v such that $v(t) \leq w(t)$, $t \in [t_0,T]$, for the IVP (1.1) we can prove the existence of solutions in the closed set

$$\Omega = \{(t,x) \mid v(t) \leq x \leq w(t), \; t \in [t_0,T]\}.$$

2.6. Theorem. *Let $v, w \in C_p\big([t_0,T], \mathbb{R}\big)$ be lower and upper solutions of the IVP (1.1), which are locally Holder continuous with exponent $\lambda > q$ such that $v(t) \leq w(t)$, $t \in [t_0,T]$ and $f \in C(\Omega, R)$. Then there exists a solution $x(t)$ of the IVP (1.1) satisfying $v(t) \leq x(t) \leq w(t)$ on $[t_0,T]$.* □

Next, we will give the explicit solution of the nonhomogeneous linear fractional differential equation. Consider the nonhomogeneous IVP

$$D^q x = \lambda x + f(t), \; x^0 = x(t)\big(t-t_0\big)^{1-q}|_{t=t_0} \tag{2.6}$$

for the linear fractional differential equation, where λ is a real number and $f \in C_p\big([t_0,T], \mathbb{R}\big)$. The equivalent Volterra fractional integral equation for $t_0 \leq t \leq T$, is

$$x(t) = x^0(t) + \frac{\lambda}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s)\, ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s)\, ds, \tag{2.7}$$

where $x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$.

When we apply the method of successive approximations to find the solution $x(t) = x(t,t_0,x^0)$ explicitly for the given nonhomogeneous IVP (2.6), we obtain

$$x(t) = x^0(t-t_0)^{q-1} E_{q,q}(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds, \tag{2.8}$$

$t \in [t_0,T]$, where $E_{q,q}$ denotes the two parameter Mittag-Leffler function.

If $f(t) \equiv 0$, we get, as the solution of the corresponding homogeneous IVP,

$$x(t) = x^0(t-t_0)^{q-1} E_{q,q}(\lambda(t-t_0)^q), \; t \in [t_0,T]. \tag{2.9}$$

3. Comparison results and existence results relative to initial time difference

In case the upper and lower solutions start at different initial times, we have following existence and comparison results. New results are dealt with after this point. We begin with a comparison result.

3.1. Theorem. *Assume that*

(i) $\alpha \in C_p\big[[t_0, t_0+T], \mathbb{R}\big]$, $t_0, T > 0$, $\beta \in C_p^*\big[[\tau_0, \tau_0+T], \mathbb{R}\big]$ *is locally Holder continuous for an exponent* $0 < \lambda < 1$, *and* $\lambda > q$, $p = 1-q$, *where*

$$C_p\big([t_0, t_0+T], \mathbb{R}\big) = \big\{u \in C\big([t_0, t_0+T], \mathbb{R}\big) \mid u(t)\cdot(t-t_0)^p \in C\big([t_0, t_0+T], \mathbb{R}\big)\big\},$$

$$C_p^*\big([\tau_0, \tau_0+T], \mathbb{R}\big) = \big\{u \in C\big([\tau_0, \tau_0+T], \mathbb{R}\big) \mid u(t)\cdot(t-\tau_0)^p \in C\big([\tau_0, \tau_0+T], \mathbb{R}\big)\big\},$$

$f \in C\big[[t_0, \tau_0+T]\times\mathbb{R}, \mathbb{R}\big]$, *and*

$$(3.1) \qquad D^q\alpha(t) \le f(t, \alpha(t)), \ t_0 \le t \le t_0+T,$$

$$(3.2) \qquad D^q\beta(t) \ge f(t, \beta(t)), \ \tau_0 \le t \le \tau_0+T,$$

$\alpha^0 \le x^0 \le \beta^0$ *where* $\alpha^0 = \alpha(t)(t-t_0)^{1-q}|_{t=t_0}$, $\beta^0 = \beta(t)(t-\tau_0)^{1-q}|_{t=\tau_0}$;

(ii) $f(t,x)$ *satisfies a Lipschitz condition such that*

$$(3.3) \qquad f(t,x) - f(t.y) \le L(x-y), \ x \ge y \text{ and } L > 0;$$

(iii) $\tau_0 > t_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x.

Then we have

(a) $\alpha(t) \le \beta(t+\eta)$, $t_0 \le t \le t_0+T$,

(b) $\alpha(t-\eta) \le \beta(t)$, $\tau_0 \le t \le \tau_0+T$, *where* $\eta = \tau_0 - t_0$.

Proof. Let $\beta_0(t) = \beta(t+\eta)$, $t \ge t_0$. Then

$$\beta_0^0 = \beta_0(t)(t-t_0)^{1-q}|_{t=t_0} = \beta(t+\eta)(t-t_0)^{1-q}|_{t=t_0}$$
$$= \beta(t)(t-\tau_0)^{1-q}|_{t=\tau_0} = \beta^0 \ge x^0 \ge \alpha^0.$$

Also, using the nondecreasing property of $f(t,x)$ in t for each x, we have

$$D^q\beta_0(t) = D^q\beta(t+\eta)$$
$$\ge f(t+\eta, \beta(t+\eta))$$
$$D^q\beta_0(t) \ge f(t, \beta_0(t)),$$

so we get that $\beta_0(t)$ is an upper solution of (1.1). Now for any $\varepsilon > 0$, we set $\beta_{0\varepsilon}(t) = \beta_0(t) + \epsilon\lambda(t)$, where $\lambda(t) = (t-t_0)^{q-1}E_{q,q}(2L(t-t_0)^q)$.

This implies that

$$\beta_{0\epsilon}^0 = \beta_{0\epsilon}(t-t_0)^{1-q}|_{t=t_0} = \beta_0(t)(t-t_0)^{1-q}|_{t=t_0} + \epsilon\lambda(t)(t-t_0)^{1-q}|_{t=t_0},$$
$$\beta_{0\epsilon}^0 = \beta_0^0 + \epsilon\lambda^0.$$

It follows that $\beta_{0\epsilon}^0 > \beta_0^0 \ge \alpha^0$ and $\beta_{0\epsilon}(t) > \beta_0(t)$, $t \ge t_0$. Then we obtain

$$D^q\beta_{0\epsilon}(t) = D^q\beta_0(t) + \epsilon D^q\lambda(t)$$
$$\ge f(t, \beta_0(t)) + 2\epsilon L\lambda(t)$$
$$\ge f(t, \beta_{0\epsilon}(t)) - \epsilon L\lambda(t) + 2\epsilon L\lambda(t),$$
$$D^q\beta_{0\epsilon}(t) > f(t, \beta_{0\epsilon}(t)), \ t_0 \le t \le t_0+T.$$

Here we have used (ii) and the fact that $\lambda(t)$ is the solution of the initial value problem

(3.4) $\quad D^q\lambda(t) = 2L\lambda(t),\ \lambda(t)(t-t_0)^{1-q}|_{t=t_0} = \lambda^0 > 0.$

Applying now Theorem 2.2 to $\beta_{0\epsilon}(t)$ and $\alpha(t)$, we get $\alpha(t) < \beta_{0\epsilon}(t)$, $t_0 \le t \le t_0 + T$.

Consequently, making $\epsilon \to 0$ we get $\alpha(t) \le \beta_{0\epsilon}(t) = \beta_0(t) = \beta(t+\eta)$ on $[t_0, t_0+T]$.

To prove (b), let $\alpha_0(t) = \alpha(t-\eta)$, $t \ge \tau_0$. It is clear that

$$\alpha_0(t)(t-\tau_0)^{1-q}|_{t=\tau_0} = \alpha(t-\eta)(t-\tau_0)^{1-q}|_{t=\tau_0} = \alpha(t)(t-t_0)^{1-q}|_{t=t_0}$$
$$\le \beta(t)(t-\tau_0)^{1-q}|_{t=\tau_0}.$$

Then we set $\alpha_{0\epsilon}(t) = \alpha_0(t) - \epsilon\lambda(t)$, and proceed as before. □

In case $t_0 > \tau_0$, one can have a dual form of Theorem 3.1. Then, the assumption (iii) must be replaced by

(iii*) $t_0 > \tau_0$ and $f(t,x)$ is nonincreasing in t for each x.

Then the the dual result, that we now state, is valid.

3.2. Theorem. *Assume that the conditions* (i)*,* (ii) *and* (iii*) *hold. Then the conclusion of Theorem 3.1 remains valid.* □

Now we give an existence result with initial time difference.

3.3. Theorem. *Assume that*

(i) $\alpha \in C_p\big[[t_0, t_0+T], \mathbb{R}\big]$, $t_0, T > 0$, $\beta \in C_p^*\big[[\tau_0, \tau_0+T], \mathbb{R}\big]$, $\tau_0 > t_0$ *is locally Holder continuous for an exponent* $0 < \lambda < 1$ *and* $\lambda > q$ *where* $f \in C\big[[t_0, \tau_0+T]\times\mathbb{R}, \mathbb{R}\big]$,

$$D^q\alpha(t) \le f(t, \alpha(t)),\ t_0 \le t \le t_0+T,$$
$$D^q\beta(t) \ge f(t, \beta(t)),\ \tau_0 \le t \le \tau_0+T,$$

and $\alpha^0 \le x^0 \le \beta^0$*, where* $\alpha^0 = \alpha(t)(t-t_0)^{1-q}|_{t=t_0}$, $\beta^0 = \beta(t)(t-\tau_0)^{1-q}|_{t=\tau_0}$*;*

(ii) $f(t,x)$ *is nondecreasing in* t *for each* x *and* $\alpha(t) \le \beta(t+\eta)$, $t_0 \le t \le t_0+T$, *where* $\eta = \tau_0 - t_0$.

Then there exists a solution $x(t)$ *of* (1.1) *with* $x^0 = x(t)\cdot(t-t_0)^{1-q}|_{t=t_0}$ *satisfying* $\alpha(t) \le x(t) \le \beta(t+\eta)$ *on* $[t_0, t_0+T]$.

Proof. Let $\beta_0(t) = \beta(t+\eta)$, $t \ge t_0$. Then $\beta_0(t)(t-t_0)^{1-q}|_{t=t_0} = \beta^0 \ge x^0 \ge \alpha^0$. So we get

$$\begin{aligned} D^q\beta_0(t) &= D^q\beta(t+\eta) \\ &\ge f(t+\eta, \beta(t+\eta)) \\ &\ge f(t, \beta_0(t)). \end{aligned}$$

Let $p : [t_0, t_0+T]\times\mathbb{R} \to \mathbb{R}$ be defined by $p(t,x) = \max[\alpha(t), \min(x(t), \beta_0(t))]$. Then $f(t, p(t,x))$ defines a continuous extension of f to $[t_0, t_0+T]\times\mathbb{R}$ which is also bounded since f is bounded on Ω, where

(3.5) $\quad \Omega = \big\{(t,x) \mid t_0 \le t \le t_0+T,\ \alpha(t) \le x(t) \le \beta_0(t)\big\}.$

Therefore

(3.6) $\quad D^q x(t) = f(t, p(t,x)),\ x(t)(t-t_0)^{1-q}\big|_{t=t_0} = x^0$

has a solution $x(t)$ on $[t_0, t_0+T]$.

For $\epsilon > 0$ consider

(3.7) $\quad \alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ and $\beta_{0\epsilon}(t) = \beta_0(t) + \epsilon\lambda(t),$

where $\lambda(t) = (t-t_0)^{q-1} E_{q,q}\big((t-t_0)^q\big)$. It is clear that

$$\alpha_\epsilon(t)(t-t_0)^{1-q}\,|_{t=t_0} = \alpha(t)(t-t_0)^{1-q}\,|_{t=t_0} - \epsilon\lambda^0$$
$$\beta_{0\epsilon}(t)(t-t_0)^{1-q}\,|_{t=t_0} = \beta_0(t)(t-t_0)^{1-q}\,|_{t=t_0} + \epsilon\lambda^0,$$

it therefore follows that $\alpha_\epsilon^0 = \alpha^0 - \epsilon\lambda^0$, $\beta_{0\epsilon}^0 = \beta^0 + \epsilon\lambda^0$ which imply $\alpha_\epsilon^0 < x^0 < \beta_{0\epsilon}^0$ in view of $\lambda^0 > 0$.

We are now to show that $\alpha_\epsilon(t) < x(t) < \beta_{0\epsilon}(t)$ on $[t_0, t_0+T]$. Suppose that it is not true, and so there exists a $t_1 \in (t_0, t_0+T]$ such that

$$x(t_1) = \beta_{0\epsilon}(t_1) \text{ and } \alpha_\epsilon(t) < x(t) < \beta_{0\epsilon}(t),\ t_0 \le t < t_1.$$

Then we have $x(t_1) > \beta_0(t_1)$ and $p(t_1, x(t_1)) = \beta_0(t_1)$ and $\alpha(t_1) \le p(t_1, x(t_1)) \le \beta_0(t_1)$.

We set $m(t) = x(t) - \beta_{0\epsilon}(t)$. Note that $m(t_1) = 0$ and $m(t) \le 0$ on $[t_0, t_1]$. Hence by Lemma 2.5 we get $D^q m(t_1) \ge 0$, which gives

$$\begin{aligned} f(t_1, \beta_0(t_1)) = f(t_1, p(t_1, x(t_1))) &= D^q x(t_1) \ge D^q \beta_{0\epsilon}(t_1) \\ &= D^q \beta_0(t_1) + \epsilon\lambda(t_1) \\ &> f(t_1, \beta_0(t_1)), \end{aligned}$$

which is a contradiction. The other case can be proved similarly. Consequently, we obtain

(3.8) $$\alpha_\epsilon(t) < x(t) < \beta_{0\epsilon}(t) \text{ on } [t_0, t_0+T],$$

and making $\epsilon \to 0$ we have the desired result

(3.9) $$\alpha(t) \le x(t) \le \beta(t+\eta) \text{ on } [t_0, t_0+T].$$

In a similar manner, a dual result of Theorem 3.3 can be proved, as before, in the case $t_0 > \tau_0$.

3.4. Theorem. *Assume that the condition* (i) *of Theorem 3.3 holds. Assume further that*

(ii)* $t_0 > \tau_0$ *and* $f(t,x)$ *is nonincreasing in* t *for each* x.

Then the conclusion of Theorem 3.3 remains valid. □

4. Monotone iterative technique with initial time difference

The purpose of this section is to apply the monotone iterative technique for the nonlinear fractional order differential equation (1.1) by choosing lower and upper solutions with initial time difference.

4.1. Theorem. *Assume that*

(A$_1$) $\alpha \in C_p\big[[t_0, t_0+T], \mathbb{R}\big]$, t_0, $T > 0$, $\beta \in C_p^*\big[[\tau_0, \tau_0+T], \mathbb{R}\big]$, $\tau_0 > t_0$, *where* $p = 1-q$, $q \in (0,1)$ *and* $f \in C\big[[t_0, \tau_0+T] \times \mathbb{R}, \mathbb{R}\big]$ *where*

$$D^q\alpha(t) \le f(t, \alpha(t)),\ t_0 \le t \le t_0+T,$$
$$D^q\beta(t) \ge f(t, \beta(t)),\ \tau_0 \le t \le \tau_0+T,$$
$$\alpha^0 \le x^0 \le \beta^0 \text{ where } \alpha^0 = \alpha(t)(t-t_0)^{1-q}\big|_{t=t_0},\ \beta^0 = \beta(t)(t-\tau_0)^{1-q}\,|_{t=\tau_0}.$$

(A$_2$) $f(t,x)$ *is nondecreasing in* t *for each* x, *and* $\alpha(t) \le \beta(t+\eta)$ *on* $[t_0, t_0+T]$, $\eta = \tau_0 - t_0$.

(A$_3$) $f(t,x) - f(t,y) \ge -M(x-y)$ *for* $\alpha \le y \le x \le \beta$, $M > 0$.

Then there exist monotone sequences $\{\widetilde{\alpha}_n\}$ *and* $\{\widetilde{\beta}_n\}$ *which converge uniformly and monotonically on* $[t_0, t_0+T]$ *such that* $\widetilde{\alpha}_n \to \rho$, $\widetilde{\beta}_n \to r$ *as* $n \to \infty$. *Also,* ρ *and* r *are minimal and maximal solutions of (1.1) respectively.*

Proof. Let $\widetilde{\beta}_0(t) = \beta(t+\eta)$ and $\widetilde{\alpha}_0(t) = \alpha(t)$ for $t_0 \le t \le t_0 + T$, where $\eta = \tau_0 - t_0$. Using the nondecreasing property of $f(t,x)$ in t for each x we have

$$\begin{aligned} D^q \tilde{\beta}_0(t) &= D^q \beta(t+\eta) \\ &\ge f(t+\eta, \beta(t+\eta)) \\ &\ge f\big(t, \tilde{\beta}_0(t)\big), \end{aligned}$$

and

$$\tilde{\beta}_0^0 = \tilde{\beta}_0(t)(t-t_0)^{1-q}\big|_{t=t_0} = \beta(t+\eta)(t-t_0)^{1-q}\big|_{t=t_0} = \beta(t)(t-\tau_0)^{1-q}\big|_{t=\tau_0} = \beta^0.$$

Also,

$$D^q \widetilde{\alpha}_0(t) = D^q \alpha(t) \le f(t, \alpha_0(t)) = f(t, \widetilde{\alpha}_0(t))$$

and $\widetilde{\alpha}_0^0 = \widetilde{\alpha}_0(t)(t-t_0)^{1-q}\big|_{t=t_0} = \alpha(t)(t-t_0)^{1-q}\big|_{t=t_0} = \alpha^0$, $\alpha^0 \le x^0 \le \beta^0$, which proves that $\tilde{\beta}_0$ and $\widetilde{\alpha}_0$ are upper and lower solutions of (1.1), respectively.

For any $\theta \in C_p\big[[t_0, t_0+T], \mathbb{R}\big]$ such that $\widetilde{\alpha}_0(t) \le \theta \le \widetilde{\beta}_0(t)$ on $[t_0, t_0+T]$, we define the following linear fractional differential equation

$$D^q x(t) = f(t,\theta) - M.(x-\theta), \ x(t)(t-t_0)^{1-q}\big|_{t=t_0} = x^0. \tag{4.1}$$

Note that unique solutions exist since the right hand side of (4.1) satisfies a Lipschitz condition. Let A be a mapping such that $A\theta = x$, and construct the sequences $\{\widetilde{\alpha}_n\}$ and $\{\widetilde{\beta}_n\}$. We must prove that

(i) $\widetilde{\alpha}_0 \le A\widetilde{\alpha}_0$, $\tilde{\beta}_0 \ge A\tilde{\beta}_0$;
(ii) A is a monotone operator on the segment

$$\big[\widetilde{\alpha}_0, \tilde{\beta}_0\big] = \big\{x \in C_p\big[[t_0, t_0+T], \mathbb{R}\big] \mid \widetilde{\alpha}_0 \le x \le \tilde{\beta}_0\big\}.$$

To prove (i), we set $A\widetilde{\alpha}_0 = \widetilde{\alpha}_1$, where $\widetilde{\alpha}_1$ is the unique solution of (4.1) with $\theta = \widetilde{\alpha}_0$. Setting $p(t) = \widetilde{\alpha}_0(t) - \widetilde{\alpha}_1(t)$, we see that

$$\begin{aligned} D^q p &= D^q \widetilde{\alpha}_0 - D^q \widetilde{\alpha}_1 \\ &\le f\big(t, \widetilde{\alpha}_0\big) - \big[f\big(t, \widetilde{\alpha}_0\big) - M\big(\widetilde{\alpha}_1 - \widetilde{\alpha}_0\big)\big] \\ &= -M \cdot p \end{aligned}$$

and $p(t) \cdot (t-t_0)^{1-q}|_{t=t_0} \le 0$, which gives by Corollary 2.4 that $p(t) \le 0$. So we have $\widetilde{\alpha}_0 \le \widetilde{\alpha}_1$ on $[t_0, t_0+T]$. Similarly we can form $p = \tilde{\beta}_0 - \tilde{\beta}_1$, where $\tilde{\beta}_1 = A\tilde{\beta}_0$. Then

$$D^q p = D^q \tilde{\beta}_0 - D^q \tilde{\beta}_1 \ge f\big(t, \tilde{\beta}_0\big) - \big[f\big(t, \tilde{\beta}_0\big) - M\big(\tilde{\beta}_1 - \tilde{\beta}_0\big)\big].$$

This implies $D^q p \ge -Mp$ and $p(t)(t-t_0)^{1-q}\big|_{t=t_0} \ge 0$, which because of Corollary 2.4 yields $p(t) \ge 0$. Thus we obtain $\tilde{\beta}_1 \le \tilde{\beta}_0$ on $[t_0, t_0+T]$.

To prove (ii), let $\theta_1, \theta_2 \in \left[\widetilde{\alpha}_0, \widetilde{\beta}_0\right]$ be such that $\theta_1 \leq \theta_2$. Also suppose that $x_1 = A\theta_1$ and $x_2 = A\theta_2$. Set $p(t) = x_1 - x_2$, then

$$\begin{aligned} D^q p(t) &= D^q x_1 - D^q x_2 \\ &= f(t,\theta_1) - M(x_1 - \theta_1) - [f(t,\theta_2) - M(x_2 - \theta_2)] \\ &\leq -M(\theta_1 - \theta_2) - M((x_1 - \theta_1)) + M(x_2 - \theta_2) \\ &= -M(x_1 - x_2) \\ &= -Mp, \end{aligned} \tag{4.2}$$

and $p(t)(t-t_0)^{1-q}\big|_{t=t_0} = 0$. We have utilized the inequality in (A_3). Hence applying Corollary 2.4 we get $A\theta_1 \leq A\theta_2$.

We are now in a position to define the sequences $\widetilde{\alpha}_n = A\widetilde{\alpha}_{n-1}$ and $\widetilde{\beta}_n = A\,\widetilde{\beta}_{n-1}$, and conclude from the foregoing arguments that

$$\widetilde{\alpha}_0 \leq \widetilde{\alpha}_1 \leq \cdots \leq \widetilde{\alpha}_n \leq \widetilde{\beta}_n \leq \cdots \leq \widetilde{\beta}_1 \leq \widetilde{\beta}_0 \text{ on } [t_0, t_0+T]. \tag{4.3}$$

It is clear that the sequences $\{\widetilde{\alpha}_n\}$ and $\{\widetilde{\beta}_n\}$ are uniformly bounded on $[t_0, t_0+T]$, and by (4.1) it follows that $\left|D^q\widetilde{\alpha}_n\right|$ and $\left|D^q\widetilde{\beta}_n\right|$ are also uniformly bounded. As a result the sequences $\{\widetilde{\alpha}_n\}$ and $\{\widetilde{\beta}_n\}$ are equicontinuos on $[t_0, t_0+T]$ and consequently by Ascoli-Arzela's theorem there exist subsequences $\{\widetilde{\alpha}_{n_k}\}$ and $\{\widetilde{\beta}_{n_k}\}$ that converge uniformly on $[t_0, t_0+T]$. In view of (4.3) it follows that the entire sequences $\{\widetilde{\alpha}_n\}$ and $\{\widetilde{\beta}_n\}$ converge uniformly and monotonically to ρ and r, respectively, as $n \to \infty$.

Now we need to show that ρ and r are solutions of (1.1). For this aim, using the corresponding Volterra integral equation for (4.1), we can have

$$\begin{aligned} \widetilde{\alpha}_n &= \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left[f(s,\widetilde{\alpha}_{n-1}(s)) - M(\widetilde{\alpha}_n(s) - \widetilde{\alpha}_{n-1}(s))\right]ds \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \widetilde{\beta}_n &= \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left[f\left(s,\widetilde{\beta}_{n-1}(s)\right) - M\left(\widetilde{\beta}_n(s) - \widetilde{\beta}_{n-1}(s)\right)\right]ds \end{aligned} \tag{4.5}$$

as $n \to \infty$ we get

$$\rho = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left[f(s,\rho(s))\right]ds \tag{4.6}$$

and

$$r = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left[f(s,r(s))\right]ds, \tag{4.7}$$

where $x^0 = x(t)(t-t_0)^{1-q}\big|_{t=t_0}$.

Finally we must show that ρ, r are the minimal and maximal solutions of the initial value problem (1.1), respectively. Let x be any solution of (1.1) such that $\widetilde{\alpha}_0 \leq x \leq \widetilde{\beta}_0$

on $[t_0, t_0+T]$, then we need to prove $\widetilde{\alpha}_0 \leq \rho \leq x \leq r \leq \widetilde{\beta}_0$ on $[t_0, t_0+T]$. Suppose that for some n, $\widetilde{\alpha}_n \leq x \leq \widetilde{\beta}_n$. Then we set $p(t) = \widetilde{\alpha}_{n+1} - x$, thus

$$\begin{aligned} D^q p(t) &= D^q \widetilde{\alpha}_{n+1} - D^q x \\ &= f(t, \widetilde{\alpha}_n) - M(\widetilde{\alpha}_{n+1} - \widetilde{\alpha}_n) - f(t, x) \\ &\leq -M(\widetilde{\alpha}_n - x) - M(\widetilde{\alpha}_{n+1} - \widetilde{\alpha}_n) \\ &= -Mp, \end{aligned}$$

and $p(t)(t-t_0)^{1-q}\big|_{t=t_0} = 0$. It follows from Corollary 2.4 that $\widetilde{\alpha}_{n+1} \leq x$. In a similar manner we can show that $x \leq \widetilde{\beta}_{n+1}$ on $[t_0, t_0+T]$. This proves by induction that $\widetilde{\alpha}_n \leq x \leq \widetilde{\beta}_n$ for all n. Taking limits as $n \to \infty$ we arrive at $\rho \leq x \leq r$ on $[t_0, t_0+T]$, and the proof is complete. □

4.2. Corollary. *If in addition to the assumptions of Theorem 4.1, we assume*

$$f(t,x) - f(t,y) \leq M(x-y), \ \alpha \leq y \leq x \leq \beta, \ M > 0. \tag{4.8}$$

Then we have unique solution of (1.1) such that $\rho = x = r$.

Proof. If we set $p = r - \rho$ then $D^q p = D^q r - D^q \rho = f(t,r) - f(t,\rho) \leq M(r-\rho)$, which gives $D^q p \leq Mp$ and $p(t)(t-t_0)^{1-q}\big|_{t=t_0} = 0$. Again from Corollary 2.4, as before, we get $p(t) \leq 0$ on $[t_0, t_0+T]$, which implies $r \leq \rho$. Also, utilizing the fact that $\rho \leq r$, we have $\rho = x = r$ is the unique solution of (1.1). □

5. Conclusion

In this paper, some existence and comparison results in terms of lower and upper solutions have been rearranged relative to initial time difference. Also the well-known monotone iterative technique has been applied in a closed set for the given fractional R-L differential equations with initial time difference.

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