Hacettepe Journal of Mathematics and Statistics
Volume 40 (1) (2011), 41 – 52

ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A CERTAIN CLASS OF NON-LINEAR SINGULAR INTEGRAL EQUATIONS

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Received 21 : 08 : 2009 : Accepted 04 : 08 : 2010

Abstract

In this study, the existence of a solution of the non-linear singular integral equation system

$$\begin{aligned} w(z) &= f_1\Big(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \\ &\qquad\qquad \Pi_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)\Big), \\ h(z) &= f_2\Big(z, w(z), h(z), T_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \\ &\qquad\qquad \Pi_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)\Big), \end{aligned}$$

has been investigated. This system is more general than the one

$$\begin{aligned} w(z) &= f_1\big(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)\big), \\ h(z) &= f_2\big(z, w(z), h(z), \Pi_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)\big)(z)), \end{aligned}$$

studied by Musayev and Duz (*Existence and uniqueness theorems for a certain class of non linear singular integral equations* SJAM **10** (1), 3–18, 2009). Here, $T_G f(z)$ and $\Pi_G f(z)$ are the Vekua integral operators defined by

$$\begin{aligned} T_G f(z) &= -\frac{1}{\pi} \iint\limits_G \frac{f(\varsigma)}{\varsigma - z}\, d\xi\, d\eta, \\ \Pi_G f(z) &= -\frac{1}{\pi} \iint\limits_G \frac{f(\varsigma)}{(\varsigma - z)^2}\, d\xi\, d\eta. \end{aligned}$$

Keywords: Singular integral equations, Holder continuity, Fixed point theorem, Contraction mapping.

2000 AMS Classification: 45 E 05, 30 G 20, 32 A 55, 30 E 20.

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1. Introduction

Let $G \subset \mathbb{C}$ be a simply connected region with smooth boundary. As known, the system of real partial differential equations of the form

$$u_x - v_y = H_1(x, y, u, v, u_x, u_y, v_x, v_y)$$
$$u_y + v_x = H_2(x, y, u, v, u_x, u_y, v_x, v_y)$$

is equivalent to the complex partial differential equation

$$\partial_{\overline{z}} w = F(z, w, \partial_z w) \tag{1.1}$$

where

$$w = u + iv, z = x + iy, \partial_{\overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

The existence of a solution of the equation (1.1) satisfying the Dirichlet boundary conditions

$$\mathrm{Re} w|_{\partial G} = g(z), g \in C^\alpha(\partial G),$$
$$\mathrm{Im} w(z_0) = c_0, z_0 \in \overline{G},$$

in Holder space $C^\alpha(\overline{G})$, under suitable conditions, had been investigated by Tutschke [4]. Let the function F in (1.1) be a complex valued scalar function defined on the region

$$D = \{(z, w, h) : z \in \overline{G}, w, h \in \mathbb{C}\} = \overline{G} \times \mathbb{C}^2,$$

and let us consider the operators

$$T_G f(z) = -\frac{1}{\pi} \iint_G \frac{f(\zeta)}{\zeta - z}\, d\xi\, d\eta,$$
$$\Pi_G f(z) = -\frac{1}{\pi} \iint_G \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta,$$

$\zeta = \xi + i\eta$, for $f \in C^\alpha(\overline{G})$. In this case, the solutions w of the equation (1.1) satisfy the system of nonlinear singular integral equations

$$\begin{aligned} w(z) &= \phi(z) + T_G F(\cdot, w(\cdot), h(\cdot))(z),\\ h(z) &= \phi'(z) + \Pi_G F(\cdot, w(\cdot), h(\cdot))(z), \end{aligned} \tag{1.2}$$

where $h = \partial_z w$ and $\phi(z)$ are arbitrary holomorphic functions defined on G. The system (1.2), under weaker conditions on the function F using a variant of the Banach fixed point principle was studied by Altun, Koca and Musayev [1]. In [3], the less restrictive nonlinear singular integral equation system

$$\begin{aligned} w(z) &= f_1(z, w(z), h(z), T_G g_1(\cdot, w(\cdot), h(\cdot))(z)),\\ h(z) &= f_2(z, w(z), h(z), \Pi_G g_2(\cdot, w(\cdot), h(\cdot))(z)), \end{aligned} \tag{1.3}$$

has been studied. In this paper, the more general nonlinear singular integral equation system

$$\begin{aligned} w(z) &= f_1(\cdot, w(\cdot), h(\cdot), T_G g_1(\cdot, w(\cdot), h(\cdot)), \Pi_G g_1(\cdot, w(\cdot), h(\cdot)))(z),\\ h(z) &= f_2(\cdot, w(\cdot), h(\cdot), T_G g_2(\cdot, w(\cdot), h(\cdot)), \Pi_G g_2(\cdot, w(\cdot), h(\cdot)))(z), \end{aligned} \tag{1.4}$$

will be discussed for given functions f_1, f_2, g_1, g_2 under some conditions.

2. Main results

In this section, we will present some theorems related to the solutions of the system (1.4) under suitable conditions.

2.1. Definition. If for every $z_1, z_2 \in \overline{G}$ there are constants $H > 0$ and α satisfying the inequality:

$$|w(z_2) - w(z_1)| \leq H|z_2 - z_1|^{\alpha},\ 0 < \alpha < 1,$$

then the function $w : \overline{G} \to \mathbb{C}$ is said to *satisfy the Holder condition in the region* $\overline{G}$, or to be *Holder continuous.*

Let us denote the class of Holder continuous functions defined on $\overline{G}$ by $C^{\alpha}(\overline{G})$. This class is a vector space. On the other hand, $C^0(\overline{G}) \equiv C(\overline{G})$ is the class of all continuous functions on $\overline{G}$, and for $w \in C(\overline{G})$ in this class, the norm is defined to be

$$\|w\|_{\infty} \equiv \|w\|_{C(\overline{G})} = \max_{\overline{G}} \left\{ |w(z)| : z \in \overline{G} \right\}.$$

On the other hand, if the norm for $w \in C^{\alpha}(\overline{G})$ is defined as

$$\|w\|_{\alpha} \equiv \|w\|_{C^{(\alpha)}(\overline{G})} = \|w\|_{\infty} + H(w,\alpha)$$

where

$$H(w,\alpha) = \sup_{\overline{G}} \left\{ |w(z_1) - w(z_2)||z_1 - z_2|^{-\alpha} : z_1 \neq z_2, z_1, z_2 \in \overline{G} \right\},$$

then the class $C^{\alpha}(\overline{G})$ becomes a Banach space with this norm.

Let us denote the Holder continuous functions defined on $\overline{G}$ and having partial derivatives of first order with respect to the variables z and $\overline{z}$ by $C^{(1,\alpha)}(\overline{G})$. This class constitutes a Banach space with norm

$$\|w\|_{1,\alpha} \equiv \|w\|_{C^{(1,\alpha)}(\overline{G})} = \max \left\{ \|w\|_{\alpha}, \|\partial_z w\|_{\alpha}, \|\partial_{\overline{z}} w\|_{\alpha} \right\}$$

for $w \in C^{(1,\alpha)}(\overline{G})$. Moreover, the vector spaces

$$C^2(\overline{G}) = C(\overline{G}) \times C(\overline{G}) = \left\{ (w,h) : w, h \in C(\overline{G}) \right\},$$
$$C^{(\alpha),2}(\overline{G}) = C^{\alpha}(\overline{G}) \times C^{\alpha}(\overline{G}) = \left\{ (w,h) : w, h \in C^{\alpha}(\overline{G}) \right\},$$

having norms

$$\|(w,h)\|_{\infty,2} \equiv \|(w,h)\|_{C^2(\overline{G})} = \max \left\{ \|w\|_{\infty}, \|h\|_{\infty} \right\}$$
$$\|(w,h)\|_{\alpha,2} \equiv \|(w,h)\|_{C^{(\alpha),2}(\overline{G})} = \max \left\{ \|w\|_{\alpha}, \|h\|_{\alpha} \right\}$$

become Banach spaces. We denote these spaces by

$$(C^2(\overline{G});\ \|(\,\cdot\,,\,\cdot\,)\|_{\infty,2}) \text{ and } (C^{(\alpha),2}(\overline{G});\ \|(\,\cdot\,,\,\cdot\,)\|_{\alpha,2}),$$

respectively. Let

$$L_p(\overline{G}) = \left\{ f : \iint_G |f(z)|^p \, dx \, dy < \infty \right\},\ 1 \leq p < \infty.$$

Then, for $w \in L_p(\overline{G})$ consider the norm

$$\|(w,h)\|_{p,2} \equiv \|(w,h)\|_{L_p^2(\overline{G})} = \max \left\{ \|w\|_p, \|h\|_p \right\},$$

defined for $(w,h) \in L_p^2(\overline{G})$, where

$$L_p^2(\overline{G}) = L_p(\overline{G}) \times L_p(\overline{G})$$

and

$$\|w\|_p \equiv \|w\|_{L_p(\overline{G})} = \left(\iint_G |w(\zeta)|^p \, d\xi \, d\eta \right)^{\frac{1}{p}}.$$

Let $d = \max_{z_1, z_2 \in \overline{G}} |z_1 - z_2|$.

2.2. Lemma. [1] *If then for* $1 < p < \infty$ *and* $0 < \varepsilon \leq d$ *we have the following inequality:*

$$\|(w,h)\|_{\infty,2} \leq 2.\varepsilon^{\alpha} \|(w,h)\|_{\alpha,2} + \frac{1}{(\pi\epsilon^2)^{\frac{1}{p}}} \|(w,h)\|_{p,2}. \qquad \square$$

2.3. Theorem. [1] *For* $(w,h) \in C^{(\alpha),2}(\overline{G})$, $0 < \alpha < 1$ *and* $1 < p < \infty$, *the following inequality holds:*

$$\|(w,h)\|_{\infty,2} \leq M(\alpha,p) \|(w,h)\|_{\alpha,2}^{\frac{2}{2+\alpha p}} \|(w,h)\|_{p,2}^{\frac{\alpha p}{2+\alpha p}}. \tag{2.1}$$

Here

$$M(\alpha,p) = \max\{M_1(\alpha,p), M_2(\alpha,p)\},$$

where

$$m(\alpha,p) = (\alpha p \sqrt[p]{\pi})^{-\frac{p}{2+\alpha p}},$$

$$M_1(\alpha,p) = 2m^{\alpha}(\alpha,p) + (\pi m^2(\alpha,p))^{-\frac{1}{p}},$$

$$M_2(\alpha,p) = \frac{2\sqrt[p]{4}}{\sqrt[p]{4}-1} m^{\alpha}(\alpha,p). \qquad \square$$

2.4. Definition. Let

$$h : \overline{D} \to \mathbb{C},$$

where $\overline{D} = \overline{G} \times \mathbb{C}^4$ be given. If for every

$$(z_1, p_1, q_1, r_1, s_1), (z_2, p_2, q_2, r_2, s_2) \in \overline{D}$$

there are positive numbers

$$l_1, l_2, l_3, l_4, l_5$$

satisfying

$$\begin{aligned} &|h(z_1, p_1, q_1, r_1, s_1) - h(z_2, p_2, q_2, r_2, s_2)| \\ &\qquad \leq l_1 |z_1 - z_2|^{\alpha} + l_2 |p_1 - p_2| + l_3 |q_1 - q_2| + l_4 |r_1 - r_2| + l_5 |s_1 - s_2| \end{aligned} \tag{2.2}$$

then the function h is said to be of class $H_{\alpha,1,1,1,1}(l_1, l_2, l_3, l_4, l_5; \overline{D})$ over $\overline{D}$, and we write $h \in H_{\alpha,1,1,1,1}(l_1, l_2, l_3, l_4, l_5; \overline{D})$.

2.5. Definition. Let $h^* : \overline{D}_1 \to \mathbb{C}$, where $\overline{D_1} = \overline{G} \times \mathbb{C}^2$. If for every $(z_1, p_1, q_1), (z_2, p_2, q_2) \in \overline{D_1}$ there are positive numbers m_1, m_2, m_3 satisfying

$$|h^*(z_1, p_1, q_1) - h^*(z_2, p_2, q_2)| \leq m_1 |z_1 - z_2|^{\alpha} + m_2 |p_1 - p_2| + m_3 |q_1 - q_2| \tag{2.3}$$

then the function h^* is said to be of class $H_{\alpha,1,1}(m_1, m_2, m_3; \overline{D_1})$ over $\overline{D}_1$, and we write $h^* \in H_{\alpha,1,1}(m_1, m_2, m_3; \overline{D_1})$.

Let us define a norm for the bounded operators T_G and Π_G as follows:

$$\|T_G\|_{\alpha} = \sup \left\{ \|T_G w\|_{\alpha} : w \in C^{(\alpha)}(\overline{G}), \|w\|_{\alpha} < 1 \right\},$$

$$\|\Pi_G\|_{\alpha} = \sup \left\{ \|\Pi_G w\|_{\alpha} : w \in C^{(\alpha)}(\overline{G}), \|w\|_{\alpha} < 1 \right\}.$$

2.6. Lemma. *Let* $f_k \in H_{\alpha,1,1,1,1}(l_{k1}, l_{k2}, l_{k3}, l_{k4}, l_{k5}; \overline{D}), g_k \in H_{\alpha,1,1}(m_{k1}, m_{k2}, m_{k3}; \overline{D_1})$, $(k = 1, 2)$, $\theta = (0, 0)$ *and* $S_\alpha(\theta, R) = \{(w, h) : \|(w, h)\|_{\alpha,2} \le R\}$. *If*

$$\begin{aligned}
&l_{0k} = \max\big\{|f_k(z, 0, 0, 0, 0)| : z \in \overline{G}\big\},\\
&m_{ok} = \max\big\{|g_k(z, 0, 0)| : z \in \overline{G}\big\},\\
&K_1 = l_{01} + l_{11} + 2(l_{12} + l_{13})R + [2m_{01} + 4m_{11} + 4(m_{12} + m_{13})R]\\
&\qquad\qquad \times \big(l_{14}\|T_G\|_\alpha + l_{15}\|\Pi_G\|_\alpha\big),\\
&K_2 = l_{02} + l_{21} + 2(l_{22} + l_{23})R + [2m_{02} + 4m_{21} + 4(m_{22} + m_{23})R]\\
&\qquad\qquad \times \big(l_{24}\|T_G\|_\alpha + l_{25}\|\Pi_G\|_\alpha\big),\\
&\max\{K_1, K_2\} \le R,
\end{aligned}$$

then for

$$\begin{aligned}
\tilde{w}(z) &= f_1(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \Pi_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z))\\
\tilde{h}(z) &= f_2(z, w(z), h(z), T_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \Pi_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z))
\end{aligned}$$

the operator

$$\begin{aligned}
&A : C^{(\alpha),2}(\overline{G}) \to C^{(\alpha),2}(\overline{G}),\ 0 < \alpha < 1,\\
&\qquad (w, h) \mapsto A(w, h) = (\tilde{w}, \tilde{h})
\end{aligned}$$

transforms the ball $S_\alpha(\theta; R)$ *into itself.*

Proof. From the definition of $\tilde{w}(z)$,

$$\begin{aligned}
|\tilde{w}(z)| &= \big|f_1(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \Pi_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z))\big|\\
&\le \big|f_1\big(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z), \Pi_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)\big)\\
&\quad - f_1(z, 0, 0, T_G g_1(\,\cdot\,, 0, 0)(z), \Pi_G g_1(\,\cdot\,, 0, 0)(z))\big|\\
&\quad + \big|f_1(z, 0, 0, T_G g_1(\,\cdot\,, 0, 0)(z), \Pi_G g_1(\,\cdot\,, 0, 0)(z)) - f_1(z, 0, 0, 0, 0)\big|\\
&\quad + |f_1(z, 0, 0, 0, 0)|.
\end{aligned}$$

From the inequality (2.2) we can write

$$\begin{aligned}
|\tilde{w}(z)| &\le l_{12}|w(z)| + l_{13}|h(z)| + l_{14}|T_G[g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z) - g_1(\,\cdot\,, 0, 0)(z)]|\\
&\quad + l_{15}|\Pi_G[g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z) - g_1(\,\cdot\,, 0, 0)(z)]| + l_{14}|T_G g_1(\,\cdot\,, 0, 0)(z)|\\
&\quad + l_{15}|\Pi_G g_1(\,\cdot\,, 0, 0)(z)| + l_{01}\\
&\le l_{12}|w(z)| + l_{13}|h(z)| + l_{14}\|T_G\|_\alpha \|g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)) - g_1(\,\cdot\,, 0, 0)\|_{C^\alpha(\overline{D_1})}\\
&\quad + l_{15}\|\Pi_G\|_\alpha \|g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)) - g_1(\,\cdot\,, 0, 0)\|_{C^\alpha(\overline{D_1})}\\
&\quad + l_{14}\|T_G\|_\alpha \|g_1(\,\cdot\,, 0, 0)\|_{C^\alpha(\overline{D_1})} + l_{15}\|\Pi_G\|_\alpha \|g_1(\,\cdot\,, 0, 0)\|_{C^\alpha(\overline{D_1})} + l_{01}
\end{aligned} \tag{2.4}$$

Now let us obtain a bound for

$$\|g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)) - g_1(\,\cdot\,, 0, 0)\|_{C^{(\alpha)}(\overline{D_1})}.$$

For every $z, z_1, z_2 \in \overline{G}$ from (2.3),

$$|g_1(z, w(z), h(z)) - g_1(z, 0, 0)| \le m_{12}|w(z)| + m_{13}|h(z)| \le (m_{12} + m_{13})R \tag{2.5}$$

and

$$\begin{aligned}
&\big|\big[g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))-g_1(\,\cdot\,,0,0)\big](z_1)-[g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))-g_1(\,\cdot\,,0,0)](z_2)\big|\\
&\quad=|g_1(z_1,w(z_1h(z_1))-g_1\big(z_2,w(z_2)h(z_2)\big)\\
&\qquad\qquad-[g_1(z_1,0,0)-g_1(z_2,0,0)]|\\
&\quad\le m_{11}|z_1-z_2|^\alpha+m_{12}|w(z_1)-w(z_2)|\\
&\qquad\qquad+m_{13}|h(z_1)-h(z_2)|+m_{11}|z_1-z_2|^\alpha\\
&\quad\le 2m_{11}|z_1-z_2|^\alpha+m_{12}\|w\|_{C^{(\alpha)}}(\overline{G})|z_1-z_2|^\alpha\\
&\qquad\qquad+m_{13}\|h\|_{C^{(\alpha)}(\overline{G})}|z_1-z_2|^\alpha\\
&\quad\le[2m_{11}+(m_{12}+m_{13})R]|z_1-z_2|^\alpha
\end{aligned}$$

from the inequality(2.5), we can write

$$\|g_1((\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))-g_1(\,\cdot\,,0,0)\|_{C^{(\alpha)}(\overline{D_1})}\le 2[m_{11}+(m_{12}+m_{13})R] \tag{2.6}$$

Now let us obtain a bound for $\|g_1(\,\cdot\,,0,0)\|_{C^{(\alpha)}(\overline{D_1})}$. For any $z_1,z_2\in\overline{G}$, since

$$\begin{aligned}
|g_1(\,\cdot\,,0,0)(z_2)-g_1(\,\cdot\,,0,0)(z_1)|&=|g_1(z_2,0,0)-g_1(z_1,0,0)|\\
&\le m_{11}|z_1-z_2|^\alpha,
\end{aligned}$$

we can write

$$\|g_1(\,\cdot\,,0,0)\|_{C^{(\alpha)}(\overline{D_1})}\le m_{11}+m_{01}. \tag{2.7}$$

Using the inequalities (2.6) and (2.7) in (2.4), for every $z\in\overline{G}$, we obtain

$$\begin{aligned}
|\tilde w(z)|&\le(l_{12}+l_{13})R+(l_{14}\|T_G\|_\alpha+l_{15}\|\Pi_G\|_\alpha)2[m_{11}+(m_{12}+m_{13})R]\\
&\quad+(l_{14}\ \|T_G\|_\alpha+l_{15}\|\Pi_G\|_\alpha)(m_{11}+m_{01})+l_{01}.
\end{aligned}$$

Now, let us obtain the Holder constant $H(\tilde w,\alpha)$. For every $z_1,z_2\in\overline{G}$,

$$\begin{aligned}
|\tilde w(z_1)-\tilde w(z_2)|&\le|f_1(z_1,w(z_1),h(z_1),T_Gg_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_1),\\
&\qquad\qquad\Pi_Gg_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,)))(z_1)\\
&\qquad-f_1(z_2,w(z_2),h(z_2),T_Gg_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_2),\\
&\qquad\qquad\Pi_Gg_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_2))|\\
&\le l_{11}|z_1-z_2|^\alpha+l_{12}|w(z_1)-w(z_2)|+l_{13}|h(z_1)-h(z_2)|\\
&\quad+l_{14}|T_G[g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_1)-g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_2)]|\\
&\quad+l_{15}|\Pi_G[g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_1)-g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))(z_2)]|\\
&\le[l_{11}+(l_{12}+l_{13})R]|z_1-z_2|^\alpha+(l_{14}\|T_G\|_\alpha\\
&\qquad+l_{15}\|\Pi_G\|_\alpha)\|g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))\|_\alpha|z_1-z_2|^\alpha\\
&=[l_{11}+(l_{12}+l_{13})R+(l_{14}\|T_G\|_\alpha\\
&\qquad+l_{15}\|\Pi_G\|_\alpha)\|g_1(\,\cdot\,,w(\,\cdot\,),h(\,\cdot\,))\|_\alpha]|z_1-z_2|^\alpha.
\end{aligned}$$

Moreover, for every $z,z_1,z_2\in\overline{G}$,

$$\begin{aligned}
|g_1(z,w(z),h(z))|&\le|g_1(z,w(z),h(z))-g_1(z,0,0)|+|g_1(z,0,0)|\\
&\le(m_{12}+m_{13})R+m_{01}.
\end{aligned}$$

From (2.3),

$$|g_1(z_1, w(z_1), h(z_1)) - g_1(z_2, w(z_2), h(z_2))| \\ \leq [m_{11} + (m_{12} + m_{13})R]|z_1 - z_2|^\alpha,$$
$$\|g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))\|_\alpha \leq [m_{01} + m_{11} + 2(m_{12} + m_{13})R].$$

Hence, for any $z_1, z_2 \in \overline{G}$,

$$\big|\tilde{w}(z_2) - \tilde{w}(z_1)\big| \leq [l_{11} + (l_{12} + l_{13})R + \big(l_{14}\|T_G\|_\alpha + l_{15}\|\Pi_G\|_\alpha\big) \\ \times (m_{01} + m_{11} + 2(m_{12} + m_{13})R)]\big|z_1 - z_2\big|^\alpha,$$

so we obtain

$$H(\tilde{w}, \alpha) = [l_{11}+(l_{12}+l_{13})R+(l_{14}\|T_G\|_\alpha+l_{15}\|\Pi_G\|_\alpha)(m_{01}+m_{11}+2(m_{12}+m_{13})R)].$$

Thus, for

$$K_1 = l_{01}+l_{11}+2(l_{12}+l_{13})R+[2m_{01}+4m_{11}+4(m_{12}+m_{13})R]\big(l_{14}\|T_G\|_\alpha+l_{15}\|\Pi_G\|_\alpha\big)$$

we get

$$\|\tilde{w}\|_\alpha \leq K_1.$$

In a similar way, for

$$K_2 = l_{02}+l_{21}+2(l_{22}+l_{23})R+[2m_{02}+4m_{21}+4(m_{22}+m_{23})R]\big(l_{24}\|T_G\|_\alpha+l_{25}\|\Pi_G\|_\alpha\big)$$

it can be shown that

$$\|\tilde{h}\|_\alpha \leq K_2.$$

Therefore,

$$\|(\tilde{w}, \tilde{h})\|_{\alpha,2} = \max\{\|\tilde{w}\|_\alpha, \|\tilde{h}\|_\alpha\} \leq \max\{K_1, K_2\}.$$

If $\max\{K_1, K_2\} \leq R$, then $\|(\tilde{w}, \tilde{h})\|_{\alpha,2} \leq R$, i.e., $A(w, h) = (\tilde{w}, \tilde{h}) \in S_\alpha(\theta, R)$. □

2.7. Lemma. [3] *The ball $S_\alpha(\theta, R)$ is compact in $(C^{(\alpha),2}(\overline{G}); \|(\,\cdot\,,\,\cdot\,)\|_{\infty,2})$.* □

2.8. Lemma. *The sphere $S_\alpha(\theta, R)$ is a complete subspace of $(C^{(\alpha),2}(\overline{G}); \|(\,\cdot\,,\,\cdot\,)\|_{\infty,2})$.* □

For $(w, h),\ (\tilde{w}, \tilde{h}) \in C^{(\alpha),2}(\overline{G}),\ (0 < \alpha < 1)$, let

$$d_{\infty,2}[(w, h), (\tilde{w}, \tilde{h})] = \|(w, h) - (\tilde{w}, \tilde{h})\|_{\infty,2},$$

and for $1 \leq p < \infty$,

$$d_{\alpha,2}[(w, h), (\tilde{w}, \tilde{h})] = \|(w, h) - (\tilde{w}, \tilde{h})\|_{\alpha,2},$$
$$d_{p,2}[(w, h), (\tilde{w}, \tilde{h})] = \|(w, h) - (\tilde{w}, \tilde{h})\|_{p,2}.$$

The transformations

$$d_{\infty,2},\ d_{p,2} : C^{(\alpha),2}(\overline{G}) \times C^{(\alpha),2}(\overline{G}) \to [0, \infty)$$

define metrics on $C^{(\alpha),2}(\overline{G})$. Thus, $(C^{(\alpha),2}(\overline{G}); d_{\infty,2})$ and $(C^{(\alpha),2}(\overline{G}); d_{p,2})$ become metric spaces.

2.9. Lemma. [1] *Let $0 < \alpha < 1$ and $1 \leq p < \infty$. Then convergence on the ball $S_\alpha(\theta, R)$ with respect to the metrics $d_{\infty,2}$ and $d_{p,2}$ are equivalent.* □

2.10. Lemma. *Let*

$$f_k \in H_{\alpha,1,1,1,1}(l_{k1}, l_{k2}, l_{k3}, l_{k4}, l_{k5}; \overline{D}) \text{ and } g_k \in H_{\alpha,1,1}(m_{k1}, m_{k2}, m_{k3}; \overline{D_1}),$$

$k = 1, 2$, $0 < \alpha < 1$ *and* $1 < p < \infty$. *In this case, for the operator* A *defined in Lemma 2.6, the inequality*

$$d_{p,2}[A(w_1, h_1), A(w_2, h_2)] \le M_3(p) d_{\infty,2}[(w_1, h_1), (w_2, h_2)] \tag{2.8}$$

is satisfied for all $(w_1, h_1), (w_2, h_2) \in S_\alpha(\theta, R)$, *where*

$$M_3(p) = (mG)^{\frac{1}{p}} \max\{l_{12} + l_{13} + \big(l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p\big)(m_{12} + m_{13}), \\ l_{22} + l_{23} + (l_{24}\|T_G\|_p + l_{25}\|\Pi_G\|_p)(m_{22} + m_{23})\}.$$

Proof. For all

$$(w_1, h_1),\ (w_2, h_2) \in S_\alpha(\theta, R)$$

and $z \in \overline{G}$, using

$$\|A(w_1, h_1) - A(w_2, h_2)\|_{p,2} = \|(\tilde{w}_1, \tilde{h}_1) - (\tilde{w}_2, \tilde{h}_2)\|_{p,2} \max\{\|\tilde{w}_1 - \tilde{w}_2\|_p, \|\tilde{h}_1 - \tilde{h}_2\|_p\}$$

let us find upper bounds for

$$\|\tilde{w}_1 - \tilde{w}_2\|_p \text{ and } \|\tilde{h}_1 - \tilde{h}_2\|_p.$$

For all $z \in \overline{G}$, since

$$\|\tilde{w}_1 - \tilde{w}_2\|_p \le l_{12}|w_1(z) - w_2(z)| + l_{13}|h_1(z) - h_2(z)| \\ + l_{14}|T_G(g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot)))(z)| \\ + l_{15}|\Pi_G(g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot)))(z)|,$$

from Minkowski's inequality and $g_1 \in H_{\alpha,1,1}(m_{11}, m_{12}, m_{13}; \overline{D_1})$, we obtain

$$\left(\iint_G |\tilde{w}_1(z) - \tilde{w}_2(z)|^p \, dx\, dy\right)^{\frac{1}{p}}$$

$$\le \left\{ \iint_G t[l_{12}|w_1(z) - w_2(z)| + l_{13}|h_1(z) - h_2(z)| \\ + l_{14}|T_G[g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot))]| \\ + l_{15}|\Pi_G[g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot))]|^p \, dx\, dy \right\}^{1/p}$$

$$\le \big(l_{12}\|w_1 - w_2\|_p + l_{13}\|h_1 - h_2\|_p \\ + l_{14}\|T_G\|_p \|g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot))\|_p. \\ + l_{15}\|\Pi_G\|_p \|g_1(\cdot, w_1(\cdot), h_1(\cdot)) - g_1(\cdot, w_2(\cdot), h_2(\cdot))\|_p\big)(mG)^{\frac{1}{p}}$$

$$\le \big((l_{12} + l_{13}) \max\{\|w_1 - w_2\|_p, \|h_1 - h_2\|_p\} \\ + l_{14}\|T_G\|_p(m_{12}\|w_1 - w_2\|_p + m_{13}\|h_1 - h_2\|_p) \\ + l_{15}\|\Pi_G\|_p(m_{12}\|w_1 - w_2\|_p + m_{13}\|h_1 - h_2\|_p)\big)(mG)^{\frac{1}{p}}$$

$$\le \big([(l_{12} + l_{13}) + (l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p)(m_{12} + m_{13})] \\ \times \max\{\|w_1 - w_2\|_p, \|h_1 - h_2\|_p\}\big)(mG)^{\frac{1}{p}}$$

$$\le (mG)^{\frac{1}{p}}[l_{12} + l_{13} + \big(l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p\big)(m_{12} + m_{13})] \\ \times d_{\infty,2}[(w_1, h_1), (w_2, h_2)].$$

Thus we get

(2.9) $$\|\tilde{w}_1 - \tilde{w}_2\|_p \le (mG)^{\frac{1}{p}}[l_{12} + l_{13} + (l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p)(m_{12} + m_{13})]\hat{d}_{\infty,2}.$$

Similarly

(2.10) $$\|\tilde{h}_1 - \tilde{h}_2\|_p \le (mG)^{\frac{1}{p}}[l_{22} + l_{23} + (l_{24}\|T_G\|_p + l_{25}\|\Pi_G\|_p)(m_{22} + m_{23})_p]\hat{d}_{\infty,2},$$

where

$$\hat{d}_{\infty,2} = d_{\infty,2}[(w_1, h_1), (w_2, h_2)].$$

The required inequality (2.8) is obtained with the help of the inequalities (2.9) and (2.10). □

2.11. Lemma. [1] *Assume the conditions of Lemma 2.6 are satisfied. Let* $\max\{K_1, K_2\} \le R$. *In this case, the operator* $A : S_\alpha(\theta; R) \to S_\alpha(\theta; R)$ *defined in Lemma 2.6, is a continuous operator with respect to the metric* $d_{\infty,2}$. □

2.12. Theorem. [3] *Let,*

$$f_k \in H_{\alpha,1,1,1,1}(l_{k1}, l_{k2}, l_{k3}, l_{k4}, l_{k5}; \overline{D}),\ \ g_k \in H_{\alpha,1,1}(m_{k1}, m_{k2}, m_{k3}; \overline{D_1}),$$

$k = 1, 2$ *and* $\max\{K_1, K_2\} \le R$. *The nonlinear singular integral equation system (1.4) has at least one solution on the sphere* $S_\alpha(\theta, R)$. □

Now, let us study the uniqueness of the solution of the system (1.4) and how to find it. For this, we use a variant of Banach's fixed point theorem:

2.13. Theorem. [2] *Assume that the following hypothses hold:*

(1) *Let* (X, ρ_1) *be a compact metric space.*
(2) *Let* ρ_2 *be another metric on* X *such that any sequence converging with respect to* ρ_1 *is also convergent in* ρ_2.
(3) *Let the operator* $A : X \to X$ *be a contraction mapping with respect to* ρ_2, *i.e. let for any* $x, y \in X$ *there exist a number* $0 \le q < 1$ *such that*

$$\rho_2(Ax, Ay) \le q\rho_2(x, y).$$

Then the equation $x = Ax$ *has a unique solution* x_* *and* $x_0 \in X$ *being any initial element, the sequence* (x_n) *defined by* $x_n = Ax_{n-1}$, $n = 1, 2, \ldots,$ *converges to* x_* *with speed*

$$\rho_2(x_n, x_*) \le \frac{q^n}{1-q}\rho_2(x_1, x_0).$$ □

2.14. Theorem. *Let the conditions*

$$f_k \in H_{\alpha,1,1,1,1}(l_{k1}, l_{k2}, l_{k3}, l_{k4}, l_{k5}; \overline{D}),\ \ g_k \in H_{\alpha,1,1}(m_{k1}, m_{k2}, m_{k3}; \overline{D_1}),$$

$k = 1, 2)$, $0 < \alpha < 1$, $\max\{K_1, K_2\} \le R$, *and*

$$l = \max\{l_{12} + l_{13} + (l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p)(m_{12} + m_{13}),$$
$$l_{22} + l_{23} + (l_{24}\|T_G\|_p + l_{25}\|\Pi_G\|_p)(m_{22} + m_{23})\} < 1$$

hold. Then the system (1.4) of nonlinear singular integral equations has a unique solution $(w_*, h_*) \in S_\alpha(\theta, R)$. *This solution is the limit of the sequence* (w_n, h_n) *defined by*

(2.11) $$\begin{aligned} w_n(z) &= f_1(z, w_{n-1}(z), h_{n-1}(z), T_G g_1(\,\cdot\,, w_{n-1}(\,\cdot\,), h_{n-1}(\,\cdot\,)), \\ &\qquad \Pi_G g_1(\,\cdot\,, w_{n-1}(\,\cdot\,), h_{n-1}(\,\cdot\,)))(z) \\ h_n(z) &= f_2(z, w_{n-1}(z), h_{n-1}(z), T_G g_2(\,\cdot\,, w_{n-1}(\,\cdot\,), h_{n-1}(\,\cdot\,)), \\ &\qquad \Pi_G g_2(\,\cdot\,, w_{n-1}(\,\cdot\,), h_{n-1}(\,\cdot\,)))(z) \end{aligned}$$

$n = 1, 2, \ldots,$ *where* $(w_0, h_0) \in S_\alpha(\theta, R)$ *is any initial element. Moreover, the inequality*

$$d_{p,2}[(w_n, h_n), (w_*, h_*)] \le \frac{l^n}{1-l} d_{p,2}[(w_1, h_1), (w_0, h_0)]$$

holds.

Proof. Let $X = S_\alpha(\theta, R)$, $\rho_1 = d_{\alpha,2}$ and $\rho_2 = d_{p,2}$ in Theorem 2.13. Let A be an operator defined in the Lemma 2.6. Since $\max\{K_1, K_2\} \le R$, the operator A transform the space $(X, d_{\alpha,2})$ into itself.

Now let us show that when $l < 1$, the operator A is a contraction operator on the sphere $S_\alpha(\theta, R)$ with respect to the metric $d_{p,2}$.

For $(w, h) \in S_\alpha(\theta, R)$,

$$A(w,h)(z) = (A_1(w,h)(z), A_2(w,h)(z)),$$

where

$$\begin{aligned} A_1(w,h)(z) &= f_1(z, w(z), h(z), T_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)), \Pi_G g_1(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)),\\ A_2(w,h)(z) &= f_2(z, w(z), h(z), T_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,)), \Pi_G g_2(\,\cdot\,, w(\,\cdot\,), h(\,\cdot\,))(z)). \end{aligned}$$

Thus for any

$$\big(w^{(1)}, h^{(1)}\big),\ \big(w^{(2)}, h^{(2)}\big) \in S^\alpha(\theta; R)$$

we can write

$$\begin{aligned} d_{p,2}\big(A\big(w^{(1)}, h^{(1)}\big), A\big(w^{(2)}, h^{(2)}\big)\big) = \max\big\{&\|A_1\big(w^{(1)}, h^{(1)}\big) - A_1\big(w^{(2)}, h^{(2)}\big)\|_p,\\ &\|A_2\big(w^{(1)}, h^{(1)}\big) - A_2\big(w^{(2)}, h^{(2)}\big)\|_p\}. \end{aligned}$$

Since $f_1 \in H_{\alpha,1,1,1,1}(l_{11}, l_{12}, l_{13}, l_{14}, l_{15}; \overline{D})$, $g_1 \in H_{\alpha,1,1}(m_{11}, m_{12}, m_{13}; \overline{D_1})$,

$$\begin{aligned} &\big\|A_1\big(w^{(1)}, h^{(1)}\big) - A_1\big(w^{(2)}, h^{(2)}\big)\big\|_p\\ &\quad= \big\|f_1\big(w^{(1)}, h^{(1)}, T_G g_1\big(\,\cdot\,, w^{(1)}, h^{(1)}\big), \Pi_G g_1\big(\,\cdot\,, w^{(1)}, h^{(1)}\big)\big)\\ &\qquad\qquad - f_1\big(w^{(2)}, h^{(2)}, T_G g_1\big(\,\cdot\,, w^{(2)}, h^{(2)}\big), \Pi_G g_1\big(\,\cdot\,, w^{(2)}, h^{(2)}\big)\big)\big\|_p\\ &\quad= l_{12}\big\|w^{(1)} - w^{(2)}\big\|_p + l_{13}\big\|h^{(1)} - h^{(2)}\big\|_p + l_{14}\big|T_G\big(g_1\big(\,\cdot\,, w^{(1)}(\,\cdot\,), h^{(1)}(\,\cdot\,)\big)\\ &\qquad\qquad - g_1\big(\,\cdot\,, w^{(2)}(\,\cdot\,), h^{(2)}(\,\cdot\,)\big)\big)(z)\big| + l_{15}\big|\Pi_G\big(g_1\big(\,\cdot\,, w^{(1)}(\,\cdot\,), h^{(1)}(\,\cdot\,)\big)\\ &\qquad\qquad\qquad - g_1(\,\cdot\,, w^{(2)}(\,\cdot\,), h^{(2)}(\,\cdot\,)))(z)|]^p\, dx\, dy\}^{\frac{1}{p}}\\ &\quad\le l_{12}\big\|w^{(1)} - w^{(2)}\big\|_p + l_{13}\big\|h^{(1)} - h^{(2)}\big\|_p + l_{14}\bigg(\iint\limits_G \big|T_G\big(g_1\big(\,\cdot\,, w^{(1)}(\,\cdot\,), h^{(1)}(\,\cdot\,)\big)\\ &\qquad\qquad\qquad - g_1\big(\,\cdot\,, w^{(2)}(\,\cdot\,), h^{(2)}(\,\cdot\,)\big)\big)(z)\big|^p\, dx\, dy\bigg)^{\frac{1}{p}}\\ &\qquad + l_{15}\bigg(\iint\limits_G \big|\Pi_G(g_1(\,\cdot\,, w^{(1)}(\,\cdot\,), h^{(1)}(\,\cdot\,))\\ &\qquad\qquad\qquad - g_1(\,\cdot\,, w^{(2)}(\,\cdot\,), h^{(2)}(\,\cdot\,)))(z)|^p\, dx\, dy\bigg)^{\frac{1}{p}}\\ &\quad\le l_{12}\big\|w^{(1)} - w^{(2)}\big\|_p + l_{13}\big\|h^{(1)} - h^{(2)}\big\|_p\\ &\qquad + l_{14}\big\|T_G\big\|_p\bigg(\iint\limits_G \big|\big(g_1(z, w^{(1)}(z), h^{(1)}(z)) - g_1\big(z, w^{(2)}(z), h^{(2)}(z)\big)\big)\big|^p\, dx\, dy\bigg)^{\frac{1}{p}}\\ &\quad\le l_1 d_{p,2}\big(\big(w^{(1)}, h^{(1)}\big), \big(w^{(2)}, h^{(2)}\big)\big) \end{aligned}$$

where

$$l_1 = l_{12} + l_{13} + (m_{12} + m_{13})(l_{14}\|T_G\|_p + l_{15}\|\Pi_G\|_p).$$

Similarly

$$\left\|A_2\big(w^{(1)}, h^{(1)}\big) - A_2\big(w^{(2)}, h^{(2)}\big)\right\|_p \le l_2 d_{p,2}\big(\big(w^{(1)}, h^{(1)}\big), \big(w^{(2)}, h^{(2)}\big)\big),$$

where

$$l_2 = l_{22} + l_{23} + \big(m_{22} + m_{23}\big)\big(l_{24}\|T_G\|_p + l_{25}\|\Pi_G\|_p\big).$$

Thus, for

$$l = \max\{l_1, l_2\}$$

we can write

$$d_{p,2}\big[A\big(w^{(1)}, h^{(1)}\big), A\big(w^{(2)}, h^{(2)}\big)\big] \le l d_{p,2}\big[\big(w^{(1)}, h^{(1)}\big), \big(w^{(2)}, h^{(2)}\big)\big].$$

Thus, when $l < 1$, the operator A is a contraction operator on the sphere $S_\alpha(\theta, R)$ with respect to the metric $d_{p,2}$.

By Theorem 2.13, the system (2.11) has at least one solution in the ball $S_\alpha(\theta, R)$. Let us show this is indeed the case. Since

$$(w_n, h_n) = A(w_{n-1}, h_{n-1}),\ \ n = 1, 2, \dots,$$

we obtain

$$\begin{aligned} d_{p,2}[(w_{n+1}, h_{n+1}), (w_n, h_n)] &= d_{p,2}[A(w_n, h_n), A(w_{n-1}, h_{n-1})] \\ &\le l d_{p,2}[(w_n, h_n), (w_{n-1}, h_{n-1})]. \end{aligned}$$

Repeating this process, it follows that

$$d_{p,2}[(w_{n+1}, h_{n+1}), (w_0, h_0)] \le l^n d_{p,2}[(w_1, h_1), (w_0, h_0)].$$

Thus, for any two natural numbers m and n we can write

$$d_{p,2}[(w_{n+m}, h_{n+m}), (w_n, h_n)] = l^n \frac{1 - l^m}{1 - l} d_{p,2}[(w_1, h_1), (w_0, h_0)]. \tag{2.12}$$

Since $\lim_{n\to\infty} l^n = 0$, the sequence $\{(w_n, h_n)\}_1^\infty$ is Cauchy by (2.12). Since $(X, d_{p,2})$ is complete, there is an element $(w_*, h_*) \in X$ such that $\lim_{n\to\infty}(w_n, h_n) = (w_*, h_*)$. On the other hand,

$$\begin{aligned} d_{p,2}[(w_{n+1}, h_{n+1}), A(w_*, h_*)] &= d_{p,2}[A(w_n, h_n), A(w_*, h_*)] \\ &\le l d_{p,2}[(w_n, h_n), (w_*, h_*)] \end{aligned}$$

and

$$\lim_{n\to\infty} d_{p,2}[(w_n, h_n), (w_*, h_*)] = 0$$

imply that

$$\lim_{n\to\infty} d_{p,2}[(w_{n+1}, h_{n+1}), A(w_*, h_*)] = 0,$$

and thus

$$\lim_{n\to\infty} d_{p,2}(w_{n+1}, h_{n+1}) = A(w_*, h_*).$$

So we get $(w_*, h_*) = A(w_*, h_*)$, and this shows that (w_*, h_*) is a solution to the equation $(w, h) = A(w, h)$.

Now let us prove the uniqueness of this solution: Let (w_{**}, h_{**}) be another solution of the system (2.11). In this case, we can write

$$d_{p,2}[(w_*, h_*), (w_{**}, h_{**})] = d_{p,2}[A(w_*, h_*), A(w_{**}, h_{**})]$$
$$\leq l d_{p,2}[(w_*, h_*), (w_{**}, h_{**})].$$

However, this is possible only if $d_{p,2}[(w_*, h_*), (w_{**}, h_{**})] = 0$. □

2.15. Remark. Since, by (2.11), the sequence $\{(w_n, h_n)\}_1^\infty$, whose terms are defined by $(w_n, h_n) = A(w_{n-1}, h_{n-1})$ is convergent to the solution (w_*, h_*) in the ball $S_\alpha(\theta, R)$ with respect to the metric $d_{p,2}$, it is also convergent with respect to the metric $d_{\infty,2}$. Thus the metrics $d_{\infty,2}$ and $d_{p,2}$ are equivalent on X.

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