Hacettepe Journal of Mathematics and Statistics
Volume 40 (1) (2011), 53 – 61

A GENERALIZATION OF JORDAN'S INEQUALITY AND AN APPLICATION

Zhen-Hong Huo*, Da-Wei Niu†, Jian Cao‡, Feng Qi§¶‖

Received 02 : 02 : 2010 : Accepted 03 : 07 : 2010

Abstract

In this article, a new generalization of Jordan's inequality

$$\sum_{k=1}^{n}\mu_k\left(\theta^t-x^t\right)^k\le\frac{\sin x}{x}-\frac{\sin\theta}{\theta}\le\sum_{k=1}^{n}\omega_k\left(\theta^t-x^t\right)^k$$

for $t\ge 2$, $n\in\mathbb{N}$ and $\theta\in(0,\pi]$ is established, where the coefficients μ_k and ω_k are defined by recursion formulas, and are the best possible. As an application, Yang's inequality is refined.

Keywords: Jordan's inequality, Yang's inequality, L'Hôspital's rule, Refinement, Application.

2000 AMS Classification: 26 D 05, 26 D 15, 30 D 35, 34 E 05, 41 A 58, 41 A 60.

1. Introduction

The well-known Jordan's inequality (see [2, 5], [3, p. 143], [7, p. 269] and [10, p. 33]) states that

$$\frac{2}{\pi}\le\frac{\sin x}{x}<1 \tag{1.1}$$

for $0<|x|\le\frac{\pi}{2}$. Equality in (1.1) is valid if and only if $x=\frac{\pi}{2}$.

*College of Science, Zhongyuan University of Technology, Zhengzhou City, Henan Province, 450007, China. E-mail: hzh568@yahoo.com.cn

†College of Information and Business, Zhongyuan University of Technology, Zhengzhou City, Henan Province, 450007, China. E-mail: nnddww@gmail.com nnddww@hotmail.com nnddww@163.com

‡Department of Mathematics, Hangzhou Normal University, Hangzhou City, Zhejiang Province, 310036, China. E-mail: 21caojian@gmail.com 21caojian@163.com

§Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China. E-mail: qifeng618@gmail.com qifeng618@hotmail.com qifeng618@qq.com

¶Corresponding Author.

‖Supported in part by the China Scholarship Council and the Science Foundation of Tianjin Polytechnic University

Jordan's inequality and its refinements have important applications in several mathematical areas such as calculus and trigonometry, where specially the theory of limits are involved in [25]. These are important tools in approximating the Riemann zeta function $\zeta(x)$ in [8], in improving Yang's inequality in [29] and its generalization, which play an important role in the theory of distribution of values of functions. Therefore, many mathematicians have struggled to refine, generalize and apply it. For more detailed information, please refer to [7, pp. 274–275] and [1, 4, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 28, 30, 33, 34, 35], especially [11, 20], and related references therein.

In [1, 9, 15, 16, 17, 18, 19], among other things, Jordan's inequality had been refined as

$$\frac{1}{\pi^3}x\big(\pi^2-4x^2\big)\le \sin x-\frac{2}{\pi}x\le \frac{\pi-2}{\pi^3}x\big(\pi^2-4x^2\big). \tag{1.2}$$

In [35], a stronger sharp double inequality for $x\in\big(0,\frac{\pi}{2}\big]$ was obtained:

$$\frac{12-\pi^2}{16\pi^5}\big(\pi^2-4x^2\big)^2\le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{1}{\pi^3}\big(\pi^2-4x^2\big)\le \frac{\pi-3}{\pi^5}\big(\pi^2-4x^2\big)^2. \tag{1.3}$$

Recently, the following general refinement of Jordan's inequality was shown in [13]:

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\big(\pi^2-4x^2\big)^k\le \frac{\sin x}{x}\le \frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\big(\pi^2-4x^2\big)^k, \tag{1.4}$$

where the constants

$$\alpha_k=\frac{(-1)^k}{(4\pi)^k k!}\sum_{i=1}^{k+1}\left(\frac{2}{\pi}\right)^i c_{i-1}^k\sin\left(\frac{k+i}{2}\pi\right) \tag{1.5}$$

and

$$\beta_k=\begin{cases}\dfrac{1-2/\pi-\sum_{i=1}^{n-1}\alpha_i\pi^{2i}}{\pi^{2n}}, & k=n\\ \alpha_k, & 1\le k<n\end{cases} \tag{1.6}$$

with

$$c_i^k=\begin{cases}(i+k-1)c_{i-1}^{k-1}+c_i^{k-1}, & 0<i\le k\\ 1, & i=0\\ 0, & i>k\end{cases} \tag{1.7}$$

are the best possible.

In [28], as a generalization of Jordan's inequality (1.1), the following sharp inequality

$$\begin{aligned}\frac{1}{2\tau^2}\bigg[(1+\lambda)\bigg(\frac{\sin\theta}{\theta}-\cos\theta\bigg)-\theta\sin\theta\bigg]\bigg(1-\frac{x^\tau}{\theta^\tau}\bigg)^2&\\ \le \frac{\sin x}{x}-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\bigg(\frac{\sin\theta}{\theta}-\cos\theta\bigg)\bigg(1-\frac{x^\lambda}{\theta^\lambda}\bigg)&\\ \le \bigg[1-\frac{\sin\theta}{\theta}-\frac{1}{\lambda}\bigg(\frac{\sin\theta}{\theta}-\cos\theta\bigg)\bigg]\bigg(1-\frac{x^\tau}{\theta^\tau}\bigg)^2&\end{aligned} \tag{1.8}$$

was obtained for $0<x\le\theta\in\big(0,\frac{\pi}{2}\big]$, $\tau\ge 2$ and $\tau\le\lambda\le 2\tau$. Equalities in (1.8) hold if and only if $x=\theta$. The coefficients of the term $\big(1-\frac{x^\tau}{\theta^\tau}\big)^2$ are the best possible. If

$1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$, then inequality (1.8) is reversed. In particular, when $\theta = \frac{\pi}{2}$, inequality (1.8) becomes

$$\begin{aligned}\frac{4\lambda+4-\pi^2}{4\tau^2\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2 &\le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{2}{\lambda\pi^{\lambda+1}}\left(\pi^\lambda-2^\lambda x^\lambda\right)\\ &\le \frac{\lambda\pi-2\lambda-2}{\lambda\pi^{2\tau+1}}\left(\pi^\tau-2^\tau x^\tau\right)^2\end{aligned} \tag{1.9}$$

for $0 < x \le \frac{\pi}{2}$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$. If $1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$, then the inequality (1.9) is reversed. If we take $(\tau, \lambda) = (2, 2)$ in (1.9), then the inequality (1.3) can be deduced.

For recent developments of refinements, generalizations and applications of Jordan's inequality, please refer to the survey paper [20] and related references therein.

The first aim of this paper is to generalize inequalities (1.4) and (1.8) as the following Theorem 1.1.

1.1. Theorem. *For $0 < x \le \theta < \pi$, $n \in \mathbb{N}$ and $t \ge 2$, the inequality*

$$\sum_{k=1}^{n}\mu_k\left(\theta^t-x^t\right)^k \le \frac{\sin x}{x}-\frac{\sin\theta}{\theta} \le \sum_{k=1}^{n}\omega_k\left(\theta^t-x^t\right)^k \tag{1.10}$$

holds with equalities if and only if $x = \theta$, where the constants

$$\mu_k = \frac{(-1)^k}{k!t^k}\sum_{i=1}^{k+1}a_{i-1}^k\theta^{k-i-kt}\sin\left(\theta+\frac{k+i-1}{2}\pi\right) \tag{1.11}$$

and

$$\omega_k = \begin{cases}\dfrac{1-\sin\theta/\theta-\sum_{i=1}^{n-1}\mu_i\theta^{ti}}{\theta^{tn}}, & k=n\\ \mu_k, & 1\le k<n\end{cases} \tag{1.12}$$

with

$$a_i^k = \begin{cases}a_i^{k-1}+[i+(k-1)(t-1)]a_{i-1}^{k-1}, & 0<i\le k\\ 1, & i=0\\ 0, & i>k\end{cases} \tag{1.13}$$

are the best possible.

1.2. Remark. Taking $t = 2$ in (1.10) yields inequality (1.4). Letting $n = 2$ in (1.10) leads to (1.8) for $\lambda = \tau = 2$.

The second aim of this paper is to apply Theorem 1.1 to refine Yang's inequality [29] as follows.

1.3. Theorem. *Let $0 \le \lambda \le 1$, $0 < x \le \theta < \pi$, $t \ge 2$ and $A_i > 0$ with $\sum_{i=1}^{n} A_i \le \pi$ for $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n \ge 2$, then*

$$L_m(n,\lambda) \le H(n,\lambda) \le R_m(n,\lambda), \tag{1.14}$$

where

$$L_m(n,\lambda)=\binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta}+\sum_{k=1}^{m}2^{-kt}\mu_k\big(2^t\theta^t-\lambda^t\pi^t\big)^k\right]^2\cos^2\Big(\frac{\lambda}{2}\pi\Big), \tag{1.15}$$

$$H(n,\lambda)=(n-1)\sum_{k=1}^{n}\cos^2(\lambda A_k)-2\cos(\lambda\pi)\sum_{1\le i<j\le n}\cos(\lambda A_i)\cos(\lambda A_j), \tag{1.16}$$

$$R_m(n,\lambda)=\binom{n}{2}\lambda^2\pi^2\left[\frac{\sin\theta}{\theta}+\sum_{k=1}^{m}2^{-kt}\omega_k\big(2^t\theta^t-\lambda^t\pi^t\big)^k\right]^2\cos^2\Big(\frac{\lambda}{2}\pi\Big), \tag{1.17}$$

and μ_k *and* ω_k *are defined by* (1.11).

2. Lemmas

To prove our main results, the following lemmas are necessary.

2.1. Lemma. *For* $x>0$, *let* $u_0(x)=\frac{\sin x}{x}$ *and* $u_k(x)=\frac{u'_{k-1}(x)}{x^r}$ *for* $k\in\mathbb{N}$ *and* $r\ge 1$. *Then*

$$u_k(x)=\sum_{i=1}^{k+1}\frac{a_{i-1}^k\sin\big(x+(i+k-1)\pi/2\big)}{x^{kr+i}}, \tag{2.1}$$

where a_i^k *is defined by* (1.13).

Proof. It is apparent that

$$u_1(x)=x^{-r}\left(\frac{\sin x}{x}\right)'=x^{-1-r}\cos x-x^{-2-r}\sin x,$$

which tells us that the formula (2.1) is valid for $k=1$.

Now assume the formula (2.1) holds for some given $k>1$. Direct computation and utilization of (1.13) gives

$$\begin{aligned}
u_{k+1}&=\sum_{i=1}^{k+1}a_{i-1}^k\left[\frac{1}{x^{kr+i+r}}\cos\left(x+\frac{k+i-1}{2}\pi\right)\right.\\
&\qquad\left.-\frac{1}{x^{kr+i+r+1}}\sin\left(x+\frac{k+i-1}{2}\pi\right)\right]\\
&=\frac{a_0^k}{x^{kr+r+1}}\cos\left(x+\frac{k}{2}\pi\right)-\frac{(kr+k+1)a_k^k}{x^{kr+r+k+2}}\sin(x+k\pi)\\
&\qquad-\sum_{i=0}^{k-1}\frac{a_i^k(kr+1+i)+a_{i+1}^k}{x^{kr+r+i+2}}\sin\left(x+\frac{k+i}{2}\pi\right)\\
&=\frac{a_0^{k+1}}{x^{kr+r+1}}\sin\left(x+\frac{k+1}{2}\pi\right)+\frac{a_{k+1}^{k+1}}{x^{kr+r+k+2}}\sin[x+(k+1)\pi]\\
&\qquad+\sum_{i=0}^{k-1}\frac{a_{i+1}^{k+1}}{x^{kr+r+i+2}}\sin\left(x+\frac{k+i+2}{2}\pi\right)\\
&=\sum_{i=1}^{k+2}\frac{a_{i-1}^{k+1}}{x^{kr+i+r}}\sin\left(x+\frac{k+i}{2}\pi\right).
\end{aligned}$$

By mathematical induction, Lemma 2.1 is proved. □

2.2. Lemma. *For $x > 0$ and $k \in \mathbb{N}$, let*

$$v_1(x) = \sum_{i=1}^{k+1} a_{i-1}^k x^{k-i+1} \sin\left(x + \frac{k+i-1}{2}\pi\right)$$

and $v_{j+1}(x) = \frac{1}{x} v_j'(x)$ for $j \in \mathbb{N}$. Then

$$v_j(x) = \sum_{i=0}^{k-j+1} b_i^j x^{k-i-j+1} \sin\left(x + \frac{k+i+j-1}{2}\pi\right) \tag{2.2}$$

is valid for $j \in \mathbb{N}$, where $b_i^1 = a_i^k$, $b_0^j = 1$ and

$$b_i^j = b_i^{j-1} - (k-i-j+3) b_{i-1}^{j-1}, \qquad 0 < i \le k-j+1,\ j > 1. \tag{2.3}$$

Proof. When $j = 1$, the formula (2.2) is clearly valid.

By induction, suppose that the formula (2.2) holds for some $j > 1$. Since $k - j + 1 > k - (j+1) + 1$, it can be deduced from (2.3) that $b_{k-j+1}^{j+1} = b_{k-j+1}^j - b_{k-j}^j = 0$. Thus,

$$\begin{aligned} v_{j+1}(x) &= \frac{1}{x}\left\{\sum_{i=0}^{k-j} b_i^j \left[(k-i-j+1) x^{k-i-j} \sin\left(x + \frac{k+i+j-1}{2}\pi\right)\right.\right. \\ &\qquad \left.\left. + x^{k-i-j+1} \cos\left(x + \frac{k+i+j-1}{2}\pi\right)\right] + b_{k-j+1}^j \cos(x + k\pi)\right\} \\ &= b_0^j x^{k-j} \sin\left(x + \frac{k+j}{2}\pi\right) \\ &\quad + \sum_{i=0}^{k-j-1} \left[b_{i+1}^j - (k-i-j+1) b_i^j\right] x^{k-i-j+1} \sin\left(x + \frac{k+i+j+1}{2}\pi\right) \\ &= b_0^j x^{k-j} \sin\left(x + \frac{k+j}{2}\pi\right) \\ &\qquad + \sum_{i=0}^{k-j-1} b_{i+1}^{j+1} x^{k-i-j+1} \sin\left(x + \frac{k+i+j+1}{2}\pi\right) \\ &= \sum_{i=0}^{k-j} b_i^{j+1} x^{k-i-j} \sin\left(x + \frac{k+i+j}{2}\pi\right). \end{aligned}$$

By mathematical induction, the formula (2.2) is proved. □

2.3. Lemma. [23] *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x) \ne 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) in (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing) in (a,b).*

2.4. Lemma. *Let $0 < x \le \theta < \pi$ and $t \ge 2$. Then the double inequality*

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}} - \frac{\cos\theta}{\theta^t}\right)(\theta^t - x^t) \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} \le \left(\frac{1}{\theta^t} - \frac{\sin\theta}{\theta^{1+t}}\right)(\theta^t - x^t) \tag{2.4}$$

holds with equalities if and only if $x = \theta$, where the constants

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}} - \frac{\cos\theta}{\theta^t}\right) \text{ and } \left(\frac{1}{\theta^t} - \frac{\sin\theta}{\theta^{1+t}}\right)$$

are the best possible.

Proof. Let

$$(2.5)\qquad \begin{aligned} f(x)&=\frac{\sin x}{x}-\frac{\sin\theta}{\theta}, \qquad g(x)=\theta^t-x^t,\\ f_1(x)&=x\cos x-\sin x, \quad g_1(x)=-tx^{1+t}. \end{aligned}$$

Then

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(0)}{g(x)-g(0)}, \qquad \frac{f'(x)}{g'(x)}=\frac{f_1(x)-f_1(0)}{g_1(x)-g_1(0)}, \qquad \frac{f_1'(x)}{g_1'(x)}=\frac{\sin x}{t(1+t)x^t}.$$

Since $\frac{\sin x}{x^t}$ is decreasing in $(0,\pi]$, then $\frac{f_1'(x)}{g_1'(x)}$ is decreasing, and so, in virtue of Lemma 2.3, the function $\frac{f'(x)}{g'(x)}$ is decreasing, and the function $\frac{f(x)}{g(x)}$ is decreasing in $(0,\pi]$, thus,

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}}-\frac{\cos\theta}{\theta^t}\right)=\lim_{x\to\theta^-}\frac{f(x)}{g(x)}\le\frac{f(x)}{g(x)}\le\lim_{x\to 0^+}\frac{f(x)}{g(x)}=\frac{1}{\theta^t}\left(1-\frac{\sin\theta}{\theta}\right)$$

and the two constants are proved to be the best possible. □

3. Proofs of theorems

3.1. Proof of Theorem 1.1. If $n=1$, the inequality (1.10) becomes (2.4).

For $n\ge 2$, let $t=r+1$ and

$$\varphi(x)=\frac{\sin x}{x}-\frac{\sin\theta}{\theta}-\sum_{k=1}^{n-1}\mu_k\left(\theta^{r+1}-x^{r+1}\right)^k, \quad \psi(x)=\left(\theta^{r+1}-x^{r+1}\right)^n,$$

$$\varphi_1(x)=\frac{\varphi(x)}{x^r}, \quad \varphi_{i+1}(x)=\frac{\varphi_i'(x)}{x^r}, \quad \psi_1(x)=\frac{\psi'(x)}{x^r}, \quad \psi_{i+1}(x)=\frac{\psi_i'(x)}{x^r},$$

where $2\le i\le n$. Then for $1\le k\le n-2$,

$$\varphi_k(x)=u_k(x)-[-(r+1)]^k k!\mu_k-\sum_{i=1}^{n-k-1}\frac{(i+k)!}{i!}\mu_{i+k}\left(\theta^{1+r}-x^{1+r}\right)^i,$$

$$\varphi_{n-1}(x)=u_{n-1}(x)-(n-1)![-(r+1)]^{n-1}\mu_{n-1},$$

and $\varphi_n(x)=u_n(x)$, where $u_k(x)$ for $1\le k\le n$ is defined by (2.1).

In view of (2.1), it is deduced that

$$[-(1+r)]^k k!\mu_k=u_k(\theta)$$

for $1\le k\le n-1$, hence $\varphi_i(\theta)=0$ for $1\le i\le n-1$. A simple calculation gives

$$\psi_i(x)=[-(1+r)]^i\prod_{\ell=0}^{i-1}(n-\ell)(\theta^{r+1}-x^{r+1})^{n-i}$$

for $1\le i\le n$, consequently $\psi_i(\theta)=0$ for $1\le i\le n-1$. As a result, for $1\le i\le n-1$,

$$\frac{\varphi(x)}{\psi(x)}=\frac{\varphi(x)-\varphi(\theta)}{\psi(x)-\psi(\theta)}, \qquad \frac{\varphi'(x)}{\psi'(x)}=\frac{\varphi_1(x)-\varphi_1(\theta)}{\psi_1(x)-\psi_1(\theta)},$$

$$\frac{\varphi_i'(x)}{\psi_i'(x)}=\frac{\varphi_{i+1}(x)-\varphi_{i+1}(\theta)}{\psi_{i+1}(x)-\psi_{i+1}(\theta)}, \qquad \frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)}=\frac{\varphi_n(x)}{\psi_n(x)}=\frac{u_n(x)}{n![-(r+1)]^n}.$$

Let $h_1(x)=x^{nr+n+1}$ and $h_{i+1}(x)=\frac{1}{x}h_i'(x)$ for $1\le i\le n$ and $n\in\mathbb{N}$. Then it is easy to see that

$$h_{i+1}(x)=\prod_{\ell=1}^{i}(nr+n-2\ell+3)x^{nr+n-2i+1}$$

for $1 \le i \le n$. Utilization of Lemma 2.1 and Lemma 2.2 leads to

$$\frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)} = \frac{\sum_{i=1}^{n+1} a_{i-1}^n x^{n-i+1} \sin\bigl(x + \frac{n+i-1}{2}\pi\bigr)}{n![-(1+r)]^n x^{rn+n+1}} = \frac{v_1(x)}{n![-(1+r)]^n h_1(x)},$$

and, since $v_i(0) = h_i(0) = 0$ for $1 \le i \le n+1$,

$$\frac{v_1(x)}{h_1(x)} = \frac{v_1(x) - v_1(0)}{h_1(x) - h_1(0)}, \qquad \frac{v_j'(x)}{h_j'(x)} = \frac{v_{j+1}(x) - v_{j+1}(0)}{h_{j+1}(x) - h_{j+1}(0)},$$
$$\frac{v_n'(x)}{h_n'(x)} = \frac{v_{n+1}(x) - v_{n+1}(0)}{h_{n+1}(x) - h_{n+1}(0)} = \frac{(-1)^n \sin x}{\prod_{\ell=1}^{i}(nr + n - 2\ell + 3)x^{nr-n+1}}$$

for $1 \le j \le n-1$. Since $\frac{\sin x}{x}$ and $x^{-n(r-1)}$ are decreasing on $(0, \pi)$, then the function $\frac{\sin x}{x^{nr-n+1}}$ is decreasing and $\frac{(-1)^n v_n'(x)}{h_n'(x)}$ is deceasing. Accordingly, from Lemma 2.3, it follows that the functions $\frac{(-1)^n v_i'(x)}{h_i'(x)}$ and $\frac{(-1)^n v_{i-1}'(x)}{h_{i-1}'(x)}$ for $2 \le i \le n$ are decreasing. Thus, the functions $\frac{(-1)^n v_1'(x)}{h_1'(x)}$ and $\frac{(-1)^n v_1(x)}{h_1(x)}$ are decreasing, and so $\frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)}$ is decreasing in $(0, \pi)$.

Utilizing Lemma 2.3 again reveals that the functions $\frac{\varphi_j'(x)}{\psi_j'(x)}$ and $\frac{\varphi_{j-1}'(x)}{\psi_{j-1}'(x)}$ for $2 \le j \le n-1$ are decreasing, which implies the decreasing monotonicity of $\frac{\varphi(x)}{\psi(x)}$ in $\bigl(0, \pi\bigr)$. By L'Hôspital's rule, it is easy to deduce that

$$\lim_{x \to \theta-} \frac{\varphi(x)}{\psi(x)} = \lim_{x \to \theta-} \frac{\varphi'(x)}{\psi'(x)} = \lim_{x \to \theta-} \frac{\varphi_i'(x)}{\psi_i'(x)} = \frac{u_n(\theta)}{n![-(1+r)]^n} = \mu_n$$

for $1 \le i \le n-1$ and $\lim_{x \to 0+} \frac{\varphi(x)}{\psi(x)} = \omega_n$, which implies $\mu_n \le \frac{\varphi(x)}{\psi(x)} \le \omega_n$, and so the constants μ_k and ω_k are the best possible.

By mathematical induction, the inequality (1.10) is proved. The proof of Theorem 1.1 is complete. □

3.2. Proof of Theorem 1.3. It was proved in [31] and [32, (2.13)] that

$$\begin{aligned} \sin^2(\lambda\pi) &\le \cos^2(\lambda A_i) + \cos^2(\lambda A_j) - 2\cos(\lambda A_i)\cos(\lambda A_j)\cos(\lambda\pi) \triangleq H_{ij} \\ &\le 4\sin^2\left(\frac{\lambda}{2}\pi\right). \end{aligned} \tag{3.1}$$

Summing up (3.1) for $1 \le i < j \le n$ yields

$$\binom{n}{2}\sin^2(\lambda\pi) \le \sum_{1 \le i < j \le n} H_{ij} = H(n, \lambda) \le 4\binom{n}{2}\sin^2\Bigl(\frac{\lambda}{2}\pi\Bigr). \tag{3.2}$$

By virtue of inequality (1.10) in Theorem 1.1,

$$4\sin^2\Bigl(\frac{\lambda}{2}\pi\Bigr) \le \lambda^2\pi^2\left[\frac{\sin\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt}\omega_k\bigl(2^t\theta^t - \lambda^t\pi^t\bigr)^k\right]^2, \tag{3.3}$$

$$\begin{aligned} \sin^2(\lambda\pi) &= 4\cos^2\Bigl(\frac{\lambda}{2}\pi\Bigr)\sin^2\Bigl(\frac{\lambda}{2}\pi\Bigr) \\ &\ge \lambda^2\pi^2\left[\frac{\sin\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt}\mu_k\bigl(2^t\theta^t - \lambda^t\pi^t\bigr)^k\right]^2\cos^2\Bigl(\frac{\lambda}{2}\pi\Bigr). \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) leads to (1.14). The proof of Theorem 1.3 is complete. □

References

[1] Abel, U. and Caccia, D. *A sharpening of Jordan's inequality*, Amer. Math. Monthly **93** (7), 568–569, 1986.

[2] Abramowitz, M. and Stegun, I. A. (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (4th printing, with corrections, Applied Mathematics Series **55**, National Bureau of Standards, Washington, 1965).

[3] Bullen, P. S. *A Dictionary of Inequalities* (Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, Harlow/Essex, 1998).

[4] Debnath, L. and Zhao, Ch.-J. *New strengthened Jordan's inequality and its applications*, Appl. Math. Lett. **16** (4), 557–560, 2003.

[5] Feng, Y.-F. *Proof without words: Jordan's inequality* $\frac{2x}{\pi} \le \sin x \le x$*,* $0 \le x \le \frac{\pi}{2}$, Math. Mag. **69**, 126, 1996.

[6] Jiang, W.-D. and Hua, Y. *Sharpening of Jordan's inequality and its applications*, J. Inequal. Pure Appl. Math. **7** (3), Art. 102, 2006; Available online at `http://www.emis.de/journals/JIPAM/article719.html?sid=719`.

[7] Kuang, J.-Ch. *Chángyòng Bùděngshì* (*Applied Inequalities*) 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè (Shandong Science and Technology Press, Ji'nan City, Shandong Province, China, 2004). (Chinese)

[8] Luo, Q.-M., Wei, Z.-L. and Qi, F. *Lower and upper bounds of* $\zeta(3)$, Adv. Stud. Contemp. Math. (Kyungshang) **6** (1), 47–51, 2003.

[9] Mercer, A. McD. *Problem E 2952*, Amer. Math. Monthly **89** (6), 424, 1982.

[10] Mitrinović, D. S. *Analytic Inequalities* (Springer-Verlag, Berlin/Heildelberg/New York, 1970).

[11] Niu, D.-W. *Generalizations of Jordan's Inequality and Applications*, Thesis supervised by Professor Feng Qi and submitted for the Master Degree at Henan Polytechnic University in June 2007. (Chinese)

[12] Niu, D.-W., Cao, J. and Qi, F. *Generalizations of Jordan's inequality and concerned relations*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (3), 85–98, 2010.

[13] Niu, D.-W., Huo, Zh.-H., Cao, J. and Qi, F. *A general refinement of Jordan's inequality and a refinement of L. Yang's inequality*, Integral Transforms Spec. Funct. **19** (3), 157–164, 2008.

[14] Özban, A. Y. *A new refined form of Jordan's inequality and its applications*, Appl. Math. Lett. **19** (2), 155–160, 2006.

[15] Qi, F. *Extensions and sharpenings of Jordan's and Kober's inequality*, Gōngkē Shùxué (Journal of Mathematics for Technology) **12** (4), 98–102, 1996. (Chinese)

[16] Qi, F., Cui, L.-H. and Xu, S.-L. *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (4), 517–528, 1999.

[17] Qi, F. and Guo, B.-N. *Extensions and sharpenings of the noted Kober's inequality*, Jiāozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute) **12** (4), 101–103, 1993. (Chinese)

[18] Qi, F. and Guo, B.-N. *On generalizations of Jordan's inequality*, Méitàn Gāoděng Jiàoyù (Coal Higher Education), Suppl., November/1993, 32–33, 1993. (Chinese)

[19] Qi, F. and Hao, Q.-D. *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (3), 116–120, 1998.

[20] Qi, F., Niu, D.-W. and Guo, B.-N. *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009**, Article ID 271923, 52 pages, 2009; Available online at `http://dx.doi.org/10.1155/2009/271923`.

[21] Redheffer, R. *Correction*, Amer. Math. Monthly **76** (4), 422, 1969.

[22] Redheffer, R. *Problem 5642*, Amer. Math. Monthly **75** (10), 1125, 1968.

[23] Vamanamurthy, M. K. and Vuorinen, M. *Inequalities for means*, J. Math. Anal. Appl. **183**, 155–166, 1994.

[24] Williams, J. P. *A delightful inequality*, Amer. Math. Monthly **76** (10), 1153–1154, 1969.

[25] Wu, Sh.-H. *On generalizations and refinements of Jordan type inequality*, Octogon Math. Mag. **12** (1), 267–272, 2004.

[26] Wu, Sh.-H. *Sharpness and generalization of Jordan's inequality and its application*, Taiwanese J. Math. **12** (2), 325–336, 2008.
[27] Wu, Sh.-H. and Debnath, L. *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality*, Appl. Math. Lett. **19** (12), 1378–1384, 2006.
[28] Wu, Sh.-H. and Debnath, L. *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality, II*, Appl. Math. Lett. **20**, 532–538, 2007.
[29] Yang, L. *Zhí Fēnbù Lǐlùn Jíqí Xīn Yánjiū* (*The Theory of Distribution of Values of Functions and Recent Researches*), Kēxué Chūbǎn Shè (Science Press, Beijing, 1982). (Chinese)
[30] Zhang, X.-H., Wang, G.-D. and Chu, Y.-M. *Extensions and sharpenings of Jordan's and Kober's inequalities*, J. Inequal. Pure Appl. Math. **7** (2), Art. 63, 2006; Avaialble online at http://www.emis.de/journals/JIPAM/article680.html?sid=680.
[31] Zhao, Ch.-J. *The extension and strength of Yang Le inequality*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory) **30** (4), 493–497, 2000. (Chinese)
[32] Zhao, Ch.-J. and Debnath, L. *On generalizations of L. Yang's inequality*, J. Inequal. Pure Appl. Math. **3** (4), Art. 56, 2002; Available online at http://www.emis.de/journals/JIPAM/article208.html?sid=208.
[33] Zhu, L. *Sharpening of Jordan's inequalities and its applications*, Math. Inequal. Appl. **9** (1), 103–106, 2006.
[34] Zhu, L. *Sharpening Jordan's inequality and the Yang Le inequality*, Appl. Math. Lett. **19** (3), 240–243, 2006.
[35] Zhu, L. *Sharpening Jordan's inequality and the Yang Le inequality, II*, Appl. Math. Lett. **19** (9), 990–994, 2006.