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DYNAMIC RELIABILITY AND PERFORMANCE EVALUATION OF MULTI-STATE SYSTEMS WITH TWO COMPONENTS

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Abstract

In this paper we study multi-state systems consisting of two components when the number of system states and the number of states of each component are the same, i.e. the systems under consideration are homogeneous multi-state systems. In particular we evaluate multi-state series and cold standby systems assuming that the degradation in their components follow a Markov process. The behaviour of systems with respect to degradation rates is also investigated in terms of stochastic ordering.

Keywords: Failure rate, Mean residual life, Reliability, Standby system, Stochastic order.

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1. Introduction

Reliability evaluation has a vital importance at all stages of processing and controlling modern engineering systems. It is a very common situation in which the system and its components have a range of performance levels from perfect functioning to complete failure. This situation can be handled and analyzed by multi-state reliability theory. For example, networks and their components perform their tasks at different performance levels. Therefore the performance degradation of the network occurs due to the degradation in components over time. One of the most common and tractable assumptions is that degradation in the system occurs according to a Markov process. The reader is referred to Lisnianski and Levitin [7] for the use of Markov processes in multi-state

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system modeling. Much attention has been paid recently to the evaluation of multi-state systems. Some recent contributions on the subject, among others, are the works of Zuo and Tian [15], Li and Zuo [5], Eryılmaz and Iscioglu [3], and the references therein.

Some dynamic reliability measures used in binary state analysis have been extended to the multi-state case in several works (see, e.g. Brunelle and Kapur [1]). Because there are more than two states in multi-state system modeling, there is a necessity to define the lifetime of a multi-state system in a specific state j or above. In fact, this lifetime random variable is the time spent by the system in state j or above. Lisnianski and Frenkel [9] studied a non-homogeneous Markov reward model for an aging multi-state system under minimal repair.

Standby redundancy is widely used in engineering design to improve system reliability. In this case, only one component is active in the system and the other components are placed in the system in the standby condition. Several types of standby redundancy have been defined and studied in the literature. The most recent results on binary standby systems are in the works of Cha *et al.* [2], Li *et al.* [6]. Some redundancy problems for multi-state systems including typical series and parallel connections, k-out-of-n and consecutive k-out-of-n structures were studied in the literature [4, 7]. Recently Lisnianski and Ding [8] discussed a new general type of redundancy which considers two interconnected multi-state systems, where one multi-state system can satisfy its own stochastic demand and can also provide abundant resource to another system.

In this paper we study multi-state series and cold standby systems under the assumption that the degradation in their components follow a Markov process. In Section 2, we present the definitions and obtain the reliability functions. In Section 3, we study how the degradation rates influences the performance of systems in terms of stochastic ordering.

2. Definitions and reliability functions

This section is devoted to the construction of the probabilistic frame for modeling the multi-state series and cold standby systems. Below we provide the notation that will be used throughout the paper.

$\{0,1,\ldots,M\}$: The state set of the system and its components, where "0" and "M" represent respectively the worst (completely failed) and best (perfect functioning) states.

$X_i(t)$: The state of component i at time t, $X_i(t) \in \{0,1,\ldots,M\}$, $i=1,2$.

$T_i^{\geq j}$: The lifetime of component i in the state subset $\{j, j+1, \ldots, M\}$ (or the time spent by the component i in state j or above), $i = 1, 2$.

$T_S^{\geq j}$: The lifetime of the multi-state series system in the state subset $\{j, j+1, \ldots, M\}$.

$T_C^{\geq j}$: The lifetime of the multi-state cold standby system in the state subset $\{j, j+1, j+2, \ldots, M\}$.

2.1. Definition. The two component multi-state series system is in state j ($j = 1, ..., M$) or above if and only if both of its components are in state j or above.

According to the above definition, the lifetime of a two component multi-state series system in the state subset $\{j, j+1, \ldots, M\}$ is given by

$$(2.1)\qquad T_S^{\geq j} = \min(T_1^{\geq j}, T_2^{\geq j}).$$

Under the assumption that the components are independent, the reliability function corresponding to $T_S^{\geq j}$ is

$$(2.2)\qquad P\{T_S^{\geq j} > t\} = P\{T_1^{\geq j} > t\} P\{T_2^{\geq j} > t\},$$

for $j = 1, \dots, M$.

In a multi-state cold standby system consisting of two components, the system is put into operation with component 1 in active operation and component 2 in standby. When the performance rate of component 1 falls below j, instantaneously component 2 is switched into operation.

2.2. Definition. A two component multi-state cold standby system is in state j ($j = 1, \dots, M$) or above as soon as its active component is in state j or above.

The lifetime of a two component multi-state cold standby system in the state subset $\{j, j+1, \dots, M\}$ is then defined by

$$T_C^{\geq j} = T_1^{\geq j} + T_2^{\geq j}. \tag{2.3}$$

The basic assumptions for such a system can be listed as below:

(1) A standby component is switched into operation as soon as the performance rate of the active component falls below j.
(2) The components degrade while they are in active state with time t without repair from the perfect state to lower states.
(3) A standby component does not degrade while the active component is working, i.e. $P\{X_2(T_1^{\geq j}) = M\} = 1$.

Under the above assumptions the system reliability function can be computed as

$$\begin{aligned} P\{T_C^{\geq j} > t\} &= P\{T_1^{\geq j} + T_2^{\geq j} > t\} \\ &= \int_t^{\infty} dP\{T_1^{\geq j} \leq x\} + \int_0^t P\{T_2^{\geq j} > t - x\}\, dP\{T_1^{\geq j} \leq x\} \\ &= P\{T_1^{\geq j} > t\} + \int_0^t P\{T_2^{\geq j} > t - x\}\, dP\{T_1^{\geq j} \leq x\}, \end{aligned} \tag{2.4}$$

for $t \geq 0$.

Assume that the degradation in the components follows a Markov process and that the components have 3 states: 0 (failed), 1 (partially working) and 2 (perfectly functioning). If the corresponding instantaneous degradation rates among the states for the components are denoted by $\lambda_{1,0}^{(i)}$ and $\lambda_{2,1}^{(i)}$, $i = 1, 2$, then from (3.19) of Lisnianski and Levitin [7] we have the following system of differential equations for the state probabilities $p_j(t) = P\{X_1(t) = j\}$, $j = 0, 1, 2$.

$$\begin{aligned} \frac{d}{dt}p_0(t) &= \lambda_{1,0}^{(1)} p_1(t) \\ \frac{d}{dt}p_1(t) &= \lambda_{2,1}^{(1)} p_2(t) - \lambda_{1,0}^{(1)} p_1(t) \\ \frac{d}{dt}p_2(t) &= -\lambda_{2,1}^{(1)} p_2(t) \end{aligned} \tag{2.5}$$

with initial conditions $p_2(0) = 1$, $p_1(0) = 0 = p_0(0)$.

After solving this system one obtains

$$P\{X_1(t)=0\}=1-\frac{1}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}\left(\lambda_{2,1}^{(1)}e^{-\lambda_{1,0}^{(1)}t}-\lambda_{1,0}^{(1)}e^{-\lambda_{2,1}^{(1)}t}\right), \tag{2.6}$$

$$P\{X_1(t)=1\}=\frac{\lambda_{2,1}^{(1)}}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}\left(e^{-\lambda_{1,0}^{(1)}t}-e^{-\lambda_{2,1}^{(1)}t}\right), \tag{2.7}$$

$$P\{X_1(t)=2\}=e^{-\lambda_{2,1}^{(1)}t} \tag{2.8}$$

and hence

$$P\{T_1^{\geq 1}>t\}=P\{X_1(t)\geq 1\}=\frac{\left(\lambda_{2,1}^{(1)}e^{-\lambda_{1,0}^{(1)}t}-\lambda_{1,0}^{(1)}e^{-\lambda_{2,1}^{(1)}t}\right)}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}, \tag{2.9}$$

$$P\{T_1^{\geq 2}>t\}=P\{X_1(t)\geq 2\}=e^{-\lambda_{2,1}^{(1)}t}, \tag{2.10}$$

for $t\geq 0$. Under the Markov process assumption defined above the reliability functions of the series system are

$$\begin{aligned}P\{T_S^{\geq 1}>t\}&=P\{T_1^{\geq 1}>t\}P\{T_2^{\geq 1}>t\}\\&=\frac{\left(\lambda_{2,1}^{(1)}e^{-\lambda_{1,0}^{(1)}t}-\lambda_{1,0}^{(1)}e^{-\lambda_{2,1}^{(1)}t}\right)}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}\frac{\left(\lambda_{2,1}^{(2)}e^{-\lambda_{1,0}^{(2)}t}-\lambda_{1,0}^{(2)}e^{-\lambda_{2,1}^{(2)}t}\right)}{\lambda_{2,1}^{(2)}-\lambda_{1,0}^{(2)}},\end{aligned} \tag{2.11}$$

and

$$P\{T_S^{\geq 2}>t\}=e^{-(\lambda_{2,1}^{(1)}+\lambda_{2,1}^{(2)})t}. \tag{2.12}$$

In the setup of a two component multi-state cold standby system, $\left(\lambda_{1,0}^{(1)},\lambda_{2,1}^{(1)}\right)$ denotes the degradation rates for the active component, and $\left(\lambda_{1,0}^{(2)},\lambda_{2,1}^{(2)}\right)$ denotes the degradation rates for the standby component while it is in the active state. The reliability of a multi-state cold standby system is found to be

$$\begin{aligned}P\{T_C^{\geq 1}>t\}&=\frac{\left(\lambda_{2,1}^{(1)}e^{-\lambda_{1,0}^{(1)}t}-\lambda_{1,0}^{(1)}e^{-\lambda_{2,1}^{(1)}t}\right)}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}+\frac{1}{(\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)})(\lambda_{2,1}^{(2)}-\lambda_{1,0}^{(2)})}\\&\quad\times\int_0^t\Big[\left(\lambda_{2,1}^{(2)}e^{-\lambda_{1,0}^{(2)}(t-x)}-\lambda_{1,0}^{(2)}e^{-\lambda_{2,1}^{(2)}(t-x)}\right)\\&\quad\times\left(\lambda_{2,1}^{(1)}\lambda_{1,0}^{(1)}e^{-\lambda_{1,0}^{(1)}x}-\lambda_{1,0}^{(1)}\lambda_{2,1}^{(1)}e^{-\lambda_{2,1}^{(1)}x}\right)\Big]\,dx\end{aligned} \tag{2.13}$$

$$\begin{aligned}&=\frac{(\lambda_{2,1}^{(1)}e^{-\lambda_{1,0}^{(1)}t}-\lambda_{1,0}^{(1)}e^{-\lambda_{2,1}^{(1)}t})}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)}}+\frac{1}{(\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(1)})(\lambda_{2,1}^{(2)}-\lambda_{1,0}^{(2)})}\\&\quad\times\Bigg[\lambda_{1,0}^{(1)}\lambda_{2,1}^{(1)}\lambda_{2,1}^{(2)}e^{-\lambda_{1,0}^{(2)}t}\left\{\frac{(1-e^{-(\lambda_{1,0}^{(1)}-\lambda_{1,0}^{(2)})t})}{\lambda_{1,0}^{(1)}-\lambda_{1,0}^{(2)}}\right.\\&\quad\left.-\frac{(1-e^{-(\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(2)})t})}{\lambda_{2,1}^{(1)}-\lambda_{1,0}^{(2)}}\right\}+\lambda_{1,0}^{(1)}\lambda_{2,1}^{(1)}\lambda_{1,0}^{(2)}e^{-\lambda_{2,1}^{(2)}t}\\&\quad\left\{\frac{(1-e^{-(\lambda_{2,1}^{(1)}-\lambda_{2,1}^{(2)})t})}{\lambda_{2,1}^{(1)}-\lambda_{2,1}^{(2)}}-\frac{(1-e^{-(\lambda_{1,0}^{(1)}-\lambda_{2,1}^{(2)})t})}{\lambda_{1,0}^{(1)}-\lambda_{2,1}^{(2)}}\right\}\Bigg],\end{aligned} \tag{2.14}$$

and

$$P\{T_C^{\geq 2} > t\} = e^{-\lambda_{2,1}^{(1)} t} + \frac{\lambda_{2,1}^{(1)}}{\lambda_{2,1}^{(1)} - \lambda_{2,1}^{(2)}} \left(e^{-\lambda_{2,1}^{(2)} t} - e^{-\lambda_{2,1}^{(1)} t} \right), \tag{2.15}$$

for $\lambda_{1,0}^{(1)} \neq \lambda_{1,0}^{(2)}$, $\lambda_{2,1}^{(1)} \neq \lambda_{2,1}^{(2)}$, $\lambda_{1,0}^{(1)} \neq \lambda_{2,1}^{(2)}$, and $\lambda_{1,0}^{(2)} \neq \lambda_{2,1}^{(1)}$.

The mean lifetime of the cold standby system in the state subsets $\{1, 2\}$ and $\{2\}$ are found to be respectively

$$\mathrm{E}(T_C^{\geq 1}) = \frac{1}{\lambda_{1,0}^{(1)}} + \frac{1}{\lambda_{2,1}^{(1)}} + \frac{1}{\lambda_{1,0}^{(2)}} + \frac{1}{\lambda_{2,1}^{(2)}}, \tag{2.16}$$

and

$$\mathrm{E}(T_C^{\geq 2}) = \frac{1}{\lambda_{2,1}^{(1)}} + \frac{1}{\lambda_{2,1}^{(2)}}. \tag{2.17}$$

3. Stochastic comparison of systems

We first adapt the definitions of some ageing and ordering properties from binary systems to multi-state systems.

3.1. Definition. The lifetime of system A is *stochastically smaller* than the lifetime of system B in the state subset $\{j, j+1, \ldots, M\}$, if

$$P\{T_A^{\geq j} > t\} \leq P\left\{T_B^{\geq j} > t\right\} \tag{3.1}$$

for all $t \geq 0$, and we write $T_A^{\geq j} \leq_{st} T_B^{\geq j}$.

3.2. Definition. A multi-state system is said to have an *increasing failure rate* (IFR) in the state subset $\{j, j+1, \ldots, M\}$ if the failure rate function defined by

$$\begin{aligned} r^{\geq j}(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\left\{T^{\geq j} \leq t + \Delta t \mid T^{\geq j} > t\right\} \\ &= -\frac{\frac{d}{dt} P\left\{T^{\geq j} > t\right\}}{P\left\{T^{\geq j} > t\right\}} \end{aligned} \tag{3.2}$$

is a nondecreasing function of t.

3.3. Definition. The lifetime of system A is *smaller* than the lifetime of system B in the state subset $\{j, j+1, \ldots, M\}$ in failure rate ordering, if

$$r_A^{\geq j}(t) \geq r_B^{\geq j}(t) \tag{3.3}$$

for all $t \geq 0$, or equivalently $P\{T_B^{\geq j} > t\}/P\{T_A^{\geq j} > t\}$ is a nondecreasing function of t, and we write $T_A^{\geq j} \leq_{fr} T_B^{\geq j}$.

3.4. Definition. A vector $\mathbf{x} = (x_1, \ldots, x_n)$ is said to *majorize* a vector $\mathbf{y} = (y_1, \ldots, y_n)$, denoted by $\mathbf{x} \succeq_m \mathbf{y}$, if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$, and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$, where $x_{(1)} \leq \cdots \leq x_{(n)}$ and $y_{(1)} \leq \cdots \leq y_{(n)}$.

3.5. Theorem. [14] *Let X_1, X_2, Y_1 and Y_2 be independent random variables such that $X_1 \leq_{fr} Y_1$ and $X_2 \leq_{fr} Y_2$. If $X_i, Y_i, i = 1, 2$ are all IFR, then*

$$X_1 + X_2 \leq_{fr} Y_1 + Y_2. \tag{3.4}$$

3.6. Lemma. *If $\alpha = (\alpha_{1,0}, \alpha_{2,1}) \succeq_m \lambda = (\lambda_{1,0}, \lambda_{2,1})$, then the function defined by*

$$g(t) = \frac{(\lambda_{2,1} - \lambda_{1,0})}{(\alpha_{2,1} - \alpha_{1,0})} \frac{(\alpha_{2,1} e^{-\alpha_{1,0} t} - \alpha_{1,0} e^{-\alpha_{2,1} t})}{(\lambda_{2,1} e^{-\lambda_{1,0} t} - \lambda_{1,0} e^{-\lambda_{2,1} t})} \tag{3.5}$$

is nondecreasing for all $t \geq 0$.

Proof. It is easy to compute that

$$
\begin{aligned}
h(t) &= \frac{d}{dt} g(t) \\
&= \frac{1}{A^2(t)} \frac{(\lambda_{2,1}-\lambda_{1,0})}{(\alpha_{2,1}-\alpha_{1,0})} \Big[\alpha_{2,1}\lambda_{2,1}(\lambda_{1,0}-\alpha_{1,0})e^{-(\alpha_{1,0}+\lambda_{1,0})t} \\
&\qquad + \alpha_{2,1}\lambda_{1,0}(\alpha_{1,0}-\lambda_{2,1})e^{-(\alpha_{1,0}+\lambda_{2,1})t} + \alpha_{1,0}\lambda_{2,1}(\alpha_{2,1}-\lambda_{1,0}) \\
&\qquad\qquad \times e^{-(\alpha_{2,1}+\lambda_{1,0})t} + \alpha_{1,0}\lambda_{1,0}(\lambda_{2,1}-\alpha_{2,1})e^{-(\alpha_{2,1}+\lambda_{2,1})t}\Big],
\end{aligned}
\tag{3.6}
$$

where $A(t) = (\lambda_{2,1}e^{-\lambda_{1,0}t} - \lambda_{1,0}e^{-\lambda_{2,1}t})$. The four possible cases we need to consider are:

(i) $\alpha_{1,0} \leq \alpha_{2,1}$ and $\lambda_{1,0} \leq \lambda_{2,1}$,
(ii) $\alpha_{1,0} \leq \alpha_{2,1}$ and $\lambda_{2,1} \leq \lambda_{1,0}$,
(iii) $\alpha_{2,1} \leq \alpha_{1,0}$ and $\lambda_{1,0} \leq \lambda_{2,1}$,
(iv) $\alpha_{2,1} \leq \alpha_{1,0}$ and $\lambda_{2,1} \leq \lambda_{1,0}$,

Consider the case (i). If $\alpha \succeq_m \lambda$, then $\alpha_{1,0} \leq \lambda_{1,0}, \alpha_{1,0}+\alpha_{2,1} = \lambda_{1,0}+\lambda_{2,1}$, which yield $\alpha_{2,1}-\alpha_{1,0} \geq 0,\ \lambda_{2,1}-\lambda_{1,0} \geq 0,\ \lambda_{1,0}-\alpha_{1,0} \geq 0,\ \alpha_{1,0}-\lambda_{2,1} \leq 0,\ \alpha_{2,1}-\lambda_{1,0} \geq 0$, and $\lambda_{2,1}-\alpha_{2,1} \leq 0$. In this case

$$A = \alpha_{2,1}\lambda_{2,1}(\lambda_{1,0}-\alpha_{1,0})e^{-(\alpha_{1,0}+\lambda_{1,0})t} \geq 0,\ B = \alpha_{2,1}\lambda_{1,0}(\alpha_{1,0}-\lambda_{2,1})e^{-(\alpha_{1,0}+\lambda_{2,1})t} \leq 0$$

and

$$C = \alpha_{1,0}\lambda_{2,1}(\alpha_{2,1}-\lambda_{1,0})e^{-(\alpha_{2,1}+\lambda_{1,0})t} \geq 0,\ D = \alpha_{1,0}\lambda_{1,0}(\lambda_{2,1}-\alpha_{2,1})e^{-(\alpha_{2,1}+\lambda_{2,1})t} \leq 0,$$

but

$$-\frac{A}{D} \geq 1, 0 \leq -\frac{C}{B} \leq 1.$$

Because $e^{-(\alpha_{1,0}+\lambda_{1,0})t} \geq e^{-(\alpha_{2,1}+\lambda_{2,1})t},\ e^{-(\alpha_{2,1}+\lambda_{1,0})t} \geq e^{-(\alpha_{2,1}+\lambda_{2,1})t},\ e^{-(\alpha_{1,0}+\lambda_{2,1})t} \geq e^{-(\alpha_{2,1}+\lambda_{2,1})t}$ we have

$$\frac{A+B+C}{D} \geq \frac{\alpha_{2,1}-\lambda_{2,1}}{\lambda_{2,1}-\alpha_{2,1}} = -1,$$

which yields $h(t) \geq 0$. For the cases (ii), (iii) and (iv) we have $-A/D \geq 1,\ -C/B \leq 1$, $-A/D \leq 1,\ -C/B \geq 1$ and $-A/D \leq 1,\ -C/B \geq 1$, respectively. The other three cases can be evaluated similarly and the proof is completed. □

In the sequel, the components of each system are assumed to have equal degradation rates.

3.7. Theorem. *Let* $T_S^{\geq j} = \min\left(T_1^{\geq j}, T_2^{\geq j}\right)$ *and* $Z_S^{\geq j} = \min\left(Z_1^{\geq j}, Z_2^{\geq j}\right)$ *denote the lifetime of two independent multi-state series systems in the state subset* $\{j, j+1, \ldots, M\}$. *Assume that* $M = 2$ *and the degradations in the first and second systems' components follow a Markov process with degradation rates* $\lambda = (\lambda_{1,0},\ \lambda_{2,1})$ *and* $\alpha = (\alpha_{1,0}, \alpha_{2,1})$, *respectively.*

(a) *If* $\alpha \succeq_m \lambda$, *then* $T_S^{\geq 1} \leq_{fr} Z_S^{\geq 1}$.
(b) *If* $\lambda_{2,1} \geq \alpha_{2,1}$, *then* $T_S^{\geq 2} \leq_{fr} Z_S^{\geq 2}$.

Proof. We only prove part (a). The other part is simple. Because $g(t) \geq 0$,

$$\frac{P\{Z_S^{\geq 1} > t\}}{P\{T_S^{\geq 1} > t\}} = \frac{(\lambda_{2,1}-\lambda_{1,0})^2}{(\alpha_{2,1}-\alpha_{1,0})^2} \left(\frac{\alpha_{2,1}e^{-\alpha_{1,0}t} - \alpha_{1,0}e^{-\alpha_{2,1}t}}{\lambda_{2,1}e^{-\lambda_{1,0}t} - \lambda_{1,0}e^{-\lambda_{2,1}t}}\right)^2 = g^2(t), \tag{3.7}$$

and from Lemma 3.6 $g(t)$ is a nondecreasing function of $t \geq 0$ for $\alpha \succeq_m \lambda$. It follows that $T_S^{\geq 1} \leq_{fr} Z_S^{\geq 1}$. □

3.8. Definition. A reliability function R is said to be a *generalized finite mixture* of the reliability functions $R_1, R_2, \dots, R_n$ if

$$(3.8)\qquad R(t) = \sum_{i=1}^{n} \omega_i R_i(t),$$

for all t, where $\omega_1, , \omega_n$ are real numbers such that $\sum_{i=1}^{n} \omega_i = 1$.

For the properties and applications of generalized mixtures in reliability modelling see, e.g. Navarro and Hernandez [10,12] and Navarro *et al.* [13].

3.9. Proposition. *Let* $T_S^{\geq 1} = \min\left(T_1^{\geq 1}, T_2^{\geq 1}\right)$ *be the lifetime of a multi-state series system in the state subset* $\{1,2\}$. *Assume that degradation in the components follow a Markov process with degradation rates* $\lambda = (\lambda_{1,0}, \lambda_{2,1})$. *Then the distribution of* $T_S^{\geq 1}$ *is a generalized mixture of three exponential distributions, i.e.*

$$(3.9)\qquad P\{T_S^{\geq 1} > t\} = \omega_1 e^{-2\lambda_{1,0} t} + \omega_2 e^{-2\lambda_{2,1} t} + \omega_3 e^{-(\lambda_{1,0}+\lambda_{2,1})t},$$

where

$$(3.10)\qquad \omega_1 = \left(\frac{\lambda_{2,1}}{\lambda_{2,1}-\lambda_{1,0}}\right)^2,\ \omega_2 = \left(\frac{\lambda_{1,0}}{\lambda_{2,1}-\lambda_{1,0}}\right)^2,\ \omega_3 = \frac{-2\lambda_{1,0}\lambda_{2,1}}{(\lambda_{2,1}-\lambda_{1,0})^2}. \qquad \square$$

3.10. Proposition. [10,11] *If* F *is a generalized mixture such that*

$$(3.11)\qquad F(t) = \sum_{i=1}^{n} \omega_i F_i(t),$$

for all t, *where* $\omega_1, \dots, \omega_n$ *are real numbers such that* $\sum_{i=1}^{n} \omega_i = 1$. *Let* $r_i(t)$ *be the failure rate function corresponding to* $F_i(t), i = 1, \dots, n$. *If*

$$(3.12)\qquad \lim_{t\to\infty} \inf \frac{r_i(t)}{r_1(t)} = \xi_i \in (1, \infty]\,,\ \lim_{t\to\infty} \sup \frac{r_i(t)}{r_1(t)} < \infty,$$

for $i = 2, 3, \dots, n$, *then* $\lim_{t\to\infty} \frac{r(t)}{r_1(t)} = 1$, *where* $r(t)$ *is the failure rate function corresponding to* $F(t)$.

The following result is a direct consequence of Propositions 3.9 and 3.10.

3.11. Proposition. *Let* $T_S^{\geq 1} = \min\left(T_1^{\geq 1}, T_2^{\geq 1}\right)$ *be the lifetime of a multi-state series system in the state subset* $\{1,2\}$. *Assume that degradation in components follow a Markov process with degradation rates* $\lambda = (\lambda_{1,0}, \lambda_{2,1})$. *Then:*

$$(3.13)\qquad \lim_{t\to\infty} r_S^{\geq 1}(t) = 2\min(\lambda_{1,0}, \lambda_{2,1}). \quad \square$$

3.12. Theorem. [12] *If the mixture representation in Proposition 3.10 holds, and the mean residual life functions* $m_1, m_2, \dots, m_n$ *of* $F_1, F_2, \dots, F_n$ *respectively, satisfy*

$$(3.14)\qquad \lim_{t\to\infty} \inf \frac{m_1(t)}{m_i(t)} > 1,\ \lim_{t\to\infty} \sup \frac{m_1(t)}{m_i(t)} < \infty,$$

for $i = 2, 3, \dots, n$, *then the mean residual life function* m *of* F *satisfies* $\lim_{t\to\infty} \frac{m(t)}{m_1(t)} = 1$. $\square$

The mean residual lifetime function of a multi-state system in the state subset $\{j, j+1, \dots, M\}$ is defined as

$$(3.15)\qquad m^{\geq j}(t) = \mathrm{E}(T^{\geq j} - t \mid T^{\geq j} > t).$$

The following result readily follows from Proposition 3.9 and Theorem 3.12.

3.13. Proposition. *Let* $T_S^{\geq 1} = \min\left(T_1^{\geq 1}, T_2^{\geq 1}\right)$ *be the lifetime of a multi-state series system in the state subset* $\{1, 2\}$ *and* $m_S^{\geq 1}(t) = \mathrm{E}(T_S^{\geq 1} - t \mid T_S^{\geq 1} > t)$*. Assume that degradation in components follow a Markov process with degradation rates* $\lambda = (\lambda_{1,0}, \lambda_{2,1})$*. Then*

$$\lim_{t\to\infty} m_S^{\geq 1}(t) = \frac{1}{2\min(\lambda_{1,0}, \lambda_{2,1})}. \tag{3.16}$$

□

3.14. Theorem. *Let* $T_C^{\geq j} = T_1^{\geq j} + T_2^{\geq j}$ *and* $Z_C^{\geq j} = Z_1^{\geq j} + Z_2^{\geq j}$ *denote the lifetime of two independent multi-state cold standby systems in the state subset* $\{j, j+1, \ldots, M\}$*. Assume that* $M = 2$ *and that the degradations in the first and second systems' components follow a Markov process while they are in active state, with degradation rates* $\lambda = (\lambda_{1,0}, \lambda_{2,1})$ *and* $\alpha = (\alpha_{1,0}, \alpha_{2,1})$*, respectively.*

(a) *If* $\alpha \succeq_m \lambda$*, then* $T_C^{\geq 1} \leq_{fr} Z_C^{\geq 1}$*.*
(b) *If* $\lambda_{2,1} \geq \alpha_{2,1}$*, then* $T_C^{\geq 2} \leq_{fr} Z_C^{\geq 2}$*.*

Proof. By Lemma 3.6

$$g(t) = \frac{P\{Z_i^{\geq 1} > t\}}{P\{T_i^{\geq 1} > t\}} = \frac{(\lambda_{2,1} - \lambda_{1,0})}{(\alpha_{2,1} - \alpha_{1,0})} \frac{(\alpha_{2,1}e^{-\alpha_{1,0}t} - \alpha_{1,0}e^{-\alpha_{2,1}t})}{(\lambda_{2,1}e^{-\lambda_{1,0}t} - \lambda_{1,0}e^{-\lambda_{2,1}t})}, \tag{3.17}$$

is nondecreasing for all $t \geq 0$, and this implies $T_i^{\geq 1} \leq_{fr} Z_i^{\geq 1}$, $i = 1, 2$. Because $T_1^{\geq 1}$, $T_2^{\geq 1}$, $Z_1^{\geq 1}$ and $Z_2^{\geq 1}$ are IFR the proof of (a) follows from Theorem 3.5. The proof of part (b) is simple and hence is omitted. □

3.15. Example. Let $\lambda = (0.23, 0.22)$ and $\alpha = (0.27, 0.18)$. Because $\alpha \succeq_m \lambda$ and $\lambda_{2,1} \geq \alpha_{2,1}$, we have $T_S^{\geq j} \leq_{fr} Z_S^{\geq j}$ and $T_C^{\geq j} \leq_{fr} Z_C^{\geq j}$, $j = 1, 2$.

We can also compute

$$\mathrm{E}(T_S^{\geq 1}) = 5.5578,\ \mathrm{E}(T_S^{\geq 2}) = 2.2727,\ \mathrm{E}(Z_S^{\geq 1}) = 5.7407,\ \mathrm{E}(Z_S^{\geq 2}) = 2.7778,$$
$$\mathrm{E}(T_C^{\geq 1}) = 17.7866,\ \mathrm{E}(T_C^{\geq 2}) = 9.0909 \text{ and } \mathrm{E}(Z_C^{\geq 1}) = 18.5185,\ \mathrm{E}(Z_C^{\geq 2}) = 11.1111.$$

For a series system having instantaneous degradation rate λ,

$$\lim_{t\to\infty} r_{S,\lambda}^{\geq 1}(t) = 2\min(\lambda_{1,0}, \lambda_{2,1}) = 0.44,$$
$$\lim_{t\to\infty} m_{S,\lambda}^{\geq 1}(t) = \frac{1}{2\min(\lambda_{1,0}, \lambda_{2,1})} = 2.2727,$$

and for a series system with instantaneous degradation rate α,

$$\lim_{t\to\infty} r_{S,\alpha}^{\geq 1}(t) = 2\min(\alpha_{1,0}, \alpha_{2,1}) = 0.36,$$
$$\lim_{t\to\infty} m_{S,\alpha}^{\geq 1}(t) = \frac{1}{2\min(\alpha_{1,0}, \alpha_{2,1})} = 2.7778.$$

4. Conclusions

In this paper, we have studied some properties of multi-state series and cold standby systems consisting of two components. The systems and components are assumed to have three states and the degradation in components occurs according to a Markov process. In the present study only minor failures occur. A minor failure is a failure that causes the component transition from state i to $i-1$. A more general model can be obtained by considering a major failure, that is a failure that causes the component transition from state i to state $j < i$ [7].

We have presented some ordering results associated with failure rates of series and cold standby systems. These results are useful for comparing the performances of multi-state systems having different instantaneous degradation rates. The limiting properties of failure rate and the mean residual life function of multi-state series system were also investigated via generalized mixtures.

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