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β-RELATIONS ON IMPLICATIVE BOUNDED HYPER BCK-ALGEBRAS

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Abstract

In this paper we consider the notion of hyper implicative bounded BCK-algebras, give some examples and introduce the relation β on them. Then we let β^* be the transitive closure of β. In hyper implicative bounded BCK-algebra theory, the fundamental relation is defined as the smallest equivalence relation so that the quotient would be the (fundamental) BCK-algebra. We show that β^* is the fundamental relation on a hyper implicative bounded BCK-algebra. Finally, we state conditions that are equivalent with the transitivity of this relation.

Keywords: Hyper BCK-algebra, Strongly regular relation, Fundamental relation.

2000 AMS Classification: 20 N 20.

1. Introduction

Imai and Iseki [3] in 1966 introduced the notion of BCK-algebra as an important tool for recent investigations in algebraic logic. This notion originated in two different ways. One of the motivations is based on set theory. In set theory, there are three elementary and fundamental operations, introduced by L. Kantorovic and E. Livenson, to make a new set from old sets. These fundamental operations are union, intersection and set difference. As a generalization of union, intersection and set difference and their properties, we have the concept of Boolean algebra that added the notion of distributivity. Moreover, if we consider the notion of union or intersection, we have the notion of an upper semilattice or lower semilattice. But, the notion of set difference was not considered systematically before K. Isaci.

Another motivation is taken from classical and non-classical propositional calculi. There are some systems which contain only the implication functor among the logical

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functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and the BCI, BCK-systems by C. AS. Meredith.

We know the following simple relations in set theory:

$$(A - B) - (A - C) \subset C - B,$$
$$A - (A - B) \subset B.$$

In propositional calculi, these relations are denoted by

$$(p \to q) \to ((q \to r) \to (p \to r)),$$
$$p \to ((p \to q) \to q).$$

From these relationships, K. Isaci introduced a new notion called a BCK-algebra. BCK-algebras arise as an algebraic counterpart of pure implicational logic containing only the logical connective implication $\to$ and the constant 1, considered as the value "true", in which the formulas

(B) $(p \to q) \to ((q \to r) \to (p \to r))$;
(C) $(p \to (q \to r)) \to (q \to (p \to r))$;
(K) $p \to (q \to p)$.

are theorems. Here (B) and (C) mean transitivity and commutativity, respectively.

The study of hyperstructures started in 1934 with Marty's paper at the 8th Congress of Scandinavian Mathematicians [7] which introduced the concept of *hypergroup*. Since then other classic hyperstructures have also been studied. A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *semi-hypergroup* if for all $x, y, z \in H$ the following condition holds:

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

A short review of hyperstructure theory appears in [2]. Recently (see [6]) Jun, Zahedi, Xin, and Borzoei applied hyperstructure theory to BCK-algebras, and introduced the concept of the *hyper K-algebra*, which is a generalization of the concept of BCK-algebra.

2. Preliminaries

2.1. Definition. A *BCK-algebra* is a non-empty set H endowed with a binary operation "$\circ$" and a constant 0 satisfying the following axioms:

(BCK 1): $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = 0$,
(BCK 2): $(x \circ (x \circ y)) \circ y = 0$,
(BCK 3): $x \circ x = 0$,
(BCK 4): $0 \circ x = 0$,
(BCK 5): $x \circ y = 0$ and $y \circ x = 0$ imply $x = y$.

Let A be a BCK-algebra. The partial ordering $\leq$ on A is defined by

$$x \leq y \text{ if and only if } x \circ y = 0.$$

If A contains an element 1 such that $a \leq 1$, i.e., $a \circ 1 = 0$, for all $a \in A$ then A is called a *bounded BCK-algebra*.

If $x \wedge y = y \wedge x$, where $x \wedge y = y \circ (y \circ x)$ for all $x, y \in A$, then A is called a *commutative BCK-algebra*.

A bounded commutative BCK-algebra A is a *distributive lattice* with respect to $\wedge$ and $\vee$, where $x \vee y = N(Nx \wedge Ny)$ and $Nx = 1 \circ x$ for all $x, y \in A$ (see [1, 4, 9]). A BCK-algebra A is called *implicative* if $x \circ (y \circ x) = x$ for all $x, y \in A$. Every implicative BCK-algebra is commutative [5, Theorem 9].

A BCK-algebra is therefore a partially ordered set A with a fixed element 0 and a binary operation "$\circ$" satisfying the following axioms:

(bck 1): $((x \circ y) \circ (x \circ z)) \leq (z \circ y)$,
(bck 2): $(x \circ (x \circ y)) \leq y$,
(bck 3): $x \leq x$,
(bck 4): $0 \leq x$,
(bck 5): $x \leq y$ and $y \leq x$ imply $x = y$ □

2.2. Theorem. [6] *Let $\langle A, \circ, 0\rangle$ be a BCK-algebra. We have the following properties:*

(1) $x \circ 0 = x$;
(2) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$;
(3) $x \leq y$ *implies* $x \circ z \leq y \circ z$;
(4) $x \leq y$ *implies* $z \circ x \leq z \circ y$;
(5) $(x \circ y) \circ z = (x \circ z) \circ y$;
(6) $(x \circ y) \leq z$ *implies* $(x \circ z) \leq y$;
(7) $x \circ y \leq x$. □

2.3. Definition. Let I be a non-empty subset of a BCK-algebra $\langle H, \circ, 0\rangle$. Then I is called an *ideal* of H if

(I1) $0 \in I$;
(I2) $x \circ y \in I$ and $y \in I$ imply that $x \in I$.

An ideal P in a commutative BCK-algebra A is called *prime* if $x \wedge y \in P$ implies that $x \in P$ or $y \in P$. Equivalently (see [9]), P is prime if and only if for any ideals I, J of A, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. The maximal and minimal ideals of BCK-algebras have the usual meaning.

2.4. Theorem. [4] *Let X be a subset of a BCK-algebra A. The set of all $a \in A$ for which $(\ldots((a \circ x_1) \circ x_2) \circ \ldots \circ x_n) = 0$ for some $x_1, x_2, \ldots, x_n \in X$ is the minimal ideal containing X, it is called the ideal generated by X and is usually denoted by $\langle X\rangle$. If $X = \{x\}$ then we denote $\langle\{x\}\rangle$ by $\langle x\rangle$, and call it as the principal ideal generated by x.*

Let I be an ideal of a BCK-algebra A. We define an equivalence relation R on A by aRb if and only if $a \circ b \in I$. Let C_a denote the equivalence class of $a \in A$, then evidently $C_0 = I$. Let A/I denote the set of all classes C_a, $a \in A$. Then A/I is a BCK-algebra with $C_a \circ C_b = C_{a \circ b}$ and $C_a \leq C_b$ if and only if $a \leq b$. A/I is called the *quotient BCK-algebra* of A determined by I.

2.5. Definition. Let H be a non-empty set, endowed with a binary hyperoperation "$\circ$" and a constant 0. Then $\langle H, \circ, 0\rangle$ is called a *quasi hyper BCK-algebra* if it satisfies the following axioms:

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x \circ H \ll x$,

and is called a *hyper BCK-algebra*, if also

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

A hyper BCK-algebra A is called *bounded* if there exists an element 1 such that $x \ll 1$ for all $x \in A$, and is called *implicative* if $x \in (x \circ y) \circ x$ for all $x, y \in A$.

2.6. Example. Let $A = \{0, a, 1\}$. Consider the following table:

$\circ$	0	a	1
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
1	{1}	{a,1}	{0,a,1}

Then $(A, \circ, 0, 1)$ is a bounded and implicative hyper BCK-algebra.

2.7. Example. Let $A = \{0, a, b, 1\}$. Consider the following table:

$\circ$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{0}	{0}
b	{b}	{b}	{0}	{0}
1	{1}	{b}	{a}	{0,1}

Then $(A, \circ, 0, 1)$ is a bounded hyper BCK-algebra, but it is not implicative.

2.8. Theorem. *Let $\langle H, \circ, 0\rangle$ be a hyper BCK-algebra, A, B any non-empty subsets of H and $x, y \in H$. We have the following properties:*

(h1) $x \circ y \ll x$;
(h2) $x \circ 0 \ll x$, $0 \circ x \ll 0$ *and* $0 \circ 0 \ll 0$;
(h3) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ *and* $0 \circ A \ll 0$;
(h4) $0 \circ 0 = \{0\}$;
(h5) $0 \ll x$;
(h6) $x \ll x$;
(h7) $A \ll A$;
(h8) $A \subseteq B$ *implies* $A \ll B$;
(h9) $0 \circ x = \{0\}$;
(h10) $0 \circ A = \{0\}$;
(h11) $A \ll 0$ *implies* $A = \{0\}$;
(h12) $x \in x \circ 0$, $(x \circ 0 = \{x\})$;
(h13) $x \circ 0 \ll y$ *implies* $x \ll y$;
(h14) $x \ll y$ *implies* $x \circ z \ll y \circ z$;
(h15) $x \circ y = 0$ *implies* $(x \circ z) \circ (y \circ z) = 0$ *and* $x \circ z \ll y \circ z$;
(h16) $A \circ 0 = 0$ *implies* $A = 0$.

In a hyper BCK-algebra H, the condition (H3) is equivalent to the condition (h1). □

2.9. Example. Let $\langle H, \circ, 0\rangle$ be a BCK-algebra and I an ideal of H. Now, we define the hyper operation "$\bar{\circ}$" on H/I as follows

$$x/I \;\bar{\circ}\; y/I := \bigcup_{a \in x/I, b \in y/I} (a \circ b)/I.$$

Then $\langle H/I, \bar{\circ}, 0/I\rangle$ is a hyper BCK-algebra.

2.10. Definition. Let I be a non-empty subset of a hyper BCK-algebra $\langle H, \circ, 0\rangle$. Then I is called a *hyper BCK-ideal* of H if

(I1) $0 \in I$;

(I2) if $x \circ y \ll I$ and $y \in I$ imply that $x \in I$.

2.11. Example. Let A be a hyper BCK-algebra as in Example 2.7. Then $I := \{0, 1\}$ is a hyper BCK-ideal.

3. The fundamental relation on implicative bounded hyper BCK-algebras

3.1. Definition. Let R be an equivalence relation on a hyper BCK-algebra $\langle H, \circ, 0, 1\rangle$ and $A, B \subseteq H$. Then

(i) ARB if and only if there exist $a \in A$ and $b \in B$ such that aRb.
(ii) $A\overline{R}B$ if and only if for all $a \in A$ there exist $b \in B$ such that aRb.
(iii) $A\overline{\overline{R}}B$ if and only if for all $a \in A$ and $b \in B$ we have aRb.
(iv) R is called *regular* to the right if aRb implies $a \circ c\overline{R}\ b \circ c$, for all $a, b, c \in H$.
(v) R is called *strongly regular* to the right if aRb implies $a \circ c\overline{\overline{R}}\ b \circ c$, for all $a, b, c \in H$.
(vi) R is called *good*, if $a \circ bR\{0\}$ and $b \circ aR\{0\}$ imply aRb, for all $a, b \in H$.

Let $< H, \circ, 0 >$ be a hyper BCK-algebra and A a subset of H. Then $\mathfrak{L}(A)$ will denote the set of all finite combinations of elements A with $\circ$.

Now, in the following, the well-known idea of β^* relation on hyperstructures [2, 8, 10] is transferred and applied to hyper BCK-algebras.

3.2. Definition. If $\langle H, \circ, 0\rangle$ is a hyper BCK-algebra, then we set:

$$\beta_1^H = \{(a, a) \mid a \in H\}$$

and, for every integer $n > 1$, β_n^H is the relation defined as follows:

$$a\beta_n^H b \iff \exists (c_1, c_2, \ldots, c_n) \in H^n,\ \exists z \in \mathfrak{L}(\{c_1, c_2, \ldots, c_n\}) : \{a, b\} \subseteq z.$$

Obviously, for every $n \geq 1$, the relations β_n^H are symmetric, and the relation $\beta^H = \bigcup_{n\geq 1} \beta_n^H$ is reflexive and symmetric. Let β_*^H be the *transitive closure* of β^H. (When it is clear from the context which hyper BCK-algebra is being considered, $\beta_n,\ \beta,\ \beta^*$ will be written in place of $\beta_n^H,\ \beta^H,\ \beta_*^H$).

3.3. Theorem. *Let $\langle H, \circ, 0, 1\rangle$ be a hyper BCK-algebra. Then β_*^H is strongly regular on H. Moreover, if H is implicative and bounded, then β_*^H is a good relation.*

Proof. Let $a\beta^*b$. Then there exist $s \in \mathbb{N}$, $(c_0, c_1, \ldots, c_s) \in H^{s+1}$ such that $c_0 = a, c_s = b$ and $q_1, q_2, \ldots, q_s \in \mathbb{N}$ such that $a = c_0\beta_{q_1}c_1\beta_{q_2}c_2 \ldots \beta_{q_s}c_s = b$, so for each $i \in \{1, 2, \ldots, s\}$ there exists $(z_1^i, z_2^i, \ldots, z_{q_i}^i) \in H^{q_i}$ such that $\{c_i, c_{i+1}\} \subseteq z^i$, where $z^i \in \mathfrak{L}(\{z_1^i, z_2^i, \ldots, z_{q_i}^i\})$. Now, let $x \in H$. It easily follows that $\forall i \in \{0, 1, \ldots, s-1\}, c_i \circ x, c_{i+1} \circ x \subseteq z^i \circ x$. Consequently if i is such that $0 \leq i \leq s$, $\forall u \in c_i \circ x, \forall v \in c_{i+1} \circ x$, one gets $u\beta v$. Therefore, $\forall u \in c_1 \circ x, \forall v \in c_s \circ x$ we have $u\beta^*v$ and thus by Definition 3.1, β^* is strongly regular.

Now, let H be implicative and bounded. We show that β^* is good. Suppose that $x \circ y\beta^*\{0\}$ and $y \circ x\beta^*\{0\}$. Since β^* is a strongly regular relation on H, we have $(x \circ y) \circ x\overline{\overline{\beta^*}}0 \circ x = \{0\}$ and $(y \circ x) \circ y\overline{\overline{\beta^*}}0 \circ y = \{0\}$. Since $x \in (x \circ y) \circ x$ then $x\beta^*0$. Similarly, we have $y\beta^*0$, so by transitivity of β^*, $x\beta^*y$. □

3.4. Lemma. *Let $\langle H, \circ, 0\rangle$ be a hyper BCK-algebra and R a regular relation on H. We denote the set of all equivalence classes of R by H/R (i.e $H/R = \{R(x) \mid x \in H\}$), and for all $R(x), R(y) \in H/R$ define*

$$R(x)\overline{\circ}R(y) := \{R(t) | t \in x \circ y\}.$$

Then $\overline{\circ}$ is well defined.

Proof. Let $R(x_1) = R(x_2)$ and $R(y_1) = R(y_2)$. Then x_1Rx_2 and y_1Ry_2. Since R is a regular relation, we have $x_1 \circ y_1 \overline{R} x_2 \circ y_2$. We must show that $R(x_1)\overline{\circ}R(y_1) = R(x_2)\overline{\circ}R(y_2)$. Let $R(t) \in R(x_1)\overline{\circ}R(y_1)$, therefore $R(t) = R(s)$ for some $s \in x_1 \circ y_1$. From $x_1 \circ y_1 \overline{R} x_2 \circ y_2$ there exists $u \in x_2 \circ y_2$ such that sRu. Thus $R(s) = R(u)$ and $R(s) = R(t)$, so $R(t) = R(u) \in R(x_2)\overline{\circ}R(y_2)$. Therefore, $R(x_1)\overline{\circ}R(y_1) \subseteq R(x_2)\overline{\circ}R(y_2)$. Similarly, we have $R(x_2)\overline{\circ}R(y_2) \subseteq R(x_1)\overline{\circ}R(y_1)$, hence $\overline{\circ}$ is well defined. □

3.5. Definition. Let $\langle H, \circ, 0\rangle$ be a hyper BCK-algebra, R a regular relation on H and $R(x), R(y) \in H/R$. Then we define

$$R(x) \ll R(y) \text{ if and only if } R(0) \in R(x)\overline{\circ}R(y).$$

3.6. Theorem. *Let R be an equivalence relation on H. Then R is a regular relation on H if and only if $(H/R, \overline{\circ}, R(0))$ is a quasi hyper BCK-algebra.*

Proof. $\Longrightarrow$ Let $a \in (R(x)\overline{\circ}R(z))\overline{\circ}(R(y)\overline{\circ}R(z))$. Then there exist $s \in x \circ z$, $t \in y \circ z$ and $h \in H$ such that $a = R(h) = R(s)\overline{\circ}R(t)$. So there exists $h' \in s \circ t \subseteq (x \circ z) \circ (y \circ z)$. Hence by condition (H1) of Definition 2.5, there exists $h'' \in x \circ y$ such that $h' \ll h''$, which means that $0 \in h' \circ h''$. Hence $R(0) \in R(h') \circ R(h'')$ and $R(h'') \in R(x) \circ R(y)$. We conclude that $R(0) \in R(h') \circ R(h'')$, and by Definition 3.3, we obtain $R(h) \ll R(h'')$ so

$$(R(x)\overline{\circ}R(z))\overline{\circ}(R(y)\overline{\circ}R(z)) \ll R(x)\overline{\circ}R(y).$$

Now, we must prove (H2) for H/R. Suppose that $R(h)$ is an element of $(R(x)\overline{\circ}R(y))\overline{\circ}R(z)$, so there exists $s \in x \circ y$ such that $R(h) \in R(s)\overline{\circ}R(z)$. Then there exists $t \in s \circ z$ such that $R(h) = R(t)$. But, we have $(x \circ y) \circ z = (x \circ z) \circ y$, hence $s \circ z \subseteq (x \circ z) \circ y$. By definition there exists $s' \in x \circ z$ such that $t \in s' \circ y$, so

$$R(t) \in R(s')\overline{\circ}R(y) \subseteq (R(x)\overline{\circ}R(z))\overline{\circ}R(y).$$

Similarly, $(R(x)\overline{\circ}R(z))\overline{\circ}(R(y) \subseteq (R(x)\overline{\circ}R(y))\overline{\circ}R(z)$, and so the condition (H2) issatisfied in H/R. Clearly, we have (H3). Therefore, H/R is a quasi hyper BCK-algebra.

$\Longleftarrow$ Let $a, b, c, d \in H$ with aRb and cRd. Then $R(a) = R(b)$ and $R(c) = R(d)$. Since H/R is a quasi hyper BCK-algebra and $\overline{\circ}$ is well defined, we have $R(a)\overline{\circ}R(c) = R(b)\overline{\circ}R(d)$. Let $R(x) \in R(a)\overline{\circ}R(c) = R(b)\overline{\circ}R(d)$. Then there exist $t \in a \circ c$ and $u \in b \circ d$ such that $R(x) = R(t) = R(u)$ so $a \circ cRb \circ d$. Therefore, R is a regular relation on H. □

3.7. Theorem. *Let R be a good strongly regular relation on H. Then $(H/R, \overline{\circ}, R(0))$ is a hyper BCK-algebra, called the quotient hyper BCK-algebra of H with respect to R.*

Proof. Let R be a good regular relation on H. By Theorem 3.6, we see that $(H/R, \overline{\circ}, R(0))$ satisfies all conditions of Definition 2.5 except (H4). Let $R(x) \ll R(y)$ and $R(y) \ll R(x)$. Since $R(x) \ll R(y)$, we have $R(0) \in R(x) \circ R(y)$. Then $R(0) = R(t)$ for some $t \in x \circ y$. This implies that $x \circ yR\{0\}$. Similarly, from $R(y) \ll R(x)$ we get that $y \circ xR\{0\}$. Therefore, we have $x \circ yR\{0\}$ and $y \circ xR\{0\}$. Since R is a good regular relation, by Definition 3.1 (vi), we have xRy. Hence, $R(x) = R(y)$. □

3.8. Definition. Let H, H' be hyper BCK-algebras. A function $f : H \longrightarrow H'$ is called a *homomorphism* if it satisfies the condition $\forall (x, y) \in H^2$, $f(x \circ y) = f(x) \circ f(y)$ and $f(0) = 0$.

3.9. Theorem. *Let H, H' be hyper BCK-algebras and $f : H \longrightarrow H'$ a homomorphism. The equivalence R associated with f (i.e $R = \{(x, y) \mid f(x) = f(y)\}$) is regular. Furthermore, the function $g : f(H) \longrightarrow H/R$, $g(f(x)) = \overline{x}$, is an isomorphism.*

Proof. Let xRy and $a \in H$, so $f(x \circ a) = f(x) \circ f(a) = f(y) \circ f(a) = f(y \circ a)$, from which it follows that $\forall u \in x \circ a$ there exists $v \in y \circ a$ such that $f(u) = f(v)$, and this implies that uRv. Regularity on the left is shown similarly, and therefore R is regular. Now, we prove that g is an isomorphism. Since $\overline{x \circ y} = \overline{x} \circ \overline{y}$ is clear, g is an monomorphism, and obviously g is onto. □

3.10. Theorem. *Let R be a good and strongly regular relation on a hyper BCK-algebra $(H, \circ, 0)$. Then*

(1) *H/R is a BCK-algebra.*
(2) *If H' is a BCK-algebra and $f : H \longrightarrow H'$ is a homomorphism, then the equivalence ρ associated with f is strongly regular.*

Proof. (1) Since R is a good and strongly regular relation on H, we have $x \circ y \overline{\overline{R}} x \circ y$ for each $x, y \in H$. It follows that $|\overline{x} \circ \overline{y}| = 1$. Therefore H/R is a BCK-algebra.

(2) Let $x\rho y$ and $a \in H$. Therefore $f(x) = f(y)$, so for every $u \in x \circ a,\ v \in y \circ a$ we obtain $f(u), f(v) \in f(x \circ a) = f(x) \circ f(a)$. Since H' is a BCK-algebra, then $|f(x \circ a)| = 1 = |f(y \circ a)|$ which implies that $u\rho v$. Thus, $x \circ a \ \overline{\overline{\rho}}\ y \circ a$. □

3.11. Theorem. *The relation β^* is the smallest equivalence relation such that the quotient H/β^* is a BCK-algebra.*

Proof. Firstly, β^* is a strongly regular relation and so H/β^* is a BCK-algebra. Let θ be an equivalence relation such that H/θ is a BCK-algebra and let $\varphi : H \longrightarrow H/\theta$ be the canonical projection. If $x\beta y$ there exist $n \in \mathbb{N}$, $(c_1, c_2, \ldots, c_n) \in H^n$ and $z \in \mathcal{L}(c_1, c_2, \ldots, c_n)$ such that $\{a, b\} \subseteq z$. Since $\varphi(a), \varphi(b) \in \varphi(z)$ and $|\varphi(z)| = 1$ then $\varphi(a) = \varphi(b)$, therefore $\theta(a) = \theta(b)$ and we have $a\ \theta\ b$, hence $\beta \subseteq \theta$. It follows that $\beta^* \subseteq \theta$. □

3.12. Corollary. *The relation β^* is the smallest strongly regular relation on a hyper BCK-algebra.*

Proof. Let θ be a strongly regular relation. Then H/θ is a BCK-algebra. But by Theorem 3.11, β^* is the smallest equivalence relation such that the quotient H/β^* is a BCK-algebra. Therefore, $\beta^* \subseteq \theta$. □

4. Transitivity conditions for β

In this section, we determine some necessary and sufficient conditions under which the relation β is transitive.

4.1. Definition. Let A be a non-empty subset of H. We say that A is a *β-part* if for every $n \in \mathbb{N}$, $i = 1, 2, \ldots, n$, $\forall k_i \in \mathbb{N}$, $\forall (a_{i1}, a_{i2}, \ldots, a_{ik_i}) \in H^{k_i}$, we have

$$\mathcal{L}(a_{i1}, a_{i2}, \ldots, a_{ik_i}) \cap A \neq \emptyset \implies \mathcal{L}(a_{i1}, a_{i2}, \ldots, a_{ik_i}) \subseteq A.$$

4.2. Lemma. *Let A be a non-empty subset of a hyper BCK-algebra H. The following conditions are equivalent:*

(1) *A is a β-part of H;*
(2) *$x \in A$, $x\beta y$ imply $y \in A$;*
(3) *$x \in A$, $x\beta^* y$ imply $y \in A$.*

Proof. (1) $\implies$ (2) Let $(x, y) \in H^2$ be such that $x \in A$ and $x\beta y$, so by definition there exist $n \in \mathbb{N}$ and $(a_1, a_2, \ldots, a_n) \in H$ such that $\{x, y\} \subseteq \mathcal{L}(a_1, a_2, \ldots, a_n)$. Since A is a β-part and $\mathcal{L}(a_1, a_2, \ldots, a_n) \cap A \neq \emptyset$, we have $y \in A$.

(2) $\Longrightarrow$ (3) Let $(x,y) \in H^2$ be such that $x \in A$ and $x\beta^* y$, so by transitivity there exist $n \in \mathbb{N}$ and $(x = a_0, a_1, \ldots, a_n = y) \in H^{n+1}$ such that $x = a_0 \beta a_1 \beta \cdots \beta a_n = y$. By applying (2) n times, we conclude that $y \in A$.

(3) $\Longrightarrow$ (1) Let $\mathcal{L}(a_1, a_2, \ldots, a_n) \cap A \neq \emptyset$, and $x \in \mathcal{L}(a_1, a_2, \ldots, a_n) \cap A$. Now, for every $y \in \mathcal{L}(a_1, a_2, \ldots, a_n)$, we have $x\beta y$. Therefore, $x\beta^* y$ and (3) implies $y \in A$. $\square$

Before proving the next theorem, we introduce the following notations. For every element x of a hyper BCK-algebra, set:

$$P_n(x) = \bigcup\{\mathcal{L}(A) | x \in \mathcal{L}(A),\ |A| = n\}$$
$$P_\sigma(x) = \bigcup_{n \geq 1} P_n(x).$$

4.3. Lemma. *For every $x \in H$, $P_\sigma(x) = \{y \in H \mid x\beta y\}$*

Proof. We have

$$\begin{aligned} x\,\beta\,y &\iff \exists\, n \in \mathbb{N}, \exists\,(z_1, z_2, \ldots, z_3) \in H^n : x \in \mathcal{L}(z_1, z_2, \ldots, z_3) \\ &\iff \exists\, n \in \mathbb{N} : y \in P_n(x) \\ &\iff y \in P_\sigma. \end{aligned}$$
$\square$

4.4. Theorem. *Let H be a hyper BCK-algebra. Then the following conditions are equivalent:*

(1) *β is transitive;*
(2) *for every $x \in H$, $\beta^*(x) = P_\sigma(x)$;*
(3) *for every $x \in H$, $P_\sigma(x)$ is a β-part of H.*

Proof. (1) $\Longrightarrow$ (2) By Lemma 4.3, for every pair (x,y) of elements of H we have:

$$y \in \beta^*(x) \iff x\ \beta^*\ y \iff x\,\beta\,y \iff y \in P_\sigma(x).$$

(2) $\Longrightarrow$ (3) By Lemma 4.2, if A is a non-empty subset of H, then A is a β-part of H if and only if it is a union of equivalence classes modulo β^*. Particularly, every equivalence class modulo β^* is a β-part of H.

(3) $\Longrightarrow$ (1) If $x\beta y$ and $y\beta z$, then there exist $(n,m) \in \mathbb{N}^* \times \mathbb{N}^*$, $(x_1, x_2, \ldots, x_n) \in H^n$, $(y_1, y_2, \ldots, y_m) \in H^m$, such that $\{x,y\} \in \mathcal{L}(x_1, x_2, \ldots, x_n)$, $\{y,z\} \in \mathcal{L}(y_1, y_2, \ldots, y_m)$. Since $P_\sigma(x)$ is a β-part of H, we have

$$\begin{aligned} x \in \mathcal{L}(x_1, x_2, \ldots, x_n) \cap P_\sigma(x) &\Longrightarrow \mathcal{L}(x_1, x_2, ..., x_n) \subseteq P_\sigma(x) \\ &\Longrightarrow y \in \mathcal{L}(y_1, y_2, \ldots, y_m) \cap P_\sigma(x) \\ &\Longrightarrow \mathcal{L}(y_1, y_2, \ldots, y_m) \subseteq P_\sigma(x) \\ &\Longrightarrow z \in P_\sigma(x) \\ &\Longrightarrow \exists\, k \in \mathcal{N}^* : z \in P_k(x) \Longrightarrow z\beta x. \end{aligned}$$

Therefore, β is transitive. $\square$

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