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ON LUCAS NUMBERS BY THE MATRIX METHOD

Fikri Köken*† and Durmus Bozkurt*

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Abstract

In this study we define the Lucas Q_L-matrix similar to the Fibonacci Q-matrix. The Lucas Q_L-matrix is different from the Fibonacci Q-matrix, but is related to it. Using this matrix representation, we have found some well-known equalities and a Binet-like formula for the Lucas numbers.

Keywords: Fibonacci numbers, Lucas numbers, Matrix method.

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1. Introduction

Fibonacci and Lucas numbers and their generalization have many interesting properties and applications to almost every field of science and art. For the prettiness and rich applications of these numbers and their relatives to science and nature one can see [1-5].

As in [4], let Q be the 2×2 matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then for an integer n with $n \geq 1$, Q^n has the form

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \tag{1.1}$$

This property provides an alternate proof of Cassini's Fibonacci formula:

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

*Selçuk University, Faculty of Art and Science, Department of Mathematics, 42075 Kampus, Konya, Turkey.
E-mail: (F. Köken) kokenfikri@gmail.com (D. Bozkurt) dbozkurt@selcuk.edu.tr
†Corresponding Author.

Also, let n and m be two integers such that $m, n \geq 1$. The following results are obtained from the identity $Q^{m+n} = Q^n Q^m$ for the matrix (1.1):

$$\begin{aligned} F_{m+n+1} &= F_{m+1}F_{n+1} + F_m F_n, \\ F_{m+n} &= F_{m+1}F_n + F_m F_{n-1}. \end{aligned}$$

These are basically similar, but could be applied to derive new Fibonacci identities, such as the following properties,

$$\begin{aligned} L_{m+n} &= F_{m+1}L_n + F_m L_{n-1}, \\ 2F_{m+n} &= F_m L_n + F_n L_m, \\ 2L_{m+n} &= L_m L_n + 5F_m F_n. \end{aligned}$$

Here, F_n denotes the nth Fibonacci number and L_n the nth Lucas number. The following properties of the Fibonacci and Lucas numbers are given in [3].

$$\begin{aligned} &F_{n+1} + F_{n-1} = L_n, \\ &L_{n+1} + L_{n-1} = 5F_n. \end{aligned}$$

In this study, we define the Lucas Q_L-matrix by

$$Q_L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{1.2}$$

It is easy to see that

$$\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = Q_L \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \text{ and } 5 \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Q_L \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix}$$

where F_n and L_n are as above. Our aim, is not to compute powers of matrices. Our aim is to find different relations between matrices containing Fibonacci and Lucas numbers. That is, we obtain relations between the Fibonacci Q matrix and the Lucas Q_L matrix in Theorem 2.1.

2. Matrix representation of the Lucas numbers

In this section, we will present a new matrix representation of the Fibonacci and Lucas numbers. We obtain Cassini's formulas and properties of these numbers by a similar matrix method to the Fibonacci Q-matrix.

2.1. Theorem. *Let Q_L be as in (1.2). Then, for integers $n \geq 1$,*

$$Q_L^n = \begin{cases} 5^{\frac{n}{2}} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} & \text{for even } n, \\ 5^{\frac{n-1}{2}} \begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} & \text{for odd } n, \end{cases} \tag{2.1}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively.

Proof. We use mathematical induction on n. First, we consider odd n. For $n = 1$,

$$Q_L = \begin{bmatrix} L_2 & L_1 \\ L_1 & L_0 \end{bmatrix},$$

since $L_2 = 3$, $L_1 = 1$ and $L_0 = 2$. So, (2.1) is indeed true for $n = 1$. Now we suppose it is true for $n = k$, that is

$$Q_L^k = 5^{\frac{k-1}{2}} \begin{bmatrix} L_{k+1} & L_k \\ L_k & L_{k-1} \end{bmatrix}.$$

Using properties of the Lucas numbers and the induction hypothesis, we can write

$$\begin{aligned} Q_L^{k+2} &= Q_L^k Q_L^2 \\ &= 5^{\frac{k+1}{2}} \begin{bmatrix} L_{k+3} & L_{k+2} \\ L_{k+2} & L_{k+1} \end{bmatrix}, \end{aligned}$$

as desired.

Secondly, let us consider even n. For $n = 2$ we can write

$$Q_L^2 = 5 \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}.$$

So, (2.1) is true for $n = 2$. Now, we suppose it is true for $n = k$, that is

$$Q_L^k = 5^{\frac{k}{2}} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}.$$

Using properties of the Fibonacci numbers and the induction hypothesis, we can write

$$\begin{aligned} Q_L^{k+2} &= Q_L^k Q_L^2 \\ &= 5^{\frac{k+2}{2}} \begin{bmatrix} F_{k+3} & F_{k+2} \\ F_{k+2} & F_{k+1} \end{bmatrix}, \end{aligned}$$

as desired. Hence, (2.1) holds for all n. □

2.2. Theorem. *Let Q_L^n be as in* (1.2). *Then the following equalities are valid for all integers $n \geq 1$:*

i) $\det(Q_L^n) = 5^n$,
ii) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$,
iii) $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n-1}$.

Proof. To establish (i) we use induction on n. Clearly $\det(Q_L) = 5^1$. If we make the induction hypothesis $\det(Q_L^k) = 5^k$, then from the multiplicative property of the determinant we have

$$\begin{aligned} \det(Q_L^{k+1}) &= \det(Q_L^k)\det(Q_L^1) \\ &= 5^{k+1}, \end{aligned}$$

which shows (i) for all $n \geq 1$.

The identities (ii) and (iii) easily seen by using (2.1) and (i) for even and odd values of n, respectively. □

2.3. Theorem. *Let n be any integer. The well-known Binet formulas for the Fibonacci and Lucas numbers are*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ (golden ratio) and $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. Let the matrix Q_L be as in (1.2). We can write the characteristic equation of Q_L as

$$\lambda^2 - 5\lambda + 5 = 0.$$

If we calculate the eigenvalues and eigenvectors of the matrix Q_L we obtain

$$\lambda_1 = \sqrt{5}\alpha,\ \lambda_2 = \sqrt{5}\beta$$

and

$$v_1 = (1,\ -\beta),\ v_2 = (1,\ -\alpha),$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, respectively. Then we can diagonalize of the matrix Q_L by

$$V = U^{-1} Q_L U,$$

where

$$U = (v_1^T, v_2^T) = \begin{bmatrix} 1 & 1 \\ -\beta & -\alpha \end{bmatrix}$$

and

$$V = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \sqrt{5}\alpha & 0 \\ 0 & \sqrt{5}\beta \end{bmatrix}.$$

From properties of similar matrices, we can write

$$V^n = U^{-1} Q_L^n U,$$

where n is any integer. Furthermore, we can obtain

$$Q_L^n = U V^n U^{-1}.$$

By (2.1) and taking the nth power of the diagonal matrix, we get

$$Q_L^n = 5^{\frac{n-1}{2}} \begin{pmatrix} \alpha^{n+1} + (-\beta)^{n+1} & \alpha^n - (-\beta)^n \\ \alpha^n - (-\beta)^n & \alpha^{n-1} + (-\beta)^{n-1} \end{pmatrix}.$$

Thus, the proof is completed. □

2.4. Theorem. *For all integers m and n, the following equalities are valid:*

i) $5F_{m+n} = L_n L_{m+1} + L_{n-1} L_m$,
ii) $F_{m+n} = F_n F_{m+1} + F_{n-1} F_m$,
iii) $L_{m+n} = F_{n+1} L_m + F_n L_{m-1}$,
iv) $5F_{m-n} = (-1)^{n-1}(L_m L_{n+1} - L_{m+1} L_n)$,
v) $F_{m-n} = (-1)^n (F_m F_{n+1} - F_{m+1} F_n)$,
vi) $L_{m-n} = (-1)^{n-1}(F_m L_{n+1} - F_{m+1} L_n)$.

Proof. Q_L^{m+n} can be written, using (2.1), as

$$Q_L^{m+n} = \begin{cases} 5^{\frac{m+n}{2}} \begin{bmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{bmatrix} & \text{for } m+n \text{ even}, \\ 5^{\frac{m+n-1}{2}} \begin{bmatrix} L_{m+n+1} & L_{m+n} \\ L_{m+n} & L_{m+n-1} \end{bmatrix} & \text{for } m+n \text{ odd}. \end{cases} \tag{2.2}$$

For the case of odd m and n,

$$Q_L^m \cdot Q_L^n = 5^{\frac{m+n}{2}-1} \begin{bmatrix} L_{n+1} L_{m+1} + L_n L_m & L_{n+1} L_m + L_n L_{m-1} \\ L_n L_{m+1} + L_{n-1} L_m & L_n L_m + L_{n-1} L_{m-1} \end{bmatrix}. \tag{2.3}$$

Comparing the entries $(1,2)$ of the matrices (2.2) and (2.3), we obtain

$$5F_{m+n} = L_{n+1} L_m + L_n L_{m-1},$$

while comparing the entries (2,1) gives

$$5F_{m+n} = L_n L_{m+1} + L_{n-1} L_m.$$

For the case of even m and n,

$$Q_L^m \cdot Q_L^n = 5^{\frac{m+n}{2}} \begin{bmatrix} F_{n+1} F_{m+1} + F_n F_m & F_{n+1} F_m + F_n F_{m-1} \\ F_n F_{m+1} + F_{n-1} F_m & F_n F_m + F_{n-1} F_{m-1} \end{bmatrix}. \tag{2.4}$$

Comparing the entries (1,2) and (2,1) for the matrices (2.2) and (2.4), we find that

$$F_{m+n} = F_{n+1}F_m + F_nF_{m-1},$$
$$F_{m+n} = F_nF_{m+1} + F_{n-1}F_m.$$

For cases of odd m and even n, or odd n and even m,

$$Q_L^m \cdot Q_L^n = 5^{\frac{m+n-1}{2}} \begin{bmatrix} F_{n+1}L_{m+1} + F_nL_m & F_nL_{m+1} + F_{n-1}L_m \\ F_{n+1}L_m + F_nL_{m-1} & F_nL_m + F_{n-1}L_{m-1} \end{bmatrix}. \tag{2.5}$$

Comparing the entries (1,2) and (2,1) for the matrices (2.2) and (2.5), we obtain the equations

$$L_{m+n} = F_nL_{m+1} + F_{n-1}L_m,$$
$$L_{m+n} = F_{n+1}L_m + F_nL_{m-1}.$$

The inverse of the matrix Q_L^n in (2.1) is given by

$$Q_L^{-n} = \begin{cases} \frac{(-1)^n}{5^{\frac{n}{2}}} \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}, & \text{for even } n, \\ \frac{(-1)^{n-1}}{5^{\frac{n+1}{2}}} \begin{bmatrix} L_{n-1} & -L_n \\ -L_n & L_{n+1} \end{bmatrix}, & \text{for odd } n. \end{cases}$$

Similarly, by computing the equality $Q_L^{m-n} = Q_L^m \cdot Q_L^{-n}$ the desired results are obtained. Indeed, for the case of odd m and n,

$$5F_{m-n} = (-1)^{n-1}(L_mL_{n+1} - L_{m+1}L_n).$$

For the case of even m and n,

$$F_{m-n} = (-1)^n(F_mF_{n+1} - F_{m+1}F_n).$$

Finally, for the cases of odd n and even m, odd m and even n,

$$L_{m-n} = (-1)^{n-1}(F_mL_{n+1} - F_{m+1}L_n).$$

□

References

[1] Gould, H. W. *A history of the Fibonacci Q-matrix and a higher-dimensional problem*, Fibonacci Quart. **19**, 7–250, 1981.

[2] Kalman, D. and Mena, R. *The Fibonacci numbers – exposed*, Math. Mag. **76**, 81–167, 2003.

[3] Koshy, T. *Fibonacci and Lucas Numbers with Applications* (Wiley - Interscience, New York, 2001).

[4] Silvester, J. R. *Fibonacci properties by matrix methods*, Mathematical Gazette **63**, 188–191, 1979.

[5] Vajda, S. *Fibonacci and Lucas numbers, and the Golden Section. Theory and applications* (Ellis Horwood Limited, Chichester, 1989).