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CR-SUBMANIFOLDS OF A LORENTZIAN PARA-SASAKIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract

We study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric metric connection. Moreover, we obtain integrability conditions of the distributions on CR-submanifolds.

Keywords: CR-submanifold, Lorentzian para-Sasakian manifold, Semi-symmetric metric connection, Integrability conditions of the distributions.

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1. Introduction

The notion of a CR-submanifold of a Kaehler manifold was introduced by A. Bejancu in [1]. Later, CR-submanifolds of Sasakian manifolds were studied by M. Kobayashi in [5]. K. Matsumoto introduced the idea of a Lorentzian para-Sasakian structure and studied several of its properties in [6]. U. C. De and A. K. Sengupta studied CR-Submanifolds of a Lorentzian para-Sasakian manifold in [3]. In this paper, we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\overline{M}$. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

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In [4, 11], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a *semi-symmetric connection* if its torsion tensor T is of the form

$$T(X,Y)=\eta(Y)X-\eta(X)Y,$$

where η is a 1-form. In [12], K. Yano studied some properties of semi-symmetric metric connections.

In this paper we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a semi-symmetric metric connection. We consider integrabilities of horizontal and vertical distributions of CR-submanifolds with a semi-symmetric metric connection. We also consider parallel horizontal distributions of CR-submanifolds.

The paper is organized as follows: In section 2, we give a brief introduction to Lorentzian para-Sasakian manifolds. In Section 3, we study CR-submanifolds of Lorentzian para-Sasakian manifolds. We find necessary conditions for the induced connection on a CR-submanifold of a Lorentzian para-Sasakian manifold with semi-symmetric metric connection to be also a semi-symmetric metric connection. We also discuss the integrability conditions of parallel horizontal distributions of CR-submanifolds.

2. LP-Sasakian manifolds

An n-dimensional differentiable manifold $\overline{M}$ admits an *almost paracontact Riemannian structure* (ϕ,η,ξ,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\overline{M}$ if

$$\phi^2X=X-\eta(X)\xi,\ \eta(\xi)=1,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for all vector fields X and Y on $\overline{M}$, see [9, 10].

On the other hand, $\overline{M}$ admits a *Lorentzian almost paracontact structure* (ϕ,η,ξ,g), where ϕ is a $(1,1)$ tensor field, ξ a vector field, η a 1-form and g a Lorentzian metric on $\overline{M}$, if ξ is a timelike unit vector field such that

$$\phi^2X=X+\eta(X)\xi,\ \eta(\xi)=-1, \tag{2.1}$$
$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.2}$$
$$g(X,\xi)=\eta(X), \tag{2.3}$$
$$g(\phi X,Y)=g(X,\phi Y), \tag{2.4}$$

for all vector fields X and Y on $\overline{M}$, see [9, 10].

For both structures mentioned above, it follows that

$$\phi\xi=0,\ \eta(\phi X)=0,\ \operatorname{rank}(\phi)=n-1.$$
$$g(X,\xi)=\eta(X),\ \nabla_X\xi=\varphi X, \tag{2.5}$$
$$g(\varphi X,Y)=g(X,\varphi Y) \tag{2.6}$$

and

$$\overline{\overline{\nabla}}_X\xi=\phi X. \tag{2.7}$$

A Lorentzian para-contact manifold $\overline{M}$ is called a *Lorentzian para-Sasakian* (briefly, LP-Sasakian) manifold if

$$(\overline{\overline{\nabla}}_X\phi)(Y)=g(X,Y)+\eta(Y)X+2\eta(X)\eta(Y)\xi, \tag{2.8}$$

for all vector fields X, Y on $\overline{M}$, where $\overline{\overline{\nabla}}$ is the Riemannian connection with respect to g, see [7, 8].

3. CR-submanifolds of LP-Sasakian manifolds

3.1. Definition. [3] An m-dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\overline{M}$ is called a *CR-submanifold* if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that

(i) D_x is invariant under ϕ, i.e. $\phi D_x \subset D_x$ for each $x \in M$.

(ii) The orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subset T_x(M)$ of the distribution D on M is totally real, i.e. $\phi D_x^{\perp}(M) \subset T_x^{\perp}(M)$, where $T_x(M)$ and $T_x^{\perp}(M)$ are the tangent space and normal space of M at $x \in M$, respectively.

3.2. Definition. [3] The distribution D (resp., $D^{\perp}$) is called the *horizontal* (resp., *vertical*) *distribution*. The pair $(D, D^{\perp})$ is called *ξ-horizontal* (resp., *ξ-vertical*) if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^{\perp}$) for each $x \in M$. The CR-submanifold is also called *ξ-horizontal* (resp., *ξ-vertical*) if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^{\perp}$) for each $x \in M$. The horizontal distribution D (resp., $D^{\perp}$) is said to be *parallel with respect to the connection* ∇ *on* M if $\nabla_X Y \in D$ (resp., $\nabla_X Y \in D^{\perp}$) for all vector fields $X, Y \in D$ (resp., $X, Y \in D^{\perp}$).

Let us denote the orthogonal complement of $\phi D^{\perp}$ in $T^{\perp}(M)$ by μ. Then we have,

$$TM = D \oplus D^{\perp}, T^{\perp}(M) = \phi D^{\perp} \oplus \mu.$$

It is obvious that $\phi\mu = \mu$ [3].

Any vector field X tangent to M can be decomposed as

$$X = PX + QX, \tag{3.1}$$

where PX and QX belong to the distribution D and $D^{\perp}$, respectively.

For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{3.2}$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN.

Now, we define a connection $\overline{\nabla}$ as

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X - g(X,Y)\xi \tag{3.3}$$

such that $\overline{\nabla}_X g = 0$ for any $X, Y \in TM$, where $\overline{\overline{\nabla}}_X$ is the Riemannian connection with respect to g on M. The connection $\overline{\nabla}$ is semisymmetric because

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y.$$

Substituting (3.3) in (2.8), we have

$$(\overline{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi - \eta(Y)\phi X - g(X,\phi Y)\xi, \tag{3.4}$$

$$\overline{\nabla}_X \xi = \phi X - X - \eta(X)\xi. \tag{3.5}$$

We denote by g the metric tensor of $\overline{M}$, as well as that induced on M. Let $\overline{\nabla}$ be the semi-symmetric metric connection on $\overline{M}$ and ∇ the induced connection on M with respect to the unit normal N. Then we have the following theorem:

3.3. Theorem.

i) *If M is ξ-horizontal, $X, Y \in D$ and D is parallel with respect to ∇, then the connection ∇ induced on a CR-submanifold of an LP-Sasakian manifold with a semi-symmetric metric connection is also a semi-symmetric metric connection.*

ii) *If M is ξ-vertical, $X, Y \in D^{\perp}$ and $D^{\perp}$ is parallel with respect to ∇ then the connection ∇ induced on a CR-submanifold of an LP-Sasakian manifold with a semi-symmetric metric connection is also a semi-symmetric metric connection.*

iii) *The Gauss formula with respect to the semi-symmetric metric connection is of the form* $\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$.

Proof. Let ∇ be the induced connection with respect to the unit normal N on a CR-submanifold of an LP-Sasakian manifold from semi-symmetric metric connection $\overline{\nabla}$. Then

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{3.6}$$

where m is a tensor field of type $(0,2)$ on the CR-submanifold M. If $\dot{\nabla}$ be the induced connection on the CR-submanifold from the Riemannian connection $\overline{\overline{\nabla}}$, then

$$\overline{\overline{\nabla}}_X Y = \dot{\nabla}_X Y + h(X,Y), \tag{3.7}$$

where h is a second fundamental form. By the definition of the semi-symmetric metric connection, we have

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + \eta(Y)X - g(X,Y)\xi.$$

Now, using the above equations, we have

$$\nabla_X Y + m(X,Y) = \dot{\nabla}_X Y + h(X,Y) + \eta(Y)X - g(X,Y)\xi.$$

Using (3.1), the above equation can be written as

$$\begin{aligned} P\nabla_X Y + Q\nabla_X Y + m(X,Y) = {} & P\dot{\nabla}_X Y + Q\cdot\nabla_X Y + h(X,Y) \\ & + \eta(Y)PX + \eta(Y)QX - g(X,Y)P\xi - g(X,Y)Q\xi. \end{aligned} \tag{3.8}$$

From (3.8), comparing the tangential and normal components on both sides, we get

$$h(X,Y) = m(X,Y), \tag{3.9}$$

$$P\nabla_X Y - \eta(Y)PX + g(X,Y)P\xi = P\dot{\nabla}_X Y, \tag{3.10}$$

$$Q\nabla_X Y - \eta(Y)QX + g(X,Y)Q\xi = Q\dot{\nabla}_X Y. \tag{3.11}$$

Using (3.9), the Gauss formula for a CR-submanifold of an LP-Sasakian manifold with semi-symmetric metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{3.12}$$

This proves (iii). In view of (3.10), if M is ξ-horizontal, $X, Y \in D$ and D is parallel with respect to ∇, then the connection induced on a CR-submanifold of an LP-Sasakian manifold with semi-symmetric metric connection is also a semi-symmetric metric connection.

Similarly, using (3.11), if M is ξ-vertical, $X, Y \in D^\perp$ and $D^\perp$ is parallel with respect to ∇ then the connection induced on a CR-submanifold of an LP-Sasakian manifold with semi-symmetric metric connection is also a semi-symmetric metric connection. $\square$

On the other hand, using (3.3), the Weingarten formula is given by

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N + \eta(N)X, \tag{3.13}$$

for $X, Y \in TM$, $N \in T^\perp M$, h (resp., A_N) is the second fundamental form (resp., tensor) of M in $\overline{M}$ and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, by [2],

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.14}$$

3.4. Lemma. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then*

$$P\nabla_X\phi PY - PA_{\phi QY}X = g(X,Y)P\xi - g(X,\phi Y)P\xi + \eta(Y)PX - \eta(Y)\phi PX + 2\eta(X)\eta(Y)P\xi + \phi P\nabla_X Y, \tag{3.15}$$

$$Q\nabla_X\phi PY - QA_{\phi QY}X = g(X,Y)Q\xi - g(X,\phi Y)Q\xi + \eta(Y)QX, + 2\eta(X)\eta(Y)Q\xi + Bh(X,Y) \tag{3.16}$$

$$h(X,\phi PY) + \nabla^{\perp}_X\phi QY = \phi Q\nabla_X Y + Ch(X,Y) - \eta(Y)\phi QX. \tag{3.17}$$

for $X, Y \in TM$.

Proof. By direct covariant differentiation, we have

$$\overline{\nabla}_X\phi Y = (\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_X)Y.$$

By virtue of (3.4), (3.12), (3.13), and (3.1), we get

$$\begin{aligned}\nabla_X\phi PY + h(X,\phi PY) + (-A_{\phi QY}X + \nabla^{\perp}_X\phi QY) &= g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi - \eta(Y)\phi X - g(X,\phi Y)\xi \\ &\quad + \phi\nabla_X Y + \phi h(X,Y).\end{aligned}$$

Using (3.1) and (3.2), we have

$$\begin{aligned}&P\nabla_X\phi PY + Q\nabla_X\phi PY + h(X,\phi PY) - PA_{\phi QY}X - QA_{\phi QY}X + \nabla^{\perp}_X\phi QY \\ &= g(X,Y)P\xi + g(X,Y)Q\xi + \eta(Y)PX + \eta(Y)QX + 2\eta(X)\eta(Y)P\xi \\ &\quad + 2\eta(X)\eta(Y)Q\xi - \eta(Y)\phi PX - \eta(Y)\phi QX - g(X,\phi Y)P\xi \\ &\quad - g(X,\phi Y)Q\xi + \phi P\nabla_X Y + \phi Q\nabla_X Y \\ &\quad + Bh(X,Y) + Ch(X,Y).\end{aligned}$$

Equations (3.15)-(3.17) follow by comparing the horizontal, vertical and normal components. □

3.5. Lemma. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then*

$$\phi P[Y,Z] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y$$

for $Y, Z \in D^{\perp}$.

Proof. For $Y, Z \in D^{\perp}$ we have

$$\overline{\nabla}_Y\phi Z = (\overline{\nabla}_Y\phi)Z + \phi(\overline{\nabla}_Y Z).$$

Using (3.4), (3.12) and (3.13), we have

$$\begin{aligned}-A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z &= g(Y,Z)\xi + \eta(Z)Y - \eta(Z)\phi Y + 2\eta(Y)\eta(Z)\xi \\ &\quad - g(Y,\phi Z)\xi + \phi(\nabla_Y Z + h(Y,Z)).\end{aligned}$$

By using (3.17), we get

$$\phi P\nabla_Y Z = -A_{\phi Z}Y - g(Y,Z)\xi - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi + g(Y,\phi Z)\xi - Bh(Y,Z).$$

Interchanging Y and Z, we have

$$\phi P\nabla_Z Y = -A_{\phi Y}Z - g(Z,Y)\xi - \eta(Y)Z - 2\eta(Y)\eta(Z)\xi + g(Z,\phi Y)\xi - Bh(Z,Y).$$

By subtracting, we obtain

$$\phi P[Y,Z] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y$$

for $Y, Z \in D^{\perp}$. □

Hence we can state the following theorem:

3.6. Theorem. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Z}Y - A_{\phi Y}Z = \eta(Y)Z - \eta(Z)Y$$

for $Y, Z \in D^{\perp}$. □

3.7. Proposition. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then*

$$\phi Ch(X,Y) = Ch(\phi X, Y) = Ch(X, \phi Y) \tag{3.18}$$

for $X, Y \in D$.

Proof. For $X, Y \in D$, from (3.16) we have

$$Q\nabla_X \phi Y = g(X,Y)Q\xi - g(X,\phi Y)Q\xi + Bh(X,Y) \tag{3.19}$$

and

$$Q\nabla_{\phi X}\phi Y = g(\phi X, Y)Q\xi - g(X,Y)Q\xi + Bh(\phi X, Y). \tag{3.20}$$

Interchanging X and Y in (3.19) we get

$$Q\nabla_Y \phi X = g(X,Y)Q\xi - g(X,\phi Y)Q\xi + Bh(Y,X). \tag{3.21}$$

Replacing X by ϕX and using (2.1), we find

$$Q\nabla_Y X = g(\phi X, Y)Q\xi - g(X,Y)Q\xi + Bh(\phi X, Y). \tag{3.22}$$

Subtracting (3.20) from (3.22) we have

$$Q\left(\nabla_{\phi X}\phi Y - \nabla_Y X\right) = 0,$$

which gives us

$$\nabla_{\phi X}\phi Y - \nabla_Y X \in D. \tag{3.23}$$

Now from (3.17), we find

$$h(X, \phi Y) = \phi Q \nabla_X Y + Ch(X,Y). \tag{3.24}$$

Replacing X by ϕX and Y by ϕY in (3.24), we get

$$h(\phi X, Y) = \phi Q(\nabla_{\phi X}\phi Y) + Ch(\phi X, \phi Y). \tag{3.25}$$

Also interchanging X and Y in (3.24), we have

$$h(\phi X, Y) = \phi Q \nabla_Y X + Ch(X,Y). \tag{3.26}$$

Subtracting (3.25) from (3.26), and using (3.23), we get

$$Ch(\phi X, \phi Y) = Ch(X,Y),$$

or equivalently

$$Ch(\phi^2 X, \phi Y) = Ch(\phi X, Y).$$

Hence, consequently,

$$Ch(X, \phi Y) = Ch(\phi X, Y).$$

From (3.19), we have

$$Q\nabla_X \phi^2 Y = g(X, \phi Y)Q\xi - g(X, \phi^2 Y)Q\xi + Bh(X, \phi Y).$$

Hence,

$$Q\nabla_X Y = g(X,\phi Y)Q\xi - g(X,Y)Q\xi + Bh(X,\phi Y). \tag{3.27}$$

Using (3.27) in (3.24), we get

$$h(X,\phi Y)=\phi Bh(X,\phi Y)+Ch(X,Y).$$

Applying ϕ on both sides we get

$$\phi h(X,\phi Y)=Bh(X,\phi Y)+\phi Ch(X,Y).$$

Using (3.2) in the above equation, we obtain

$$\phi Ch(X,Y)=Ch(X,\phi Y),$$

which completes the proof. □

3.8. Theorem. *Let M be a ξ-horizontal CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then the distribution D is integrable if and only if*

$$h(X,\phi Y)=h(Y,\phi X)$$

for $Y,Z\in D$.

Proof. The proof is similar to the proof of Theorem 3.2 in [3]. □

3.9. Proposition. *Let M be a ξ-horizontal CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then the distribution D is parallel if and only if*

$$h(X,\phi Y)=h(Y,\phi X)=\phi h(X,Y) \tag{3.28}$$

for all $X,Y\in D$.

Proof. The proof follows by similar computations to those used for Proposition 4.1 in [3]. □

3.10. Proposition. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with symmetric metric connection. Then the distribution $D^{\perp}$ is parallel with respect to the connection ∇ on M if and only if $A_N X\in D^{\perp}$ for each $X\in D^{\perp}$ and $N\in TM^{\perp}$.*

Proof. Let $X,Y\in D^{\perp}$. Then, using (3.4), (3.12) and (3.13), we have

$$\begin{aligned}-A_{\phi Y}X+\nabla^{\perp}_X\phi Y&=g(X,Y)\xi+\eta(Y)X-\eta(Y)\phi X+2\eta(X)\eta(Y)\xi\\&\quad-g(X,\phi Y)\xi+\phi(\nabla_X Y+h(X,Y)).\end{aligned} \tag{3.29}$$

Taking the inner product of the last equation with $Z\in D$, we find

$$g(A_{\phi Y}X,Z)=g(\nabla_X Y,\phi Z).$$

Therefore, $\nabla_X Y\in D^{\perp}$ if and only if $A_{\phi Y}X\in D^{\perp}$ for all $X\in D^{\perp}$. Our assertion follows from this. □

3.11. Definition. [3] A CR-submanifold M of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection is said to be *mixed totally geodesic* if $h(X,Y)=0$ for $X\in D$ and $Y\in D^{\perp}$.

It follows immediately that a CR-submanifold M of an LP-Sasakian manifold $\overline{M}$ is mixed totally geodesic if and only if $A_N X\in D$ for each $X\in D$ and $N\in T^{\perp}M$.

Let $X\in D$ and $Y\in\phi D^{\perp}$. For a mixed totally geodesic ξ-vertical CR-submanifold M of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection, from (3.4) we have

$$(\overline{\nabla}_X\phi)N=0.$$

Since $\overline{\nabla}_X \phi N = (\overline{\nabla}_X \phi)N + \phi(\overline{\nabla}_X N)$, we have $\overline{\nabla}_X \phi N = \phi(\overline{\nabla}_X N)$. Hence in view of (3.12), we get

$$\overline{\nabla}_X \phi N = -A_{\phi N} X + \nabla^{\perp}_X \phi N = -\phi A_N X + \phi \nabla^{\perp}_X N.$$

As $A_N X \in D$, $\phi A_N X \in D$. So $\nabla^{\perp}_X N = 0$ if and only if $\overline{\nabla}_X \phi N \in D$. Thus we have the following proposition.

3.12. Proposition. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of an LP-Sasakian manifold $\overline{M}$ with semi-symmetric metric connection. Then the normal section $N \in \phi D^{\perp}$ is D parallel if and only if $\overline{\nabla}_X \phi N \in D$ for $X \in D$.* □

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