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STATISTICAL FUZZY APPROXIMATION TO FUZZY DIFFERENTIABLE FUNCTIONS BY FUZZY LINEAR OPERATORS

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Abstract

In this paper, we obtain fuzzy approximations to fuzzy differentiable functions by means of fuzzy linear operators whose positivity condition and classical limits fail. In order to get more powerful results than the classical approach we investigate the effects of matrix summability methods on the fuzzy approximation. So, we mainly use the notion of A-statistical convergence from summability theory instead of the usual convergence.

Keywords: Korovkin theorem, Statistical convergence, Fuzzy approximation, Fuzzy continuity, Fuzzy differentiable function.

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1. Introduction

The classical Korovkin theory is mainly based on two conditions: *positivity of linear operators* and *existence of their classical limits* (see [1, 28]). So far, the former has been weakened by considering monotonicity and convexity of the functions being approximated (see, for instance, [2, 8, 9, 26]). Furthermore, by using the notion of *statistical convergence*, some Korovkin–type approximation theorems have been obtained when the classical limit of a sequence of positive linear operators fails (see [6, 10, 11, 12, 14, 15, 16, 17, 25]). Their fuzzy analogs have also been introduced in [5, 13, 21]. In this study, we obtain various fuzzy approximation results without these two conditions as mentioned above. We use the notions of statistical convergence of sequences of fuzzy numbers and fuzzy differentiability of fuzzy real valued functions to get our theorems. When proving our results, we use not only classical techniques from

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approximation theory but also new methods from summability theory and fuzzy logic theory.

We first recall some basic concepts used in the present paper.

The (asymptotic) density of a subset K of $\mathbb{N}$, the set of all natural numbers, is defined by

$$\delta(K) := \lim_j \frac{\#\{n \le j : n \in K\}}{j},$$

provided the limit exists, where the symbol $\#\{B\}$ denotes the cardinality of the set B. Then, we say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *statistically convergent to a number* L (see [18]) if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\}$ has density zero, i.e.,

$$\delta\left(\{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\}\right) = \lim_j \frac{\#\{n \le j : |x_n - L| \ge \varepsilon\}}{j} = 0.$$

Now let $A = (a_{jn})$ be an infinite non-negative regular summability matrix. Recall that A is said to be *regular* if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$, where $Ax := ((Ax)_j)$ denotes the A-transform of x given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$, provided the series converges for each j (see, for instance, [7]). Then, the A-density of a subset K is defined by

$$\delta_A(K) := \lim_j \sum_{n \in K} a_{jn}$$

provided the limit exists. Observe that if we take $A = C_1 = (c_{jn})$, the Cesáro matrix of order one, defined by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \le n \le j, \\ 0, & \text{otherwise,} \end{cases}$$

then C_1-density coincides with (asymptotic) density. With the help of A-density, Freedman and Sember [19] introduced the notion of A-statistical convergence, which is a more general method of statistical convergence. Recall that the sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *A-statistically convergent to* L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\} = 0$; or equivalently

$$\lim_j \sum_{n:\, |x_n - L| \ge \varepsilon} a_{jn} = 0.$$

This limit is denoted by $\mathrm{st}_A\text{-}\lim_n x_n = L$. It is not hard to see that if we take $A = C_1$, then C_1-statistical convergence coincides with the statistical convergence mentioned above. If A is replaced by the identity matrix, then we get the ordinary convergence of number sequences. We also note that if $A = (a_{jn})$ is any non-negative regular summability matrix for which $\lim_j \max_n\{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence (see [27]). Actually, every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Some other results regarding statistical and A-statistical convergence may be found in the papers [20, 30].

As usual, a fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $\mathrm{supp}(\mu)$ is compact, where $\mathrm{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. We denote the set of all fuzzy numbers by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \text{ and } [\mu]^r := \{x \in \mathbb{R} : \mu(x) \ge r\},\ (0 < r \le 1).$$

Then, it is well-known [22] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum

$u \oplus v$ and the product $\lambda \odot v$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \ (0 \le r \le 1).$$

Now denote the interval $[u]^r$ by $[u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \le u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \iff u_-^{(r)} \le v_-^{(r)} \text{ and } u_+^{(r)} \le v_+^{(r)} \text{ for all } 0 \le r \le 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}.$$

In this case, it is known [32] that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space on $\mathbb{R}_{\mathcal{F}}$ with the properties

$$D(u \oplus w, v \oplus w) = D(u,v) \text{ for all } u, v, w \in \mathbb{R}_{\mathcal{F}},$$
$$D(\lambda \odot u, \lambda \odot v) = |\lambda| \, D(u,v) \text{ for all } u, v \in \mathbb{R}_{\mathcal{F}} \text{ and } \lambda \in \mathbb{R},$$
$$D(u \oplus v, w \oplus z) \le D(u,w) + D(v,z) \text{ for all } u, v, w, z \in \mathbb{R}_{\mathcal{F}}.$$

Let $f, g : [a,b] \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy valued functions. Then, the distance between f and g on $[a,b]$ is given by

$$D^*(f,g) = \sup_{x \in [a,b]} \sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)}(x) - g_-^{(r)}(x) \right|, \left| f_+^{(r)}(x) - g_+^{(r)}(x) \right| \right\}.$$

Now let $(\mu_n)_{n \in \mathbb{N}}$ be a fuzzy number valued sequence. Then, Nuray and Savaş [31] introduced the fuzzy analog of statistical convergence by using the fuzzy metric D instead of the classical absolute value in the above definition. So, by a similar idea, one can obtain the following definition of A-statistical convergence of fuzzy valued sequences. We say that a sequence $(\mu_n)_{n \in \mathbb{N}}$ of fuzzy numbers is *A-statistically convergent to* $\mu \in \mathbb{R}_{\mathcal{F}}$, which is denoted by $\text{st}_A\text{-}\lim_n D(\mu_n, \mu) = 0$, if for every $\varepsilon > 0$, $\delta_A \left(\{ n \in \mathbb{N} : D(\mu_n, \mu) \ge \varepsilon \} \right) = 0$, i.e.,

$$\lim_j \sum_{n : D(\mu_n, \mu) \ge \varepsilon} a_{jn} = 0$$

holds. Of course, the case of $A = C_1$ immediately reduces to the statistical convergence of fuzzy valued sequences. Also, replacing A with the identity matrix, we get the classical fuzzy convergence introduced by Matloka [29].

2. Statistical Fuzzy Approximation

A fuzzy valued function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to be *fuzzy continuous* at $x_0 \in [a,b]$ provided that, whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to 0$ as $n \to \infty$. Also, we say that f is *fuzzy continuous on* $[a,b]$ if it is fuzzy continuous at every point $x \in [a,b]$. The set of all fuzzy continuous functions on $[a,b]$ is denoted by $C_{\mathcal{F}}[a,b]$ (see, for instance, [3, 4]). Notice that $C_{\mathcal{F}}[a,b]$ is only a cone, not a vector space.

Following [24, 33], a fuzzy valued function $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is said to *satisfy the condition* (H) *on* $[a,b]$ if, for any $x, y \in [a,b]$ satisfying $x \le y$, there exists $u \in \mathbb{R}_{\mathcal{F}}$ such that

$$f(y) = f(x) + u.$$

In this case, we call u the *H-difference* (or, *Henstock-difference*) of $f(y)$ and $f(x)$, and denote it by $f(y) - f(x)$. For brevity, throughout this paper, when we use the "$-$" operation of fuzzy numbers, we always assume that the condition (H) is satisfied.

Assume that $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ is a fuzzy valued function. Then f is called *fuzzy differentiable* at $x\in(a,b)$ if there exists a $f'(x)\in\mathbb{R}_{\mathcal{F}}$ such that the following limits

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h},\quad \lim_{h\to 0^+}\frac{f(x)-f(x-h)}{h}$$

exist and are equal to $f'(x)$. Also, if $x=a$ or $x=b$, then we use the following

$$f'(a):=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \text{ and } f'(b):=\lim_{h\to 0^+}\frac{f(b)-f(b-h)}{h}$$

provided that the above limits exist. Observe that the limits are taken in the metric space $(\mathbb{R}_{\mathcal{F}},D)$. If f is (fuzzy) differentiable at every point $x\in[a,b]$, then we say that f is (fuzzy) differentiable on $[a,b]$ with derivative f' (see [32]). It follows from [23] that a function $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ is fuzzy differentiable at $x\in[a,b]$, and $f'(x)$ is the fuzzy derivative, if and only if for every $\varepsilon>0$ there exists a $\delta>0$ such that, for any interval $[x_1,x_2]\subset(x-\delta,x+\delta)$ we have

$$D\left(\frac{f(x_2)-f(x_1)}{x_2-x_1},f'(x)\right)<\varepsilon.$$

Similarly, we can define higher order fuzzy derivatives. Also, by $C^m_{\mathcal{F}}[a,b]$, $(m\in\mathbb{N})$, we mean the set of all fuzzy valued functions from $[a,b]$ into $\mathbb{R}_{\mathcal{F}}$ that are m-times continuously differentiable in the fuzzy sense. However, in this paper, we only discuss the cases $m=0,1,2$.

Using these definitions, Kaleva [24] proved the following result.

2.1. Lemma. [24] *Let $f:[a,b]\subset R\to R_{\mathcal{F}}$ be fuzzy differentiable, and $x\in[a,b]$, $0\le r\le 1$. Then, clearly*

$$[f(x)]^r=[(f(x))^{(r)}_-,(f(x))^{(r)}_+]\subset\mathbb{R}.$$

Then $(f(x))^{(r)}_\pm$ is differentiable and

$$[f'(x)]^r=\left[((f(x))^{(r)}_-)',((f(x))^{(r)}_+)'\right],$$

i.e.,

$$(f')^{(r)}_\pm=\left(f^{(r)}_\pm\right)' \text{ for any } r\in[0,1].\quad\square$$

Also, for higher order fuzzy derivatives, Anastassiou [4] gave the similar result:

2.2. Lemma. [4] *Let $k\in\mathbb{N}$ and $f\in C^k_{\mathcal{F}}[a,b]$. Then, we have $f^{(r)}_\pm\in C^k[a,b]$ (for any $r\in[0,1]$) and*

$$[f^{(i)}(x)]^r=\left[\left((f(x))^{(r)}_-\right)^{(i)},\left((f(x))^{(r)}_+\right)^{(i)}\right]$$

for $i=0,1,\ldots,k$, and, in particular, we get

$$\left(f^{(i)}\right)^{(r)}_\pm=\left(f^{(r)}_\pm\right)^{(i)} \text{ for any } r\in[0,1] \text{ and } i=0,1,\ldots,k.\quad\square$$

In this paper, for simplicity, we use the unit interval $[0,1]$ instead of $[a,b]$. Now let $L:C_{\mathcal{F}}[0,1]\to C_{\mathcal{F}}[0,1]$ be an operator. Then L is said to be *fuzzy linear* if, for every $\lambda_1,\lambda_2\ge 0$, $f_1,f_2\in C_{\mathcal{F}}[0,1]$, and $x\in[0,1]$,

$$L(\lambda_1\odot f_1\oplus\lambda_2\odot f_2;x)=\lambda_1\odot L(f_1;x)\oplus\lambda_2\odot L(f_2;x)$$

holds. Let k be a non-negative integer. As usual, by $C^k[0,1]$ we denote the space of all k-times continuously differentiable functions (in the usual sense) on $[0,1]$, endowed with the sup-norm $\|\cdot\|$. Then, throughout the paper, we consider the following function spaces:

$$\mathcal{A} := \left\{ f \in C^2[0,1] : f \geq 0 \right\},$$
$$\mathcal{B} := \left\{ f \in C^2[0,1] : f'' \geq 0 \right\},$$
$$\mathcal{C} := \left\{ f \in C^2[0,1] : f'' \leq 0 \right\},$$
$$\mathcal{D} := \left\{ f \in C^1[0,1] : f \geq 0 \right\},$$
$$\mathcal{E} := \left\{ f \in C^1[0,1] : f' \geq 0 \right\},$$
$$\mathcal{F} := \{ f \in C[0,1] : f \geq 0 \}.$$

We also consider the test functions

$$e_i(y) = y^i \ (i = 0, 1, 2, \ldots) \text{ for every } y \in [0,1].$$

Then we have following results.

2.3. Theorem. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and $\{L_n\}_{n\in\mathbb{N}}$ a sequence of fuzzy linear operators from $C^2_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^2[0,1]$ onto itself for which the following conditions hold:*

$$(2.1) \qquad L_n(f;x)^{(r)}_{\pm} = \tilde{L}_n\left(f^{(r)}_{\pm};x\right) \ (\textit{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N})$$

and

$$(2.2) \qquad \delta_A \left\{ n \in \mathbb{N} : \tilde{L}_n\left(\mathcal{A} \cap \mathcal{B}\right) \subset \mathcal{A} \right\} = 1.$$

If

$$(2.3) \qquad \text{st}_A\text{-}\lim_n \left\| \tilde{L}_n(e_i) - e_i \right\| = 0 \textit{ for } i = 0, 1, 2,$$

then we have

$$(2.4) \qquad \text{st}_A\text{-}\lim_n D^*\left(L_n(f), f\right) = 0 \textit{ for all } f \in C^2_{\mathcal{F}}[0,1].$$

Proof. Let $x \in [0,1]$ be fixed, and $f \in C^2_{\mathcal{F}}[0,1]$ and $r \in [0,1]$. Then we may write that, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every $y \in [0,1]$,

$$D\left(f(y), f(x)\right) < \varepsilon$$

holds. So, for every $y, r \in [0,1]$ and for any $\beta \geq 1$, we have

$$(2.5) \qquad -\varepsilon - \frac{2M^{(r)}_{\pm}\beta}{\delta^2}\varphi_x(y) \leq f^{(r)}_{\pm}(y) - f^{(r)}_{\pm}(x) \leq \varepsilon + \frac{2M^{(r)}_{\pm}\beta}{\delta^2}\varphi_x(y),$$

where $M^{(r)}_{\pm} = \left\| f^{(r)}_{\pm} \right\|$ and $\varphi_x(y) = (y-x)^2$. Then, by (2.5), we obtain that

$$\left(g^{(r)}_{\beta}\right)_{\pm}(y) := \frac{2M^{(r)}_{\pm}\beta}{\delta^2}\varphi_x(y) + \varepsilon - f^{(r)}_{\pm}(y) + f^{(r)}_{\pm}(x) \geq 0$$

and

$$\left(h^{(r)}_{\beta}\right)_{\pm}(y) := \frac{2M^{(r)}_{\pm}\beta}{\delta^2}\varphi_x(y) + \varepsilon + f^{(r)}_{\pm}(y) - f^{(r)}_{\pm}(x) \geq 0$$

hold for all $y \in [0,1]$. So, the functions $\left(g^{(r)}_{\beta}\right)_{\pm}$ and $\left(h^{(r)}_{\beta}\right)_{\pm}$ belong to $\mathcal{A}$. On the other hand, it is clear that, for all $y \in [0,1]$,

$$\left(g^{(r)}_{\beta}\right)_{\pm}(y) = \frac{4M^{(r)}_{\pm}\beta}{\delta^2} - \left[f^{(r)}_{\pm}\right]''(y)$$

and

$$\left(h_{\beta}^{(r)}\right)''_{\pm}(y) = \frac{4M_{\pm}^{(r)}\beta}{\delta^2} + \left[f_{\pm}^{(r)}\right]''(y).$$

If we choose the number β such that

$$\beta \geq \max\left\{1, \frac{\left\|\left[f_{\pm}^{(r)}\right]''\right\|\delta^2}{4M_{\pm}^{(r)}}\right\}, \tag{2.6}$$

we observe that (2.5) holds for such β's and also the functions $\left(g_{\beta}^{(r)}\right)_{\pm}$ and $\left(h_{\beta}^{(r)}\right)_{\pm}$ belong to $\mathcal{B}$ because of $\left(g_{\beta}^{(r)}\right)''_{\pm}(y) \geq 0$ and $\left(h_{\beta}^{(r)}\right)''_{\pm}(y) \geq 0$ for all $y \in [0,1]$. So, we have $\left(g_{\beta}^{(r)}\right)_{\pm}, \left(h_{\beta}^{(r)}\right)_{\pm} \in \mathcal{A} \cap \mathcal{B}$ under the condition (2.6). Let

$$K := \{n \in \mathbb{N} : \tilde{L}_n(\mathcal{A} \cap \mathcal{B}) \subset \mathcal{A}\}.$$

By (2.2), it is clear that $\delta_A\{K\} = 1$, and so

$$\delta_A\{\mathbb{N} \setminus K\} = 0. \tag{2.7}$$

Then, we may write that

$$\tilde{L}_n\Big(\left(g_{\beta}^{(r)}\right)_{\pm}; x\Big) \geq 0 \text{ and } \tilde{L}_n\Big(\left(h_{\beta}^{(r)}\right)_{\pm}; x\Big) \geq 0 \text{ for any } n \in K.$$

Now using the fact that $\varphi_x \in \mathcal{A} \cap \mathcal{B}$, and considering the linearity of $\tilde{L}_n$, we obtain, for every $n \in K$, that

$$\frac{2M_{\pm}^{(r)}\beta}{\delta^2}\tilde{L}_n(\varphi_x; x) + \varepsilon\tilde{L}_n(e_0; x) - \tilde{L}_n\big(f_{\pm}^{(r)}; x\big) + f_{\pm}^{(r)}(x)\tilde{L}_n(e_0; x) \geq 0$$

and

$$\frac{2M_{\pm}^{(r)}\beta}{\delta^2}\tilde{L}_n(\varphi_x; x) + \varepsilon\tilde{L}_n(e_0; x) + \tilde{L}_n\big(f_{\pm}^{(r)}; x\big) - f_{\pm}^{(r)}(x)\tilde{L}_n(e_0; x) \geq 0,$$

or equivalently

$$\begin{aligned}
-\frac{2M_{\pm}^{(r)}\beta}{\delta^2}\tilde{L}_n(\varphi_x; x) - \varepsilon\tilde{L}_n(e_0; x) + f_{\pm}^{(r)}(x)\big(\tilde{L}_n(e_0; x) - e_0\big)& \\
\leq \tilde{L}_n\big(f_{\pm}^{(r)}; x\big) - f_{\pm}^{(r)}(x)& \\
\leq \frac{2M_{\pm}^{(r)}\beta}{\delta^2}\tilde{L}_n(\varphi_x; x) + \varepsilon\tilde{L}_n(e_0; x) + f_{\pm}^{(r)}(x)\big(\tilde{L}_n(e_0; x) - e_0\big)&.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\big|\tilde{L}_n(f_{\pm}^{(r)}; x) - f_{\pm}^{(r)}(x)\big| &\leq \varepsilon + \frac{2M_{\pm}^{(r)}\beta}{\delta^2}\tilde{L}_n\big(\varphi_x; x\big) \\
&\quad + \big(\varepsilon + \big|f_{\pm}^{(r)}(x)\big|\big)\big|\tilde{L}_n(e_0; x) - e_0\big|
\end{aligned}$$

for every $n \in K$. The last inequality gives that, for every $\varepsilon > 0$ and $n \in K$,

$$\begin{aligned}
\big\|\tilde{L}_n(f_{\pm}^{(r)}) - f_{\pm}^{(r)}\big\| &\leq \varepsilon + (\varepsilon + M_{\pm}^{(r)})\big\|\tilde{L}_n(e_0) - e_0\big\| \\
&\quad + \frac{2M_{\pm}^{(r)}\beta}{\delta^2}\big\|\tilde{L}_n(e_2) - e_2\big\| + \frac{4M_{\pm}^{(r)}\beta}{\delta^2}\big\|\tilde{L}_n(e_1) - e_1\big\| \\
&\quad + \frac{2M_{\pm}^{(r)}\beta}{\delta^2}\big\|\tilde{L}_n(e_0) - e_0\big\|.
\end{aligned}$$

Hence, we get, for any $n \in K$, that

$$\left\|\tilde{L}_n(f_{\pm}^{(r)}) - f_{\pm}^{(r)}\right\| \leq \varepsilon + C_{\pm}^{(r)}(\varepsilon)\sum_{k=0}^{2}\left\|\tilde{L}_n(e_k) - e_k\right\|, \tag{2.8}$$

where $C_{\pm}^{(r)}(\varepsilon) := \max\left\{\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}\beta}{\delta^2}, \frac{4M_{\pm}^{(r)}\beta}{\delta^2}\right\}$. Now it follows from (2.1) that

$$\begin{aligned} D^*\big(L_n(f), f\big) &= \sup_{x\in[0,1]} D\big(L_n(f;x), f(x)\big) \\ &= \sup_{x\in[0,1]}\sup_{r\in[0,1]}\max\Big\{\big|\tilde{L}_n\big(f_-^{(r)};x\big) - f_-^{(r)}(x)\big|, \\ &\qquad\qquad \big|\tilde{L}_n\big(f_+^{(r)};x\big) - f_+^{(r)}(x)\big|\Big\} \\ &= \sup_{r\in[0,1]}\max\Big\{\big\|\tilde{L}_n\big(f_-^{(r)}\big) - f_-^{(r)}\big\|, \ \big\|\tilde{L}_n\big(f_+^{(r)}\big) - f_+^{(r)}\big\|\Big\}. \end{aligned}$$

Combining the above equality with (2.8), we have, for any $n \in K$,

$$D^*\left(L_n(f), f\right) \leq \varepsilon + C(\varepsilon)\sum_{k=0}^{2}\left\|\tilde{L}_n(e_k) - e_k\right\|, \tag{2.9}$$

where $C(\varepsilon) = \sup_{r\in[0,1]}\max\left\{C_-^{(r)}(\varepsilon), C_+^{(r)}(\varepsilon)\right\}$, Now, for a given $\varepsilon' > 0$, choose an $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$, and define the following sets:

$$F := \left\{n \in \mathbb{N} : D^*\left(L_n(f), f\right) \geq \varepsilon'\right\},$$

$$F_k := \left\{n \in \mathbb{N} : \left\|\tilde{L}_n(e_k) - e_k\right\| \geq \frac{\varepsilon' - \varepsilon}{3C(\varepsilon)}\right\}, \ k = 0, 1, 2.$$

Then, it follows from (2.9) that

$$F \cap K \subset \bigcup_{k=0}^{2}(F_k \cap K),$$

which yields, for every $j \in \mathbb{N}$, that

$$\sum_{n\in F\cap K} a_{jn} \leq \sum_{k=0}^{2}\Bigg(\sum_{n\in F_k\cap K} a_{jn}\Bigg) \leq \sum_{k=0}^{2}\Bigg(\sum_{n\in F_k} a_{jn}\Bigg). \tag{2.10}$$

Now, taking the limit as $j \to \infty$ on both-sides of (2.10), and using (2.3), we immediately see that

$$\lim_j \sum_{n\in F\cap K} a_{jn} = 0. \tag{2.11}$$

Furthermore, since

$$\begin{aligned} \sum_{n\in F} a_{jn} &= \sum_{n\in F\cap K} a_{jn} + \sum_{n\in F\cap(\mathbb{N}\setminus K)} a_{jn} \\ &\leq \sum_{n\in F\cap K} a_{jn} + \sum_{n\in(\mathbb{N}\setminus K)} a_{jn} \end{aligned}$$

holds for every $j \in \mathbb{N}$, taking again the limit as $j \to \infty$ in the last inequality, and using (2.7), (2.11), we obtain

$$\lim_j \sum_{n\in F} a_{jn} = 0,$$

which means that

$$\text{st}_A\text{-}\lim_n D^*\left(L_n(f), f\right) = 0.$$

The theorem is proved. □

2.4. Theorem. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and $\{L_n\}_{n\in\mathbb{N}}$ a sequence of fuzzy linear operators from $C^2_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^2[0,1]$ onto itself for which the following conditions hold:*

$$\left[\{L_n(f)\}^{(r)}_{\pm}\right]''(x) = \left[\tilde{L}_n\big(f^{(r)}_{\pm}\big)\right]''(x) \ \big(\text{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N}\big) \tag{2.12}$$

and

$$\delta_A\big\{n \in \mathbb{N} : \tilde{L}_n\left(\mathcal{A} \cap \mathcal{C}\right) \subset \mathcal{C}\big\} = 1. \tag{2.13}$$

If

$$\text{st}_A\text{-}\lim_n \left\|[\tilde{L}_n(e_i)]'' - e''_i\right\| = 0 \text{ for } i = 0,1,2,3,4, \tag{2.14}$$

then, we have

$$\text{st}_A\text{-}\lim_n D^*\big([L_n(f)]'', f''\big) = 0 \text{ for all } f \in C^2_{\mathcal{F}}[0,1]. \tag{2.15}$$

Proof. We should remark that the derivatives in (2.12) and (2.14) are in the usual sense while the derivatives in (2.15) are in the fuzzy sense. Now let $f \in C^2_{\mathcal{F}}[0,1]$, $x \in [0,1]$ and $r \in [0,1]$ be fixed. By Lemmas 2.1 and 2.2, and as in the proof of Theorem 2.3, we can write that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$-\varepsilon + \frac{2U^{(r)}_{\pm}\beta}{\delta^2}\sigma''_x(y) \le \big[f^{(r)}_{\pm}\big]''(y) - \big[f^{(r)}_{\pm}\big]''(x) \le \varepsilon - \frac{2U^{(r)}_{\pm}\beta}{\delta^2}\sigma''_x(y) \tag{2.16}$$

holds for all $y \in [0,1]$ and for any $\beta \ge 1$, where $\sigma_x(y) = -\dfrac{(y-x)^4}{12} + 1$ and $U^{(r)}_{\pm} = \left\|\big[f^{(r)}_{\pm}\big]''\right\|$. Then, define the following functions on $[0,1]$:

$$\left(u^{(r)}_{\beta}\right)_{\pm}(y) := \frac{2U^{(r)}_{\pm}\beta}{\delta^2}\sigma_x(y) + f^{(r)}_{\pm}(y) - \frac{\varepsilon}{2}y^2 - \frac{\big[f^{(r)}_{\pm}\big]''(x)}{2}y^2,$$

and

$$\left(v^{(r)}_{\beta}\right)_{\pm}(y) := \frac{2U^{(r)}_{\pm}\beta}{\delta^2}\sigma_x(y) - f^{(r)}_{\pm}(y) - \frac{\varepsilon}{2}y^2 + \frac{\big[f^{(r)}_{\pm}\big]''(x)}{2}y^2.$$

It follows from (2.16) that

$$\left(u^{(r)}_{\beta}\right)''_{\pm}(y) \le 0 \text{ and } \left(v^{(r)}_{\beta}\right)''_{\pm}(y) \le 0 \text{ for all } y \in [0,1],$$

which implies that the functions $\left(u^{(r)}_{\beta}\right)_{\pm}$ and $\left(v^{(r)}_{\beta}\right)_{\pm}$ belong to $\mathcal{C}$. Observe that $\sigma_x(y) \ge \dfrac{11}{12}$ for all $y \in [0,1]$. Then

$$\frac{\left(\pm f^{(r)}_{\pm}(y) - \frac{\varepsilon}{2}y^2 \pm \frac{\left[f^{(r)}_{\pm}\right]''(x)}{2}y^2\right)\delta^2}{2U^{(r)}_{\pm}\sigma_x(y)} \le \frac{\big(M^{(r)}_{\pm} + U^{(r)}_{\pm} + \varepsilon\big)\delta^2}{U^{(r)}_{\pm}}$$

holds for all $y \in [0,1]$, where $M_{\pm}^{(r)} = \left\| f_{\pm}^{(r)} \right\|$ and $U_{\pm^{(r)}} = \left\| \left[f_{\pm}^{(r)} \right]'' \right\|$ as stated before. Now, if we choose β such that

$$\beta \geq \max \left\{ 1, \frac{(M_{\pm}^{(r)} + U_{\pm}^{(r)} + \varepsilon)\delta^2}{U_{\pm}^{(r)}} \right\}, \tag{2.17}$$

then inequality (2.16) holds for such β's and

$$\left(u_{\beta}^{(r)} \right)_{\pm} (y) \geq 0 \text{ and } \left(v_{\beta}^{(r)} \right)_{\pm} (y) \geq 0 \text{ for all } y \in [0,1].$$

Hence, we also get $\left(u_{\beta}^{(r)} \right)_{\pm}$, $\left(v_{\beta}^{(r)} \right)_{\pm} \in \mathcal{A}$, which gives that the functions $\left(u_{\beta}^{(r)} \right)_{\pm}$ and $\left(v_{\beta}^{(r)} \right)_{\pm}$ belong to $\mathcal{A} \cap \mathcal{C}$ under the condition (2.17). Now let

$$K := \{ n \in \mathbb{N} : \tilde{L}_n (\mathcal{A} \cap \mathcal{C}) \subset \mathcal{C} \}.$$

Then, by (2.13), we have

$$\delta_A(\mathbb{N} \setminus K) = 0. \tag{2.18}$$

Also we get, for every $n \in K$,

$$\left[\tilde{L}_n \big(v_{\beta}^{(r)} \big)_{\pm} \right]'' \leq 0 \text{ and } \left[\tilde{L}_n \big(v_{\beta}^{(r)} \big)_{\pm} \right]'' \leq 0.$$

Then, we obtain, for every $n \in K$, that

$$\frac{2U_{\pm}^{(r)}\beta}{\delta^2} \left[\tilde{L}_n(\sigma_x) \right]'' + \left[\tilde{L}_n(f_{\pm}^{(r)}) \right]'' - \frac{\varepsilon}{2} [\tilde{L}_n(e_2)]'' - \frac{\left[f_{\pm}^{(r)} \right]''(x)}{2} [\tilde{L}_n(e_2)]'' \leq 0$$

and

$$\frac{2U_{\pm}^{(r)}\beta}{\delta^2} [\tilde{L}_n(\sigma_x)]'' - \left[\tilde{L}_n(f_{\pm}^{(r)}) \right]'' - \frac{\varepsilon}{2} [\tilde{L}_n(e_2)]'' + \frac{\left[f_{\pm}^{(r)} \right]''(x)}{2} [\tilde{L}_n(e_2)]'' \leq 0.$$

These inequalities yield that

$$\begin{aligned} \frac{2U_{\pm}^{(r)}\beta}{\delta^2} [\tilde{L}_n(\sigma_x)]''(x) - \frac{\varepsilon}{2} [\tilde{L}_n(e_2)]''(x) + \frac{\left[f_{\pm}^{(r)} \right]''(x)}{2} [\tilde{L}_n(e_2)]''(x) - \left[f_{\pm}^{(r)} \right]''(x) & \\ \leq \left[\tilde{L}_n(f_{\pm}^{(r)}) \right]''(x) - \left[f_{\pm}^{(r)} \right]''(x) & \\ \leq -\frac{2U_{\pm}^{(r)}\beta}{\delta^2} [\tilde{L}_n(\sigma_x)]''(x) + \frac{\varepsilon}{2} [\tilde{L}_n(e_2)]''(x) & \\ + \frac{\left[f_{\pm}^{(r)} \right]''(x)}{2} [\tilde{L}_n(e_2)]''(x) - \left[f_{\pm}^{(r)} \right]''(x). & \end{aligned}$$

Observe now that $[\tilde{L}_n(\sigma_x)]'' \leq 0$ on $[0,1]$ for every $n \in K$ because of $\sigma_x \in \mathcal{A} \cap \mathcal{C}$. Using this, the last inequality gives, for every $n \in K$, that

$$\begin{aligned} \left| \left[\tilde{L}_n(f_{\pm}^{(r)}) \right]''(x) - \left[f_{\pm}^{(r)} \right]''(x) \right| \leq -\frac{2U_{\pm}^{(r)}\beta}{\delta^2} \left[\tilde{L}_n(\sigma_x) \right]''(x) + \frac{\varepsilon}{2} \left| \left[\tilde{L}_n(e_2) \right]''(x) \right| & \\ + \frac{\left| \left[f_{\pm}^{(r)} \right]''(x) \right|}{2} \left| \left[\tilde{L}_n(e_2) \right]''(x) - 2 \right|, & \end{aligned}$$

and hence

$$\begin{aligned} \left| \left[\tilde{L}_n(f_{\pm}^{(r)}) \right]''(x) - f''(x) \right| \leq \varepsilon + \frac{\varepsilon + \left| \left[f_{\pm}^{(r)} \right]''(x) \right|}{2} \left| \left[\tilde{L}_n(e_2) \right]''(x) - e_2''(x) \right| & \\ + \frac{2U_{\pm}^{(r)}\beta}{\delta^2} \left[\tilde{L}_n(-\sigma_x) \right]''(x). & \end{aligned} \tag{2.19}$$

Now we compute the quantity $[L_n(-\sigma_x)]''$ in inequality (2.19). Observe that

$$\begin{aligned}
\big[\tilde{L}_n(-\sigma_x)\big]''(x) &= \Big[\tilde{L}_n\Big(\frac{(y-x)^4}{12}-1\Big)\Big]''(x)\\
&= \frac{1}{12}\big[\tilde{L}_n(e_4)\big]''(x) - \frac{x}{3}\big[\tilde{L}_n(e_3)\big]''(x) + \frac{x^2}{2}\big[\tilde{L}_n(e_2)\big]''(x)\\
&\qquad - \frac{x^3}{3}\big[\tilde{L}_n(e_1)\big]''(x) + \Big(\frac{x^4}{12}-1\Big)\big[\tilde{L}_n(e_0)\big]''(x)\\
&= \frac{1}{12}\Big\{\big[\tilde{L}_n(e_4)\big]''(x) - e_4''(x)\Big\} - \frac{x}{3}\Big\{\big[\tilde{L}_n(e_3)\big]''(x) - e_3''(x)\Big\}\\
&\quad + \frac{x^2}{2}\Big\{\big[\tilde{L}_n(e_2)\big]''(x) - e_2''(x)\Big\} - \frac{x^3}{3}\Big\{\big[\tilde{L}_n(e_1)\big]''(x) - e_1''(x)\Big\}\\
&\quad + \Big(\frac{x^4}{12}-1\Big)\Big\{\big[\tilde{L}_n(e_0)\big]''(x) - e_0''(x)\Big\}.
\end{aligned}$$

Combining this with (2.19), for every $\varepsilon > 0$ and $n \in K$, we have

$$\begin{aligned}
\Big|\big[\tilde{L}_n(f_\pm^{(r)})\big]''(x) - \big[f_\pm^{(r)}\big]''(x)\Big| &\le \varepsilon + \left(\frac{\varepsilon + \big|\big[f_\pm^{(r)}\big]''(x)\big|}{2} + \frac{U_\pm^{(r)}\beta_\pm x^2}{\delta^2}\right)\\
&\qquad \times \Big|\big[L_n(e_2)\big]''(x) - e_2''(x)\Big|\\
&\quad + \frac{U_\pm^{(r)}\beta}{6\delta^2}\Big|\big[\tilde{L}_n(e_4)\big]''(x) - e_4''(x)\Big|\\
&\quad + \frac{2U_\pm^{(r)}\beta x}{3\delta^2}\Big|\big[\tilde{L}_n(e_3)\big]''(x) - e_3''(x)\Big|\\
&\quad + \frac{2U_\pm^{(r)}\beta x^3}{3\delta^2}\Big|\big[\tilde{L}_n(e_1)\big]''(x) - e_1''(x)\Big|\\
&\quad + \frac{2U_\pm^{(r)}\beta}{3\delta^2}\Big(1 - \frac{x^4}{12}\Big)\Big|\big[\tilde{L}_n(e_0)\big]''(x) - e_0''(x)\Big|.
\end{aligned}$$

Therefore, we obtain, for every $\varepsilon > 0$ and $n \in K$, that

$$(2.20)\qquad \Big\|\big[\tilde{L}_n(f_\pm^{(r)})\big]'' - \big[f_\pm^{(r)}\big]''\Big\| \le \varepsilon + E_\pm^{(r)}(\varepsilon)\sum_{k=0}^{4}\Big\|\big[\tilde{L}_n(e_k)\big]'' - e_k''\Big\|,$$

where $E_\pm^{(r)}(\varepsilon) := \dfrac{\varepsilon + U_\pm^{(r)}}{2} + \dfrac{U_\pm^{(r)}\beta}{\delta^2}$ and $U_\pm^{(r)} = \Big\|\big[f_\pm^{(r)}\big]''\Big\|$, as stated before. On the other hand, by (2.12), since

$$\begin{aligned}
D^*\big(\big[L_n(f)\big]'', f''\big) &= \sup_{x\in[0,1]} D\big(\big[L_n(f)\big]''(x), f''(x)\big)\\
&= \sup_{x\in[0,1]}\sup_{r\in[0,1]}\max\Big\{\Big|\big[\tilde{L}_n(f_-^{(r)})\big]''(x) - \big[f_-^{(r)}\big]''(x)\Big|,\\
&\qquad\qquad \Big|\big[\tilde{L}_n(f_+^{(r)})\big]''(x) - \big[f_+^{(r)}\big]''(x)\Big|\Big\}\\
&= \sup_{r\in[0,1]}\max\Big\{\Big\|\big[\tilde{L}_n(f_-^{(r)})\big]'' - \big[f_-^{(r)}\big]''\Big\|,\\
&\qquad\qquad \Big\|\big[\tilde{L}_n(f_+^{(r)})\big]'' - \big[f_+^{(r)}\big]''\Big\|\Big\},
\end{aligned}$$

we may write from (2.20) that

$$(2.21)\qquad D^*\big(\big[L_n(f)\big]'', f''\big) \le \varepsilon + E(\varepsilon)\sum_{k=0}^{4}\Big\|\big[\tilde{L}_n(e_k)\big]'' - e_k''\Big\|$$

holds for all $n \in K$, where $E(\varepsilon) = \sup_{r\in[0,1]} \max\{E_-^{(r)}(\varepsilon), E_+^{(r)}(\varepsilon)\}$. Now, for a given $\varepsilon' > 0$, choose an ε such that $0 < \varepsilon < \varepsilon'$, and consider the following sets:

$$G := \left\{ n \in \mathbb{N} : D^*\big(\big[L_n(f)\big]'', f''\big) \geq \varepsilon' \right\},$$
$$G_k := \left\{ n \in \mathbb{N} : \left\| \big[\tilde{L}_n(e_k)\big]'' - e_k'' \right\| \geq \frac{\varepsilon' - \varepsilon}{5E(\varepsilon)} \right\}, \ \ k = 0, 1, 2, 3, 4.$$

In this case, by (2.21),

$$G \cap K \subset \bigcup_{k=0}^{4} (G_k \cap K),$$

which yields, for every $j \in \mathbb{N}$, that

$$\sum_{n\in G\cap K} a_{jn} \leq \sum_{k=0}^{4} \Bigg(\sum_{n \in G_k \cap K} a_{jn} \Bigg) \leq \sum_{k=0}^{4} \Bigg(\sum_{n\in G_k} a_{jn} \Bigg). \tag{2.22}$$

Letting $j \to \infty$ on both sides of (2.22), and using (2.14), we immediately see that

$$\lim_j \sum_{n\in G\cap K} a_{jn} = 0. \tag{2.23}$$

Furthermore, if we use the inequality

$$\begin{aligned} \sum_{n\in G} a_{jn} &= \sum_{n\in G\cap K} a_{jn} + \sum_{n\in G\cap(\mathbb{N}\setminus K)} a_{jn} \\ &\leq \sum_{n\in G\cap K} a_{jn} + \sum_{n\in(\mathbb{N}\setminus K)} a_{jn} \end{aligned}$$

and take the limit as $j \to \infty$, then it follows from (2.18) and (2.23) that

$$\lim_j \sum_{n\in G} a_{jn} = 0.$$

Thus, we get

$$\mathrm{st}_A\text{-}\lim_n D^*\big(\big[L_n(f)\big]'', f''\big) = 0.$$

The theorem is proved. □

2.5. Theorem. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and $\{L_n\}_{n\in\mathbb{N}}$ a sequence of fuzzy linear operators from $C^1_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^1[0,1]$ onto itself for which the following conditions hold:*

$$\big[\{L_n(f)\}_{\pm}^{(r)}\big]'(x) = \big[\tilde{L}_n\big(f_{\pm}^{(r)}\big)\big]'(x) \ (\textit{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N}) \tag{2.24}$$

and

$$\delta_A\big\{n \in \mathbb{N} : \tilde{L}_n\,(\mathcal{D} \cap \mathcal{E}) \subset \mathcal{E}\big\} = 1. \tag{2.25}$$

If

$$\mathrm{st}_A\text{-}\lim_n \left\| \big[\tilde{L}_n(e_i)\big]' - e_i' \right\| = 0 \textit{ for } i = 0, 1, 2, 3, \tag{2.26}$$

then we have

$$\mathrm{st}_A\text{-}\lim_n D^*\big(\big[L_n(f)\big]', f'\big) = 0 \textit{ for all } f \in C^1[0,1]. \tag{2.27}$$

Proof. Let $f \in C^1_{\mathcal{F}}[0,1]$, $x \in [0,1]$ and $r \in [0,1]$ be fixed. Then, for every $\varepsilon > 0$, there exists a positive number δ such that

$$(2.28)\quad -\varepsilon - \frac{2V_{\pm}^{(r)}\beta}{\delta^2} w_x'(y) \le \big[f_{\pm}^{(r)}\big]'(y) - \big[f_{\pm}^{(r)}\big]'(x) \le \varepsilon + \frac{2V_{\pm}^{(r)}\beta}{\delta^2} w_x'(y)$$

holds for all $y \in [0,1]$ and for any $\beta \ge 1$, where $w_x(y) := \dfrac{(y-x)^3}{3} + 1$ and $V_{\pm}^{(r)} := \left\| \big[f_{\pm}^{(r)}\big]' \right\|$, Now considering the functions defined by

$$\big(\theta_{\beta}^{(r)}\big)_{\pm}(y) := \frac{2V_{\pm}^{(r)}\beta}{\delta^2} w_x(y) - f_{\pm}^{(r)}(y) + \varepsilon y + y\big[f_{\pm}^{(r)}\big]'(x)$$

and

$$\big(\lambda_{\beta}^{(r)}\big)_{\pm}(y) := \frac{2V_{\pm}^{(r)}\beta}{\delta^2} w_x(y) + f_{\pm}^{(r)}(y) + \varepsilon y - y\big[f_{\pm}^{(r)}\big]'(x),$$

we can easily check that $\big(\theta_{\beta}^{(r)}\big)_{\pm}$ and $\big(\lambda_{\beta}^{(r)}\big)_{\pm}$ belong to $\mathcal{E}$ for any $\beta \ge 1$. Also, observe that $w_x(y) \ge \dfrac{2}{3}$ for all $y \in [0,1]$. Then

$$\frac{\big(\pm f_{\pm}^{(r)}(y) - \varepsilon y \pm \big[f_{\pm}^{(r)}\big]'(x) y\big)\delta^2}{2V_{\pm}^{(r)} w_x(y)} \le \frac{(M_{\pm}^{(r)} + V_{\pm}^{(r)} + \varepsilon)\delta^2}{V_{\pm}^{(r)}}$$

holds for all $y \in [0,1]$, where $M_{\pm}^{(r)} = \|f_{\pm}^{(r)}\|$, as stated before. Now, if we choose a number β such that

$$(2.29)\quad \beta \ge \max\left\{1, \frac{(M_{\pm}^{(r)} + V_{\pm}^{(r)} + \varepsilon)\delta^2}{V_{\pm}^{(r)}}\right\},$$

then inequality (2.28) holds for such β's and

$$\big(\theta_{\beta}^{(r)}\big)_{\pm}(y) \ge 0 \text{ and } \big(\lambda_{\beta}^{(r)}\big)_{\pm}(y) \ge 0 \text{ for all } y \in [0,1],$$

which yields that $\big(\theta_{\beta}^{(r)}\big)_{\pm}$, $\big(\lambda_{\beta}^{(r)}\big)_{\pm} \in \mathcal{D}$. Thus, we get $\big(\theta_{\beta}^{(r)}\big)_{\pm}$, $\big(\lambda_{\beta}^{(r)}\big)_{\pm} \in \mathcal{D} \cap \mathcal{E}$ for any β satisfying (2.29). Let

$$K := \{n \in \mathbb{N} : \tilde{L}_n\left(\mathcal{D} \cap \mathcal{E}\right) \subset \mathcal{E}\}.$$

Then, by (2.25), we have

$$(2.30)\quad \delta_A\{\mathbb{N} \setminus K\} = 0.$$

Also we get, for every $n \in K$,

$$\big[\tilde{L}_n\big(\theta_{\beta}^{(r)}\big)_{\pm}\big]' \ge 0 \text{ and } \big[\tilde{L}_n\big(\lambda_{\beta}^{(r)}\big)_{\pm}\big]' \ge 0.$$

Hence we obtain, for every $n \in K$, that

$$\frac{2V_{\pm}^{(r)}\beta}{\delta^2}[\tilde{L}_n(w_x)]' - \big[\tilde{L}_n\big(f_{\pm}^{(r)}\big)\big]' + \varepsilon\big[\tilde{L}_n(e_1)\big]' + \big[f_{\pm}^{(r)}\big]'(x)\big[\tilde{L}_n(e_1)\big]' \ge 0$$

and

$$\frac{2V_{\pm}^{(r)}\beta}{\delta^2}[\tilde{L}_n(w_x)]' + \big[\tilde{L}_n(f_{\pm}^{(r)})\big]' + \varepsilon\big[\tilde{L}_n(e_1)\big]' - \big[f_{\pm}^{(r)}\big]'(x)\big[\tilde{L}_n(e_1)\big]' \ge 0.$$

Then, we may write that

$$\begin{aligned}
-\frac{2V_{\pm}^{(r)}\beta}{\delta^2}[\tilde{L}_n(w_x)]'(x) - \varepsilon\big[\tilde{L}_n(e_1)\big]'(x) + \big[f_{\pm}^{(r)}\big]'(x)\big[\tilde{L}_n(e_1)\big]'(x) - \big[f_{\pm}^{(r)}\big]'(x)& \\
\leq \big[\tilde{L}_n\big(f_{\pm}^{(r)}\big)\big]'(x) - \big[f_{\pm}^{(r)}\big]'(x)& \\
\leq \frac{2V_{\pm}^{(r)}\beta}{\delta^2}\big[\tilde{L}_n(w_x)\big]'(x) + \varepsilon\big[\tilde{L}_n(e_1)\big]'(x)& \\
+ \big[f_{\pm}^{(r)}\big]'(x)\big[\tilde{L}_n(e_1)\big]'(x) - \big[f_{\pm}^{(r)}\big]'(x),&
\end{aligned}$$

and hence

$$\begin{aligned}
\Big|\big[\tilde{L}_n(f_{\pm}^{(r)})\big]'(x) - \big[f_{\pm}^{(r)}\big]'(x)\Big| \leq \varepsilon + \Big(\varepsilon + \Big|\big[f_{\pm}^{(r)}\big]'(x)\Big|\Big)\Big|\big[\tilde{L}_n(e_1)\big]'(x) - e_1'(x)\Big|& \\
+ \frac{2V_{\pm}^{(r)}\beta}{\delta^2}\big[\tilde{L}_n(w_x)\big]'(x)&
\end{aligned} \tag{2.31}$$

holds for every $n \in K$ because of the fact that the function w_x belongs to $\mathcal{D} \cap \mathcal{E}$. Since

$$\begin{aligned}
\big[\tilde{L}_n(w_x)\big]'(x) &= \Big[\tilde{L}_n\Big(\frac{(y-x)^3}{3} + 1\Big)\Big]'(x) \\
&= \frac{1}{3}\big[\tilde{L}_n(e_3)\big]'(x) - x\big[\tilde{L}_n(e_2)\big]'(x) + x^2\big[\tilde{L}_n(e_1)\big]'(x) \\
&\qquad + \Big(1 - \frac{x^3}{3}\Big)\big[\tilde{L}_n(e_0)\big]'(x) \\
&= \frac{1}{3}\big\{\big[\tilde{L}_n(e_3)\big]'(x) - e_3'(x)\big\} - x\big\{\big[\tilde{L}_n(e_2)\big]'(x) - e_2'(x)\big\} \\
&\qquad + x^2\big\{\big[\tilde{L}_n(e_1)\big]'(x) - e_1'(x)\big\} \\
&\qquad + \Big(1 - \frac{x^3}{3}\Big)\big\{\big[\tilde{L}_n(e_0)\big]'(x) - e_0'(x)\big\},
\end{aligned}$$

it follows from (2.31) that

$$\begin{aligned}
\Big|\big[\tilde{L}_n(f_{\pm}^{(r)})\big]'(x) - \big[f_{\pm}^{(r)}\big]'(x)\Big| \leq \varepsilon + \left(\varepsilon + \Big|\big[f_{\pm}^{(r)}\big]'(x)\Big| + \frac{2V_{\pm}^{(r)}\beta_{\pm}x^2}{\delta^2}\right)& \\
\times \Big|\big[\tilde{L}_n(e_1)\big]'(x) - e_1'(x)\Big|& \\
+ \frac{2V_{\pm}^{(r)}\beta}{3\delta^2}\Big|\big[\tilde{L}_n(e_3)\big]'(x) - e_3'(x)\Big|& \\
+ \frac{2V_{\pm}^{(r)}\beta x}{\delta^2}\Big|\big[\tilde{L}_n(e_2)\big]'(x) - e_2'(x)\Big|& \\
+ \frac{2V_{\pm}^{(r)}\beta}{\delta^2}\Big(1 - \frac{x^3}{3}\Big)\Big|\big[\tilde{L}_n(e_0)\big]'(x) - e_0'(x)\Big|&.
\end{aligned}$$

Thus, we deduce from the last inequality that

$$\Big\|\big[\tilde{L}_n(f_{\pm}^{(r)})\big]' - \big[f_{\pm}^{(r)}\big]'\Big\| \leq \varepsilon + F_{\pm}^{(r)}(\varepsilon)\sum_{k=0}^{3}\Big\|\big[\tilde{L}_n(e_k)\big]' - e_k'\Big\| \tag{2.32}$$

holds for any $n \in K$, where $F_{\pm}^{(r)}(\varepsilon) := \varepsilon + V_{\pm}^{(r)} + \dfrac{2V_{\pm}^{(r)}\beta_{\pm}}{\delta^2}$. On the other hand, by (2.24), since

$$D^*\big([L_n(f)]', f'\big) = \sup_{x\in[0,1]} D\big([L_n(f)]'(x), f'(x)\big)$$
$$= \sup_{x\in[0,1]} \sup_{r\in[0,1]} \max\Big\{ \Big|\big[\tilde{L}_n\big(f_-^{(r)}\big)\big]'(x) - \big[f_-^{(r)}\big]'(x)\Big|,$$
$$\Big|\big[\tilde{L}_n\big(f_+^{(r)}\big)\big]'(x) - \big[f_+^{(r)}\big]'(x)\Big| \Big\}$$
$$= \sup_{r\in[0,1]} \max\Big\{ \Big\|\big[\tilde{L}_n\big(f_-^{(r)}\big)\big]' - \big[f_-^{(r)}\big]'\Big\|,$$
$$\Big\|\big[\tilde{L}_n\big(f_+^{(r)}\big)\big]' - \big[f_+^{(r)}\big]'\Big\| \Big\},$$

we may write from (2.32) that

$$D^*\big([L_n(f)]', f'\big) \le \varepsilon + F(\varepsilon) \sum_{k=0}^{3} \Big\|\big[\tilde{L}_n(e_k)\big]' - e_k'\Big\| \tag{2.33}$$

holds for all $n \in K$, where $F(\varepsilon) = \sup_{r\in[0,1]} \max\big\{F_-^{(r)}(\varepsilon), F_+^{(r)}(\varepsilon)\big\}$. Now, for a given $\varepsilon' > 0$, choose an ε such that $0 < \varepsilon < \varepsilon'$, and consider the following sets:

$$J := \big\{n \in \mathbb{N} : D^*\big([L_n(f)]', f'\big) \ge \varepsilon'\big\},$$
$$J_k := \Big\{n \in \mathbb{N} : \Big\|\big[\tilde{L}_n(e_k)\big]' - e_k'\Big\| \ge \frac{\varepsilon' - \varepsilon}{4G(\varepsilon)}\Big\},\ k = 0, 1, 2, 3.$$

In this case, by (2.33),

$$J \cap K \subset \bigcup_{k=0}^{3} (J_k \cap K),$$

which yields, for every $j \in \mathbb{N}$, that

$$\sum_{n\in J\cap K} a_{jn} \le \sum_{k=0}^{3} \Bigg(\sum_{n\in J_k\cap K} a_{jn} \Bigg) \le \sum_{k=0}^{3} \Bigg(\sum_{n\in J_k} a_{jn} \Bigg) \tag{2.34}$$

Letting $j \to \infty$ on both sides of (2.34), and also using (2.26), we immediately see that

$$\lim_j \sum_{n\in J\cap K} a_{jn} = 0. \tag{2.35}$$

Now, using the fact that

$$\sum_{n\in J} a_{jn} = \sum_{n\in J\cap K} a_{jn} + \sum_{n\in J\cap(\mathbb{N}\setminus K)} a_{jn}$$
$$\le \sum_{n\in J\cap K} a_{jn} + \sum_{n\in(\mathbb{N}\setminus K)} a_{jn},$$

and taking the limit as $j \to \infty$, then it follows from (2.30) and (2.35) that

$$\lim_j \sum_{n\in J} a_{jn} = 0.$$

Thus, we get

$$\mathrm{st}_A\text{-}\lim_n D^*\big([L_n(f)]', f'\big) = 0,$$

whence the result. □

2.6. Theorem. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and $\{L_n\}_{n\in\mathbb{N}}$ a sequence of fuzzy linear operators from $C_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C[0,1]$ onto itself for which the following conditions hold:*

$$L_n(f;x)_{\pm}^{(r)} = \tilde{L}_n\big(f_{\pm}^{(r)};x\big) \ \big(\text{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N}\big) \tag{2.36}$$

and

$$\delta_A\big\{n \in \mathbb{N} : \tilde{L}_n\big(\mathcal{F}\big) \subset \mathcal{F}\big\} = 1. \tag{2.37}$$

If

$$\mathrm{st}_A\text{-}\lim_n \|\tilde{L}_n(e_i) - e_i\| = 0 \text{ for } i = 0,1,2, \tag{2.38}$$

then we have

$$\mathrm{st}_A\text{-}\lim_n D^*\big(\big[L_n(f)\big]', f'\big) \text{ for all } f \in C[0,1]. \tag{2.39}$$

Proof. If $\tilde{L}_n(\mathcal{F}) \subset \mathcal{F}$ holds for every $n \in \mathbb{N}$ instead of (2.37), i.e., the corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ consists of positive linear operators on $C[0,1]$, then the proof follows immediately from [5, Theorem 2.1]. However, our previous proofs show that even when the weaker condition (2.37) holds, the property (2.38) implies (2.39). □

3. Concluding Remarks

If we replace the non-negative regular matrix A by the identity matrix, then our Theorems 2.3-2.6 reduce to the following results which are also new in the literature, except for the last one.

3.1. Corollary. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy linear operators from $C^2_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^2[0,1]$ onto itself for which the following conditions hold:*

$$L_n(f;x)_{\pm}^{(r)} = \tilde{L}_n\big(f_{\pm}^{(r)};x\big) \ \big(\text{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N}\big)$$

and

$$\tilde{L}_n\,(\mathcal{A} \cap \mathcal{B}) \subset \mathcal{A} \ \big(\text{for } n \in \mathbb{N}).$$

If the sequence $\{\tilde{L}_n(e_i)\}$, $(i = 0,1,2)$, is uniformly convergent to e_i on $[0,1]$ then, for all $f \in C^2_{\mathcal{F}}[0,1]$,

$$\lim_n D^*\big(L_n(f), f\big) = 0.$$ □

3.2. Corollary. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy linear operators from $C^2_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^2[0,1]$ onto itself for which the following conditions hold:*

$$\big[\{L_n(f)\}_{\pm}^{(r)}\big]''(x) = \big[\tilde{L}_n\big(f_{\pm}^{(r)}\big)\big]''(x) \ \big(\text{for } x \in [0,1],\ r \in [0,1],\ n \in \mathbb{N}\big)$$

and

$$\tilde{L}_n\,(\mathcal{A} \cap \mathcal{C}) \subset \mathcal{C} \ \big(\text{for } n \in \mathbb{N}\big).$$

If the sequence $\big\{\big[\tilde{L}_n(e_i)\big]''\big\}$, $(i = 0,1,2,3,4)$, is uniformly convergent to e_i'' on $[0,1]$ then, for all $f \in C^2_{\mathcal{F}}[0,1]$,

$$\lim_n D^*\big(\big[L_n(f)\big]'', f''\big) = 0.$$ □

3.3. Corollary. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy linear operators from $C^1_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of linear operators from $C^1[0,1]$ onto itself for which the following conditions hold:*

$$\big[\{L_n(f)\}^{(r)}_{\pm}\big]'(x) = \big[\tilde{L}_n\big(f^{(r)}_{\pm}\big)\big]'(x)\ \big(\text{for } x\in[0,1],\ r\in[0,1],\ n\in\mathbb{N}\big)$$

and

$$\tilde{L}_n\left(\mathcal{D}\cap\mathcal{E}\right)\subset\mathcal{E}\ \big(\text{for } n\in\mathbb{N}\big).$$

If the sequence $\big\{\big[\tilde{L}_n(e_i)\big]'\big\}$, $(i=0,1,2,3)$, is uniformly convergent to e_i' on $[0,1]$ then, for all $f\in C^1_{\mathcal{F}}[0,1]$,

$$\lim_n D^*\big(\big[L_n(f)\big]', f'\big) = 0. \qquad \square$$

3.4. Corollary. [3] *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy linear operators from $C_{\mathcal{F}}[0,1]$ onto itself. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C[0,1]$ onto itself for which the following condition holds:*

$$L_n(f;x)^{(r)}_{\pm} = \tilde{L}_n\big(f^{(r)}_{\pm};x\big)\ \big(\text{for } x\in[0,1],\ r\in[0,1],\ n\in\mathbb{N}).$$

If the sequence $\{\tilde{L}_n(e_i)\}$ $(i=0,1,2)$ is uniformly convergent to e_i on $[0,1]$ then, for all $f\in C_{\mathcal{F}}[0,1]$,

$$\lim_n D^*\big(L_n(f), f\big) = 0. \qquad \square$$

Finally, we give an application satisfying all the conditions of our Theorem 2.3.

3.5. Application. We first take $A = C_1$, the Cesáro matrix. Then, we know that C_1-statistical convergence reduces to the concept of statistical convergence; in this case, we use the notation st-lim instead of st_{C_1}-lim. Also, we have $\delta_{C_1}\equiv\delta$. Assume now that $\{L_n\}$ is a sequence of linear operators on $C^2_{\mathcal{F}}[0,1]$ whose corresponding sequence $\{\tilde{L}_n\}$ of linear operators on $C^2[0,1]$ is given by

$$\text{(3.1)}\qquad \tilde{L}_n(f^{(r)}_{\pm};x) = \begin{cases} -x^2, & \text{if } n=m^2\ \big(m\in\mathbb{N}\big),\\ B_n(f^{(r)}_{\pm};x), & \text{if } n\neq m^2,\end{cases}$$

where $x\in[0,1]$, $r\in[0,1]$, $n\in\mathbb{N}$, and $\{B_n\}$ denotes the sequence of classical Bernstein polynomials on $C[0,1]$. Then, observe that

$$\begin{aligned}\delta\big(\big\{n\in\mathbb{N} : \tilde{L}_n\big(\mathcal{A}\cap\mathcal{B}\big)\subset\mathcal{A}\big\}\big) &= \delta\big(\big\{n\neq m^2 : m\in\mathbb{N}\big\}\big)\\ &= 1.\end{aligned}$$

Also we have, for each $i=0,1,2$,

$$\text{st-}\lim_n \big\|\tilde{L}_n(e_i) - e_i\big\| = 0.$$

Then, it follows from Theorem 2.3 that, for all $f\in C^2_{\mathcal{F}}[0,1]$,

$$\text{st-}\lim_n D^*\big(L_n(f), f\big) = 0.$$

However, for the function $e_0 = 1$, since

$$\tilde{L}_n(e_0;x) := \begin{cases} -x^2 & \text{if } n=m^2\ \big(m\in\mathbb{N}\big)\\ 1 & \text{otherwise,}\end{cases}$$

we get, for all $x\in[0,1]$, that the sequence $\{\tilde{L}_n(e_0;x)\}$ is non-convergent in the ordinary sense. Furthermore, we see that infinitely many terms of $\tilde{L}_n$ do not satisfy the inclusion $\tilde{L}_n\big(\mathcal{A}\cap\mathcal{B}\big)\subset\mathcal{A}$. Hence, this example clearly shows that our statistical fuzzy approximation theorems obtained in this paper have a wider range of applications than the classical fuzzy approximation results.

3.6. Remark. In this paper, we focus on fuzzy approximation only for the one dimensional case. However, by simple manipulations, all our results are also valid for multi dimensional cases. However, we omit their details.

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