Hacettepe Journal of Mathematics and Statistics
Volume 39 (4) (2010), 527 – 534

CERTAIN APPLICATIONS OF SUBORDINATION ASSOCIATED WITH NEIGHBORHOODS

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Received 26 : 10 : 2009 : Accepted 05 : 04 : 2010

Abstract

In this paper we introduce the classes $T_n^p(\lambda, A, B)$ and $K_n^p(\lambda, \mu, A, B)$, and derive coefficient bounds and distortion inequalities for functions belonging to the class $T_n^p(\lambda, A, B)$. Further, we make use of the (n, δ)-neighborhoods of functions in both classes $T_n^p(\lambda, A, B)$ and $K_n^p(\lambda, \mu, A, B)$.

Keywords: Analytic function, Multivalent function, Subordination, (n, δ)-neighborhoods, Coefficient bounds, Distortion inequalities, Non-homegenous differential equations.

2000 AMS Classification: 30 C 45.

1. Introduction and definitions

Let T_n^p denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \ (a_k \geq 0, \ n, p \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic and multivalent in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Given two functions f and g, which are analytic in U, the function f is said to be *subordinate to* g, written

$$f \prec g \text{ and } f(z) \prec g(z) \tag{1.2}$$

if there exists a Schwarz function w analytic in U, with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in U)$$

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and such that

(1.3) $\quad f(z)=g(w(z))\ (z\in U).$

Earlier investigations have been carried out by Goodman [7] and Rucheweyh [9] (see also [1, 2, 3, 4, 5] and [8]). We define the (n,δ)-*neighborhoods* of functions $f\in T_n^p$ by

(1.4) $$N_{n,\delta}(f;g)=\left\{g\in T_n^p : g(z)=z^p-\sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|a_k-b_k|\le\delta\right\},$$

so that, obviously,

(1.5) $$N_{n,\delta}(h;g)=\left\{g\in T_n^p : g(z)=z^p-\sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k|b_k|\le\delta\right\},$$

where

(1.6) $\quad h(z)=z^p\ (p\in\mathbb{N},\ q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}).$

We also let $T_n^p(\lambda,A,B)$ denote the subclass of T_n^p consisting of functions $f(z)$ which satisfy the following relation

(1.7) $$\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\prec p\frac{1+Az}{1+Bz},$$

where $0\le\lambda\le1$, $p\in\mathbb{N}$, $-1\le A<B\le1$. The classes $T_n^p(0,A,B)$ and $T_n^p(1,A,B)$ are studied in [6].

Finally, let $K_n^p(\lambda,\mu,A,B)$ denote the subclass of the general class T_n^p consisting of functions $f\in T_n^p$ which satisfy the following non-homegenous Cauchy-Euler differential equation:

(1.8) $$z^2\frac{d^2w}{dz^2}+2(\mu+1)z\frac{dw}{dz}+\mu(\mu+1)w=(p+\mu)(p+\mu+1)g,$$

where $w=f(z)\in T_n^p$, $g=g(z)\in T_n^p(\lambda,A,B)$ and $\mu>-p$.

In this paper, we obtain coefficient bounds, distortion inequalities and (n,δ)-neighborhoods of functions $f\in T_n^p$ in both the classes $T_n^p(\lambda,A,B)$ and $K_n^p(\lambda,\mu,A,B)$.

2. Coefficient bounds and distortion inequalities

We begin with the following lemmas.

2.1. Lemma. *Let the function $f\in T_n^p$ be defined by* (1.1). *Then $f(z)$ is in the class $T_n^p(\lambda,A,B)$ if and only if*

(2.1) $$\sum_{k=n+p}^{\infty}(k-p-pA+kB)(\lambda k+1-\lambda)a_k\le p(B-A)(\lambda p+1-\lambda)$$

$$(0\le\lambda\le1,\ p\in\mathbb{N},\ -1\le A<B\le1).$$

The result is sharp for the function $f(z)$ given by

(2.2) $$f(z)=z^p-\frac{p(B-A)(\lambda p+1-\lambda)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda]}z^{n+p}.$$

Proof. Let $f\in T_n^p(\lambda,A,B)$ and

(2.3) $\quad F(z)=\lambda z f'(z)+(1-\lambda)f(z)$

we find from (1.7) that

(2.4) $$\frac{zF'(z)}{F(z)}\prec p\frac{1+Az}{1+Bz}$$

and

$$\left| \frac{\frac{zF'(z)}{F(z)} - p}{B\frac{zF'(z)}{F(z)} - Ap} \right| < 1, \ (z \in U). \tag{2.5}$$

Since $|\Re(z)| \leq |z|$ for all z, we have

$$\Re \frac{\sum_{k=n+p}^{\infty} (k-p)(\lambda k + 1 - \lambda) a_k z^k}{p(B-A)(\lambda p + 1 - \lambda) z^p + \sum_{k=n+p}^{\infty} (pA - kB)(\lambda k + 1 - \lambda) a_k z^k} < 1. \tag{2.6}$$

By letting $z \to 1^-$ along the real axis, we get

$$\sum_{k=n+p}^{\infty} (k - p - pA + kB) a_k \leq p(B-A)(\lambda k + 1 - \lambda). \tag{2.7}$$

Conversely, let $|z| = 1$ in (2.5). Then

$$\begin{aligned} &\left| \frac{zF'(z)}{F(z)} - p \right| - \left| B\frac{zF'(z)}{F(z)} - Ap \right| \\ &\qquad = \left| \sum_{k=n+p}^{\infty} (k-p)(\lambda k + 1 - \lambda) a_k \right| \\ &\qquad\qquad - \left| p(B-A)(\lambda p + 1 - \lambda) + \sum_{k=n+p}^{\infty} (pA - kB)(\lambda k + 1 - \lambda) a_k \right| \\ &\qquad \leq \sum_{k=n+p}^{\infty} (k - p - pA + kB)(\lambda k + 1 - \lambda) a_k - p(B-A)(\lambda p + 1 - \lambda) \\ &\qquad \leq 0. \end{aligned}$$

Hence, by the principle of maximum modulus, we have $f \in T_n^p(\lambda, A, B)$, which completes the proof of Lemma 2.1. □

2.2. Lemma. *Let the function* $f(z) \in T_n^p$ *defined by* (1.1) *be in the class* $T_n^p(\lambda, A, B)$. *Then*

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{p(B-A)(\lambda p + 1 - \lambda)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]} \tag{2.8}$$

and

$$\sum_{k=n+p}^{\infty} k a_k \leq \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}. \tag{2.9}$$

Proof. By using Lemma 2.1, we find from (2.1) that

$$\begin{aligned} &[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda] \sum_{k=n+p}^{\infty} a_k \\ &\qquad \leq \sum_{k=n+p}^{\infty} (k - p - pA + kB)(\lambda k + 1 - \lambda) a_k \\ &\qquad \leq p(B-A)(\lambda p + 1 - \lambda), \end{aligned}$$

which immediately yields the first assertion (2.8) of Lemma 2.2.

Next, by appealing to (2.1), we also have

$$(\lambda(n+p)+1-\lambda)\left[\sum_{k=n+p}^{\infty}(1+B)ka_k - p(1+A)\sum_{k=n+p}^{\infty}a_k\right] \leq p(B-A)(\lambda p+1-\lambda),$$

or

$$(1+B)\sum_{k=n+p}^{\infty}ka_k \leq \frac{p\,(B-A)\,(\lambda p+1-\lambda)}{\lambda\,(n+p)+1-\lambda} + p\,(1+A)\sum_{k=n+p}^{\infty}a_k.$$

Thus, in the light of (2.8), the above inequality immediately yields the second assertion (2.9) of Lemma 2.2. □

2.3. Theorem. *If the function $f \in T_n^p$ is in the class $T_n^p(\lambda, A, B)$, then*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p+1-\lambda)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda]}|z|^{n+p} \tag{2.10}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p+1-\lambda)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda]}|z|^{n+p}. \tag{2.11}$$

Also,

$$|f'(z)| \leq p|z|^{p-1} + \frac{p(B-A)(\lambda p+1-\lambda)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda]}|z|^{n+p-1}, \tag{2.12}$$

and

$$|f'(z)| \geq |z|^{p-1} - \frac{p(B-A)(\lambda p+1-\lambda)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda]}|z|^{n+p-1}. \tag{2.13}$$

Proof. Suppose that $f \in T_n^p$ is in the class $T_n^p(\lambda, A, B)$, Then, from (1.1) we have

$$|f(z)| \leq |z|^p + \sum_{k=n+p}^{\infty} a_k z^k \leq |z|^p + |z|^{n+p}\sum_{k=n+p}^{\infty} a_k \tag{2.14}$$

and

$$|f(z)| \geq |z|^p - |z|^{n+p}\sum_{k=n+p}^{\infty} a_k. \tag{2.15}$$

Using (2.8), the first assertion of Lemma 2.2, in (2.14) and (2.15), we get (2.10) and (2.11).

Similarly, using (2.8) in the following inequality

$$|f'(z) - p|z|^{p-1}| \leq (n+p)\sum_{k=n+p}^{\infty} a_k|z|^{n+p-1}$$

we have (2.12) and (2.13). □

By setting $\lambda = 0$, $n = 1$ in Lemma 2.1 we get following result.

2.4. Corollary. (See Goel *at al.* [6, Theorem 1]) *If $f(z) \in T_1^p(0, A, B)$ then*

$$\sum_{n=1}^{\infty}[(1+B)n + p(B-A)]a_{n+p} \leq p(B-A). \tag{2.16}$$ □

By setting $\lambda = 0$, $n = 1$ in Theorem 2.3, we have the following result.

2.5. Corollary. (See Goel *at al.* [6, Theorem 3]) *If* $f(z) \in T_1^p(0, A, B)$ *then*

$$|z|^p - \frac{p(B-A)}{1+B+p(B-A)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p(B-A)}{1+B+p(B-A)}|z|^{p+1}. \tag{2.17}$$ □

Similarly, letting $\lambda = 0$, $n = 1$ in Theorem—2.3, we get:

2.6. Corollary. (See Goel *at al.* [6, Theorem 3]) *If* $f(z) \in T_1^p(0, A, B)$ *then*

$$\begin{aligned} p|z|^{p-1} - \frac{p(B-A)(1+p)}{1+B+p(B-A)}|z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{p(B-A)p}{1+B+p(B-A)}|z|^p. \end{aligned} \tag{2.18}$$ □

2.7. Corollary. *If* $f(z) \in T_n^p(1, A, B)$ *then*

$$\begin{aligned} &|z|^p - \frac{p^2(B-A)}{[(n+p)(1+B)-p(1+A)](n+p)}|z|^{n+p} \\ &\leq |f(z)| \\ &\leq |z|^p + \frac{p^2(B-A)}{[(n+p)(1+B)-p(1+A)](n+p)}|z|^{n+p}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &p|z|^{p-1} - \frac{p^2(B-A)}{[(n+p)(1+B)-p(1+A)]}|z|^{n+p-1} \\ &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{p^2(B-A)}{[(n+p)(1+B)-p(1+A)]}|z|^{n+p-1}. \end{aligned} \tag{2.20}$$ □

The distortion inequalities for functions in the class $K_n^p(\lambda, \mu, A, B)$ are given in Theorem 2.8 below.

2.8. Theorem. *If the function* $f \in T_n^p$ *is in the class* $K_n^p(\lambda, \mu, A, B)$, *then*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p+1-\lambda)(p+\mu)(p+\mu+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda](n+p+\mu)}|z|^{n+p} \tag{2.21}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p+1-\lambda)(p+\mu)(p+\mu+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)+1-\lambda](n+p+\mu)}|z|^{n+p}. \tag{2.22}$$

Proof. Suppose that the function $f(z) \in T_n^p$ is given by (1.1). Also that the function $g(z) \in T_n^p(\lambda, A, B)$ occurring in the non-homegenous differential equation (1.8) is given as in the definitions (1.4) and (1.5), with of course

$$b_k \geq 0, \ (k = n+p, n+p+1, n+p+2, \ldots).$$

Then we readily find from (1.8) that

$$a_k = \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k, \ (k = n+p, n+p+1, n+p+2, \ldots), \tag{2.23}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k z^k, \ (z \in U)$$

and

$$|f(z)| \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}b_k. \tag{2.24}$$

Since $g(z) \in T_n^p(\lambda, A, B)$, using the first assertion (2.8) of Lemma 2.2 we have the following inequality:

$$b_k \leq \frac{p(B-A)(\lambda p + 1 - \lambda)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}. \tag{2.25}$$

Together with (2.24) and (2.25), this yields that

$$\begin{aligned} |f(z)| \leq |z|^p + & \frac{p(B-A)(\lambda p + 1 - \lambda)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]} \\ & \times \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} |z|^{n+p}, \end{aligned} \tag{2.26}$$

by using the following identity

$$\sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} = \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)} - \frac{1}{(k+\mu+1)} = \frac{1}{(n+p+\mu)}, \tag{2.27}$$

where $\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, \ldots\}$.

The assertion (2.21) of Theorem 2.8 follows at once from (2.26) together with (2.27). The assertion (2.22) of Theorem 2.8 can be proven similarly. ☐

2.9. Corollary. *If $f(z) \in K_n^p(\lambda, \mu, -1, 1)$, then*

$$|f(z)| \leq |z|^p + \frac{p(\lambda p + 1 - \lambda)(p+\mu)(p+\mu+1)}{(n+p)[\lambda(n+p) + 1 - \lambda](n+\mu+1)} |z|^{n+p}, \tag{2.28}$$

and

$$|f(z)| \leq |z|^p - \frac{p(\lambda p + 1 - \lambda)(p+\mu)(p+\mu+1)}{(n+p)[\lambda(n+p) + 1 - \lambda](n+\mu+1)} |z|^{n+p}. \tag{2.29}$$ ☐

By setting $A = -1$, $B = 1$ in Theorem 2.8, we have the result [3, Theorem 1] of Altıntaş *at al.* on letting $\alpha = 0$.

3. Neighborhoods for the classes $T_n^p(\lambda, A, B)$ and $K_n^p(\lambda, \mu, A, B)$

In this section, we find inclusion relations for the classes $T_n^p(\lambda, A, B)$ and $K_n^p(\lambda, \mu, A, B)$ involving the (n, δ)-neighborhoods defined by (1.4) and (1.5).

3.1. Theorem. *If $f \in T_n^p$ is in the class $T_n^p(\lambda, A, B)$, then*

$$T_n^p(\lambda, A, B) \subset N_{n,\delta}(h; f), \tag{3.1}$$

where $h(z)$ is given by (1.6) *and*

$$\delta = \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}. \tag{3.2}$$

Proof. The relation (3.1) asserted by Theorem 3.1 follows easily from the definition (1.5) of $N_{n,\delta}(h; f)$ with $g(z)$ replaced by $f(z)$, and the second assertion (2.9) of Lemma 2.2. ☐

3.2. Theorem. *If $f \in T_n^p$ is in the class $K_n^p(\lambda, \mu, A, B)$ then*

$$K_n^p(\lambda, \mu, A, B) \subset N_{n,\delta}(g; f), \tag{3.3}$$

where $g(z)$ is given by (1.8) *and*

$$\delta = \frac{p(B-A)(\lambda p + 1 - \lambda)[n + (p+\mu)(p+\mu+2)](n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda](n+p+\mu)}.$$

Proof. Suppose that $f(z) \in K_n^p(\lambda, \mu, A, B)$. Then, upon substituting from (2.23) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k, \ (a_k \geq 0, \ b_k \geq 0) \tag{3.4}$$

we easily obtain

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} kb_k. \tag{3.5}$$

Since $g(z) \in T_n^p(\lambda, A, B)$, the second assertion (2.9) of Lemma 2.2 yields

$$kb_k \leq \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}. \tag{3.6}$$

By making use of (2.9) as well as (3.6) on the right hand side of (3.5), we find that

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]} \times \left(1 + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}\right),$$

which, by virtue of the teloscopic sum (2.27), immediately yields

$$\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \frac{p(B-A)(\lambda p + 1 - \lambda)[n + (p+\mu)(p+\mu+2)](n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]((n+p+\mu))} = \delta.$$

So, in definition (1.4) with $g(z)$ interchanged by $f(z)$, we conclude that

$$f \in N_{n,\delta}(g; f).$$

This completes the proof of Theorem 3.2. □

If we let $A = -1$, $B = 1$ in Theorem 3.2 we have the following corollary.

3.3. Corollary. *If $f(z) \in K_n^p(\lambda, \mu, -1, 1)$, then*

$$K_n^p(\lambda, \mu, -1, 1) \subset N_{n,\delta}(g; f),$$

where $g(z)$ is given by (1.8) *and*

$$\delta = \frac{p(\lambda p + 1 - \lambda)[n + (p+\mu)(p+\mu+2)]}{[\lambda(n+p) + 1 - \lambda]((n+p+\mu))}.$$

This result was given in Altıntaş *et al.* [3, Theorem 3] for $\alpha = 0$.

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