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OSCILLATION OF FOURTH-ORDER DYNAMIC EQUATIONS

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Abstract

In this paper we shall reduce the problem of the oscillation of all solutions of certain nonlinear fourth-order dynamic equations to the problem of oscillation of two second-order dynamic equations, which are discussed intensively in the literature. Further oscillation criteria of fourth-order equations are given and proved using integration and Taylor's formula on time scales. Some conditions are presented that ensure that all bounded solutions of the equation are oscillatory.

Keywords: Oscillation, Fourth-order, Dynamic equation, Time scales.

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1. Introduction

Consider the fourth-order nonlinear dynamic equation

$$x^{\Delta^4}(t) + q(t)x^{\lambda}(t) = 0, \tag{1.1}$$

where λ is the ratio of two positive odd integers and q is a real-valued positive and rd-continuous function on a time scale $\mathbb{T} \subset \mathbb{R}$ with $\sup \mathbb{T} = \infty$. Fourth-order differential equations (i.e., $\mathbb{T} = \mathbb{R}$) and difference equations (i.e., $\mathbb{T} = \mathbb{N}$) have been deeply investigated in the literature, see e.g., [7, 14, 16, 18, 21] for differential equations and [6, 13, 19, 22–24] for difference equations.

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We recall that a solution of equation (1.1) is said to be *oscillatory* on $[t_0, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be *oscillatory* in case all of its solutions are oscillatory. For even-order differential equations (1.1), this is sometimes referred to as *Property A* (see, e.g., [18, Definition 1.1]) or *strong oscillation* (see, e.g., [14, Definition 8.22]).

The study of dynamic equations on time scale is a fairly new topic, and work in this area is rapidly growing. In the last years there has been much research activity concerning the oscillation of solutions of some dynamic equations on time scales. For recent contributions we refer the reader to the books [2–5, 10, 11], the papers [1, 8, 12, 15, 17, 20] and the references cited therein. However, most of the results obtained have centered around second-order dynamic equations on time scales, and there are very few results dealing with the qualitative behavior of solutions of higher-order dynamic equations on time scales.

The main purpose of this paper is to establish new criteria for the oscillation of equation (1.1) via comparison with two second-order dynamic equations whose oscillatory character are known.

2. Main results

Consider the inequalities

$$x^{\Delta\Delta}(t) + q(t)x^{\lambda}(t) \leq 0, \tag{2.1}$$

$$x^{\Delta\Delta}(t) + q(t)x^{\lambda}(t) \geq 0, \tag{2.2}$$

and the equation

$$x^{\Delta\Delta}(t) + q(t)x^{\lambda}(t) = 0, \tag{2.3}$$

where q and λ are as in equation (1.1).

We shall first prove the following lemma.

2.1. Lemma. *If inequality* (2.1) *(inequality* (2.2)*) has an eventually positive (negative) solution, then equation* (2.3) *also has an eventually positive (negative) solution.*

Proof. Let x be an eventually positive solution of (2.1), say $x(t) > 0$ for all $t \geq t_0$. Then $x^{\Delta\Delta}(t) < 0$ for all $t \geq t_0$ so that x^{Δ} is decreasing. Hence $x^{\Delta}(t) > 0$ for all $t \geq t_0$ or there exists $t_1 \geq t_0$ such that $x^{\Delta}(t) < 0$ for all $t \geq t_1$. In the latter case,

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(\tau)\Delta\tau \leq x(t_1) + \int_{t_1}^{t} x^{\Delta}(t_1)\Delta\tau = x(t_1) + (t - t_1)x^{\Delta}(t_1),$$

for all $t \geq t_1$, yielding a contradiction to the positivity of x. Hence $x^{\Delta}(t) > 0$ for all $t \geq t_0$. Let

$$y(t) = x^{\Delta}(t),$$

and integrating this equation from t_0 to $t \geq t_0$, we get

$$x(t) = x(t_0) + \int_{t_0}^{t} y(s)\Delta s$$

so that (2.1) becomes

$$y^{\Delta}(t) + q(t)\left(x(t_0) + \int_{t_0}^{t} y(s)\Delta s\right)^{\lambda} \leq 0 \text{ for } t \geq t_0. \tag{2.4}$$

Integrating (2.4) from t to $u \geq t \geq t_0$ and letting $u \to \infty$, we have

$$y(t) \geq \int_t^\infty q(s) \left(x(t_0) + \int_{t_0}^s y(\tau)\Delta\tau \right)^\lambda \Delta s.$$

Now we define a sequence of successive approximations $\{z_j(t)\}_{j\in\mathbb{N}_0}$ by

$$z_0(t) = y(t)$$

$$z_{j+1}(t) = \int_t^\infty q(s) \left(x(t_0) + \int_{t_0}^s z_j(\tau)\Delta\tau \right)^\lambda \Delta s, \ j \in \mathbb{N}_0, \ t \geq t_0.$$

By induction, we can easily prove that

$$0 < z_j(t) \leq y(t) \ \text{ and } \ z_{j+1}(t) \leq z_j(t), \ j \in \mathbb{N}_0, \ t \geq t_0.$$

Thus, the sequence $\{z_j(t)\}_{j\in\mathbb{N}_0}$ is positive and nonincreasing in j for each $t \geq t_0$. This means we may define

$$z(t) = \lim_{j\to\infty} z_j(t) > 0.$$

Since $0 < z(t) \leq z_j(t) \leq y(t)$ for all $j \in \mathbb{N}_0$, we see that

$$\left(x(t_0) + \int_{t_0}^t z_j(s)\Delta s \right)^\lambda \leq \left(x(t_0) + \int_{t_0}^t y(s)\Delta s \right)^\lambda.$$

Now, by the Lebesgue dominated convergence theorem [8], one can easily obtain

$$z(t) = \int_t^\infty q(s) \left(x(t_0) + \int_{t_0}^s z(\tau)\Delta\tau \right)^\lambda \Delta s,$$

and, by differentiation,

$$z^\Delta(t) = -q(t) \left(x(t_0) + \int_{t_0}^t z(s)\Delta s \right)^\lambda = -q(t)v^\lambda(t), \tag{2.5}$$

where

$$v(t) = x(t_0) + \int_{t_0}^t z(s)\Delta s.$$

Then $v(t) > 0$ for $t \geq t_0$ and

$$v^\Delta(t) = z(t). \tag{2.6}$$

From (2.5) and (2.6), we see that

$$v^{\Delta\Delta}(t) + q(t)v^\lambda(t) = 0.$$

Hence equation (2.3) has the positive solution v. For the case (2.2), the argument is similar and hence is omitted. This completes the proof. □

We shall employ the following notation. Let (see [10, Section 1.6])

$$h_1(t,s) = t - s \ \text{ and } \ h_2(t,s) = \int_s^t (\tau - s)\Delta\tau \ \text{ with } \ t, s \in \mathbb{T},$$

assume

$$\int_{t_0}^\infty q(s)\Delta s < \infty, \tag{2.7}$$

and define

$$Q_1(t) := \int_t^\infty \int_s^\infty q(\tau)\Delta\tau\Delta s$$

and

$$Q_2(t) := \left(\frac{\alpha - t_0}{t - t_0} h_2(t,\alpha)\right)^{\lambda} q(t) \text{ for } \alpha \in \mathbb{T}^{\kappa},\ t \in \mathbb{T} \text{ and } t \geq \alpha > t_0.$$

Note (using [10, Theorem 1.117 and Corollary 1.68(iii)]) that Q_1 is well defined iff

$$\int_t^{\infty} h_1(\sigma(s),t)q(s)\Delta s < \infty \text{ for all } t \geq t_0,$$

which holds (taking the derivative with respect to t and noticing that it is negative) iff

$$\int_{t_0}^{\infty} h_1(\sigma(s),t_0)q(s)\Delta s < \infty,$$

which holds (assuming (2.7) and using the definition of h_1) iff

$$\int_{t_0}^{\infty} \sigma(s)q(s)\Delta s < \infty. \tag{2.8}$$

Now, we present our main result.

2.2. Theorem. *Assume* (2.7) *and* (2.8). *If both second-order dynamic equations*

$$y^{\Delta\Delta}(t) + Q_1(t)y^{\lambda}(t) = 0 \tag{2.9}$$

and

$$z^{\Delta\Delta}(t) + Q_2(t)z^{\lambda}(t) = 0 \tag{2.10}$$

are oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0$. Since $x^{\Delta^4}(t) < 0$ for $t \geq t_0$, there exists $t_1 \geq t_0$ such that $x^{\Delta}(t)$, $x^{\Delta\Delta}(t)$, $x^{\Delta^3}(t)$ each is of constant sign on $[t_1,\infty)$. There are 8 different sign combination for these functions. Since (similarly as was shown in the proof of Lemma 2.1) it is not possible that $x^{\Delta^i}(t) > 0$, $x^{\Delta^{i+1}}(t) < 0$, $x^{\Delta^{i+2}}(t) < 0$ and since also $x^{\Delta^i}(t) < 0$, $x^{\Delta^{i+1}}(t) > 0$, $x^{\Delta^{i+2}}(t) > 0$ is not possible ($i \in \{0,1,2\}$), we are left only with the following two possibilities:

(I) $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) > 0$, $x^{\Delta^3}(t) > 0$, and $x^{\Delta^4}(t) < 0$ for $t \geq t_1$,
(II) $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x^{\Delta^3}(t) > 0$, and $x^{\Delta^4}(t) < 0$ for $t \geq t_0$.

Case (I). Suppose $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) > 0$, $x^{\Delta^3}(t) > 0$, and $x^{\Delta^4}(t) < 0$ for $t \geq t_1$. Then

$$x^{\Delta\Delta}(t) = x^{\Delta\Delta}(t_1) + \int_{t_1}^{t} x^{\Delta^3}(s)\Delta s > (t - t_1)x^{\Delta^3}(t) \text{ for } t \geq t_1.$$

Thus

$$\left(\frac{x^{\Delta\Delta}}{h_1(\cdot,t_1)}\right)^{\Delta}(t) = \frac{(t-t_1)x^{\Delta^3}(t) - x^{\Delta\Delta}(t)}{(t-t_1)(\sigma(t)-t_1)} < 0 \text{ for all } t \geq t_1. \tag{2.11}$$

Therefore, the function $\frac{x^{\Delta\Delta}}{h_1(\cdot,t_1)}$ is decreasing on $[t_1,\infty)$. By applying Taylor's formula (i.e., [10, Theorem 1.113] with $n = 3$), for $t_2 \in \mathbb{T}^{\kappa}$, $t_2 > t_1$ and $t \in \mathbb{T}$,

$$x(t) \geq h_2(t,t_2)x^{\Delta\Delta}(t_2) \text{ for all } t \geq t_2. \tag{2.12}$$

Using (2.11) and (2.12), we have

$$x(t) \geq h_2(t,t_2)\frac{t_2 - t_1}{t - t_1}x^{\Delta\Delta}(t) \text{ for all } t \geq t_2 > t_1. \tag{2.13}$$

From equation (1.1), we see that

$$(2.14)\quad -x^{\Delta^4}(t) = q(t)x^\lambda(t) \ge \left(\frac{t_2 - t_1}{t - t_1}\right)^\lambda ((h_2(t,t_2))^\lambda q(t)(x^{\Delta\Delta}(t))^\lambda = Q_2(t)(x^{\Delta\Delta}(t))^\lambda.$$

Setting $x^{\Delta\Delta}(t) = y(t)$ in (2.14), we get

$$y^{\Delta\Delta}(t) + Q_2(t)y^\lambda(t) \le 0 \ \text{ for } \ t \ge t_2.$$

By Lemma 2.1, equation (2.10) has an eventually positive solution, which contradicts the hypothesis.

Case (II). Suppose $x(t) > 0$, $x^\Delta(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x^{\Delta^3}(t) > 0$, and $x^{\Delta^4}(t) < 0$ for $t \ge t_0$. Integrating equation (1.1) twice from $t \ge t_0$ to $u \ge t$ and letting $u \to \infty$, we have

$$-x^{\Delta\Delta}(t) \ge \left(\int_t^\infty\int_s^\infty q(\tau)\Delta\tau\Delta s\right) x^\lambda(t) = Q_1(t)x^\lambda(t) \ \text{ for } \ t \ge t_0.$$

By Lemma 2.1, equation (2.9) has an eventually positive solution, which contradicts the hypothesis.

This completes the proof. □

We note that the oscillatory behavior of second-order dynamic equations of the form (2.9), (2.10) are studied intensively in the literature, and for recent contributions, we refer the reader to the papers [1, 12, 15, 20] and the references cited therein.

2.3. Example. Let $\mathbb{T} = \mathbb{R}$. If $q(t) = 6/t^4$ for all $t \ge 1$, then $Q_1(t) = 1/t^2$ and thus (2.9) is oscillatory as it is well known that

$$x'' + \frac{\gamma}{t^2}x = 0 \ \text{ is oscillatory if } \ \gamma > \frac{1}{4}.$$

2.4. Remark. If $\mathbb{T} = \mathbb{R}$ and $\lambda = 1$, then Theorem 2.2 implies [14, Theorem 8.32]. More precisely, (2.9) asserts in this case

$$Q_1(t) = \int_t^\infty (s-t)q(s)\mathrm{d}s < \infty,$$

and thus the condition assumed in [14, Theorem 8.32 with $n = 4$] holds.

Next, we let

$$h_3(t,s) = \int_s^t h_2(\tau,s)\Delta\tau \ \text{ with } \ t, s \in \mathbb{T}$$

and

$$Q(t) := \int_t^\infty q(s)\Delta s, \quad Q^*(t) := \int_t^\infty\int_s^\infty Q(\tau)\Delta\tau\Delta s \ \text{ for } \ t \ge t_0.$$

Note that Q is well defined iff (2.7) holds and Q^* is well defined (using [10, Theorem 1.117, Corollary 1.68(iii), and (1.9)]) iff

$$\int_t^\infty h_2(t,\sigma(s))q(s)\Delta s < \infty \ \text{ for all } \ t \ge t_0,$$

which holds (taking the derivative with respect to t and noticing that it is negative) iff

$$(2.15)\quad \int_{t_0}^\infty h_2(t_0,\sigma(s))q(s)\Delta s < \infty.$$

Now, we obtain the following interesting result.

2.5. Theorem. *Assume* (2.7) *and* (2.15). *If* $\lambda = 1$, *and for* $t \geq t_0$,

$$\limsup_{t\to\infty} \{h_3(t,t_0)Q(t)\} > 1 \tag{2.16}$$

and

$$\limsup_{t\to\infty} \{h_1(t,t_0)Q^*(t)\} > 1, \tag{2.17}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0$. As in the proof of Theorem 2.2, only the following two cases are possible:

Case (I). Suppose $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) > 0$, $x^{\Delta^3}(t) > 0$, and $x^{\Delta^4}(t) < 0$ for $t \geq t_1$. Then

$$x^{\Delta\Delta}(t) = x^{\Delta\Delta}(t_1) + \int_{t_1}^{t} x^{\Delta^3}(s)\Delta s \geq (t-t_1)x^{\Delta^3}(t) \text{ for } t \geq t_1. \tag{2.18}$$

Integrating (2.18) twice from t_1 to $t \geq t_1$, we get

$$x(t) \geq h_3(t,t_1)x^{\Delta^3}(t) \text{ for all } t \geq t_1. \tag{2.19}$$

Integrating equation (1.1) from $t \geq t_1$ to $u \geq t$ and letting $u \to \infty$, we have

$$x^{\Delta^3}(t) \geq \int_t^\infty q(s)x(s)\Delta s \geq Q(t)x(t) \text{ for } t \geq t_1. \tag{2.20}$$

Using (2.20) in (2.19), we obtain

$$x(t) \geq h_3(t,t_1)x^{\Delta^3}(t) \geq h_3(t,t_1)Q(t)x(t) \text{ for } t \geq t_1 \tag{2.21}$$

so that

$$1 \geq h_3(t,t_1)Q(t) \text{ for } t \geq t_1. \tag{2.22}$$

Taking the limit superior on both sides of (2.22) as $t \to \infty$, we obtain a contradiction to condition (2.16).

Case (II). Suppose $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x^{\Delta^3}(t) > 0$ and $x^{\Delta^4}(t) < 0$ for $t \geq t_0$. Integrating equation (1.1) thrice from $t \geq t_0$ to $u \geq t$ and letting $u \to \infty$, we obtain for $t \geq t_0$

$$x^{\Delta^3}(t) \geq Q(t)x(t), \quad -x^{\Delta\Delta}(t) \geq \left(\int_t^\infty Q(s)\Delta s\right)x(t),$$

and

$$x^{\Delta}(t) \geq \left(\int_t^\infty\int_s^\infty Q(\tau)\Delta\tau\Delta s\right)x(t) = Q^*(t)x(t). \tag{2.23}$$

Also, we see that

$$x(t) = x(t_0) + \int_{t_0}^{t} x^{\Delta}(s)\Delta s \geq (t-t_0)x^{\Delta}(t) \text{ for } t \geq t_0. \tag{2.24}$$

Using (2.23) in (2.24), we get

$$x(t) \geq (t-t_0)x^{\Delta}(t) \geq (t-t_0)Q^*(t)x(t) \text{ for } t \geq t_0 \tag{2.25}$$

so that

$$1 \geq (t-t_0)Q^*(t) \text{ for } t \geq t_0. \tag{2.26}$$

Taking the limit superior on both sides of (2.26) as $t \to \infty$, we obtain a contradiction to condition (2.17).

This completes the proof. $\square$

We may combine conditions (2.16) and (2.17) in one by letting

$$H(t,t_0)=\min\{h_3(t,t_0)Q(t),h_1(t,t_0)Q^*(t)\},\quad t\ge t_0. \tag{2.27}$$

Now Theorem 2.2 takes the following form.

2.6. Corollary. *Assume* (2.7) *and* (2.15). *If* $\lambda=1$ *and*

$$\limsup_{t\to\infty} H(t,t_0)>1, \tag{2.28}$$

then equation (1.1) *is oscillatory.*

2.7. Example. Let $\mathbb{T}$ be a time scale with $\sup\mathbb{T}=\infty$ and assume there exists $\alpha>1$ with $\sigma(t)\le\alpha t$ for all $t\in\mathbb{T}$. Let $t_0\in\mathbb{T}$ and define

$$q(t)=\frac{2h_2(t,t_0)}{h_3(t,t_0)h_3(\sigma(t),t_0)}\ \text{ for all }\ t\ge t_0.$$

Then

$$Q(t)=2\int_t^\infty \frac{h_2(\tau,t_0)}{h_3(\tau,t_0)h_3(\sigma(\tau),t_0)}\Delta\tau=2\int_t^\infty\left(\frac{1}{h_3(\cdot,t_0)}\right)^{\Delta}(\tau)\Delta\tau=\frac{2}{h_3(t,t_0)}$$

so that (2.7) is satisfied. Next, using [9, Theorem 4.1] and [10, Theorem 1.24],

$$\begin{aligned}
Q_1(t)&=2\int_t^\infty\frac{\Delta\tau}{h_3(\tau,t_0)}\ge 12\int_t^\infty\frac{\Delta\tau}{(\tau-t_0)^3}\\
&=6\int_t^\infty\left\{\frac{1}{(\tau-t_0)(\tau-t_0)^2}+\frac{1}{(\tau-t_0)^2(\tau-t_0)}\right\}\Delta\tau\\
&\ge 6\int_t^\infty\left\{\frac{1}{(\tau-t_0)(\sigma(\tau)-t_0)^2}+\frac{1}{(\tau-t_0)^2(\sigma(\tau)-t_0)}\right\}\Delta\tau=\frac{6}{(t-t_0)^2}
\end{aligned}$$

and

$$Q^*(t)\ge 6\int_t^\infty\frac{\Delta\tau}{(\tau-t_0)^2}\ge 6\int_t^\infty\frac{\Delta\tau}{(\tau-t_0)(\sigma(\tau)-t_0)}=\frac{6}{t-t_0}.$$

Moreover, (2.8) and (2.15) are also satisfied according to our assumption on $\mathbb{T}$ and [9, Theorem 4.2]. Now we find

$$H(t,t_0)=\min\{h_3(t,t_0)Q(t),h_1(t,t_0)Q^*(t)\}=\min\{2,6\}=2>1,$$

so by Corollary 2.6, all solutions of (1.1) are oscillatory.

2.8. Remark. If $\mathbb{T}=\mathbb{R}$, then Theorem 2.5 (or Corollary 2.6) implies the second sufficiency statement of [18, Theorem 1.6 with $n=4$]. More precisely, (2.16) implies [18, (1.58) with $n=4$], i.e.,

$$\limsup_{t\to\infty}\left\{t\int_t^\infty s^2q(s)\mathrm{d}s\right\}>6. \tag{2.29}$$

To see this, note that (2.16) for $\mathbb{T}=\mathbb{R}$ and $t_0=0$ means

$$\limsup_{t\to\infty}\left\{\frac{t^3}{6}\int_t^\infty q(s)\mathrm{d}s\right\}>1. \tag{2.30}$$

Since

$$t\int_t^\infty s^2q(s)\mathrm{d}s\ge t\int_t^\infty t^2q(s)\mathrm{d}s=t^3\int_t^\infty q(s)\mathrm{d}s,$$

(2.30) implies (2.29).

The following result is concerned with the oscillation of all bounded solutions of equation (1.1).

2.9. Theorem. *Assume* (2.7) *and* (2.15). *If* $\lambda = 1$ *and condition* (2.17) *holds, then all bounded solutions of equation* (1.1) *are oscillatory.*

Proof. Let x be a bounded nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_0$. Then we see that Case (I) is disregarded. Therefore, we consider only Case (II). The rest of the proof is similar to the proof of Case (II) in Theorem 2.5 and hence is omitted. This completes the proof. □

From the proof of Theorem 2.2, we observe that if $x(t) > 0$ for $t \geq t_0$ holds for a solution x of equation (1.1), then x satisfies (I) or (II) and so, x may be unbounded, and in the case when x satisfies (II) only, x may be bounded. We also see that the inequalities (2.21) and (2.25) when $\lambda \neq 1$ take the form

$$x(t) \geq h_3(t, t_1)Q(t)x^{\lambda}(t) \quad \text{for} \quad t \geq t_1$$

and

$$x(t) \geq h_1(t, t_0)Q^*(t)x^{\lambda}(t) \quad \text{for} \quad t \geq t_0.$$

Now, we obtain the following oscillation criteria when x is an unbounded solution of equation (1.1) and satisfies (I) or (II).

2.10. Theorem. *Assume* (2.7) *and* (2.15). *If* $\lambda > 1$ *and*

$$\limsup_{t\to\infty}\{h_3(t, t_0)Q(t)\} > 0 \quad \textit{and} \quad \limsup_{t\to\infty}\{h_1(t, t_0)Q^*(t)\} > 0$$

or

$$\limsup_{t\to\infty} H(t, t_0) > 0,$$

then equation (1.1) *is oscillatory.* □

2.11. Theorem. *Assume* (2.7) *and* (2.15). *If* $0 < \lambda < 1$ *and*

$$\limsup_{t\to\infty}\{h_3(t, t_0)Q(t)\} = \infty \quad \textit{and} \quad \limsup_{t\to\infty}\{h_1(t, t_0)Q^*(t)\} = \infty$$

or

$$\limsup_{t\to\infty} H(t, t_0) = \infty,$$

then equation (1.1) *is oscillatory.* □

The following result deals with the oscillation of all bounded solutions of equation (1.1).

2.12. Theorem. *Assume* (2.7) *and* (2.15). *If for every constant* $c > 0$,

$$\limsup_{t\to\infty}\{h_1(t, t_0)Q^*(t)\} > c,$$

then all bounded solutions of equation (1.1) *with* $\lambda \neq 1$ *are oscillatory.* □

2.13. Remark. We may employ many types of time scale e.g., $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^p$ with $p > 1$ etc., see [10, 11]. The details are left to the reader.

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