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AN APPLICATION OF RITT-WU'S ZERO DECOMPOSITION ALGORITHM TO NULL BERTRAND TYPE CURVES IN MINKOWSKI 3-SPACE

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Abstract

Bertrand curves were first studied using a computer by W. -T. Wu in (*A mechanization method of geometry and its applications II. Curve pairs of Bertrand type*, Kexue Tongbao **32**, 585–588, 1987). The same problem was studied using an improved version of Ritt-Wu's decomposition algorithm by S. -C. Chao and X. -S. Gao (*Automated reasoning in differential geometry and mechanics: Part 4: Bertrand curves*, System Sciences and Mathematical Sciences **6** (2), 186–192, 1993).

In this paper, we investigate the same problem for null Bertrand type curves in Minkowski 3-space $\mathbb{E}_1^3$ by using the well known algorithm given by Chao and Gao, and obtain new results for null Bertrand type curves in Minkowski 3-space $\mathbb{E}_1^3$.

Keywords: Mechanical theorem proving, Ritt-Wu's method, Bertrand curves, Mannheim curves, Minkowski 3-space, Null curves.

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1. Introduction

The general theory of curves in an Euclidean space (or more generally, in a Riemannian manifold) has been developed a long time ago and we have a deep knowledge of its local geometry as well as its global geometry. In the theory of curves in Euclidean space, one of the important and interesting problems is the characterizations of a regular curve. In the solution of the problem, the curvature functions k_1 (or $\varkappa$) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic,

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or if $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then the curve is a circle with radius $1/k_1$, etc. Thus we can determine the shape and size of a regular curve by using its curvatures.

Another approach to the solution of the problem is considering the relationship between the Frenet vectors of the curves (see [10]). For instance, for Bertrand curves:

In 1845, Saint Venant (see [18]) proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by Bertrand in 1850 in a paper (see [2]) in which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote the first and second curvatures of the given curve by k_1 and k_2 respectively, then for $\lambda, \mu \in \mathbb{R}$ we have $\lambda k_1 + \mu k_2 = 1$. Since the time of Bertrand's paper, pairs of curves of this kind have been called *Conjugate Bertrand Curves*, or more commonly just *Bertrand Curves* (see [10]). Bertrand curves have been studied in Euclidean and non-Euclidean spaces by many authors (for example, see [1, 3, 8]).

Another interesting example is that of Mannheim curves:

If there exists a corresponding relationship between the space curves α and β such that, at corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of β, then α is called a *Mannheim curve*, β is called the *Mannheim partner curve* of α. Mannheim partner curves were studied by Liu and Wang (see [11]) in Euclidean 3-space and in Minkowski 3-space.

Euclidean geometry is geometry in an affine space arising from the existence of a positive definite inner product among its vectors. When such an inner product is replaced by an nondegenerate inner product of signature $(-, +, +, \ldots, +)$, what results is called Lorentzian geometry (see [10, 12]). It is well known that Lorentzian geometry of 4 dimensions (also known Minkowski space-time) is the most appropriate mathematical model for the special theory of relativity. The theory of curves in Minkowski 3-space is more interesting than the Euclidean case.

Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike curves or timelike curves can be studied using a similar approach to that used in positive definite Riemannian geometry (see [5, 12]). However, null curves have many properties very different from spacelike or timelike curves. In other words, null curve theory has many results which have no Riemannian analogues. In the geometry of null curves, difficulties arise since the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. The importance of the study of null curves and its presence in the physical theories is clear from the fact that the classical relativistic string is a surface or world-sheet in Minkowski space which satisfies the Lorentzian analogue of the minimal surface equation.

Null curves have been studied in Minkowski 3-space, Minkowski spacetime, Lorentzian space and Lorentzian manifolds, and semi-Riemannian spaces with index 2 by many authors [3, 4, 6, 7, 8]. Null Bertrand curves (in the classical sense, i.e. at the corresponding points of the given two curves, the principal normal lines of one curve coincides with the principal normal lines of the other curve) have been studied in Minkowski 3-space by Balgetir, Bektaş and Inoguchi in [1]. They showed that null Bertrand curves are null geodesic or Cartan framed null curves with constant second curvature.

1.1. An improved version of Ritt-Wu's decomposition algorithm. Proving theorems in differential geometry mechanically was initiated by Professor Wen-Tsun Wu, following the mechanical thought of ancient Chinese mathematics. Wu began to work on mechanical theorem proving in geometry in 1976 (see [19, 21, 23]), and published his first paper the year after. He extended the characteristic set method, a method developed by J. F. Ritt [13] in algebraic geometry and differential algebra, to a well-ordering principle that can be used for mechanical theorem proving, and discovering in differential geometry and mechanics. This method is now widely known as Wu's method. Wu's method is capable of proving and discovering theorems in differential geometry and mechanics mechanically and efficiently. For example, the theorems of Bertrand, Mannheim and Schell (see [22]) may be proved or even discovered automatically and so may Newton's laws be derived from Kepler's laws using an implementation of this method [20].

An improved version of Ritt-Wu's decomposition algorithm was obtained by Chou and Gao [14]. They improved the original algorithm in two aspects. First, by using a weak ascending chain and W-prem, the sizes of the differential polynomials occurring in the decomposition can be reduced. Second, by using a special reduction procedure, the number of branches in the decomposition can be controlled effectively. A detailed description of the improved version of Ritt-Wu's decomposition algorithm and its applications can be found in the papers of Chou and Gao [14, 15, 16, 17].

The Bertrand curves problem was first studied using a computer by Wu [22]. The same problem was studied using the improved version of Ritt-Wu's decomposition algorithm by Chou and Gao [17]. They studied 18 types of Betrand curves in metric and affine differential geometry in Euclidean 3-space. By using the algorithm, pseudo null Bertrand curves were studied by the present authors in [9].

In this paper, we investigate the null Bertrand type curves by using the improved version of Ritt-Wu's decomposition in Minkowski 3-space. We show that the algorithm works successfully for null curves in Minkowski 3-space, and we give previously unknown results for such curves in the same space.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3 \setminus \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null* (*lightlike*) if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ [12]. A null curve α has unit speed, if $g(\alpha''(s), \alpha''(s)) = \pm 1$.

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields, respectively. If α is a null curve, the Frenet equations are given by [5, 8]:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \kappa_2 & 0 & -\kappa_1 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where $g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0$ and $g(N,N) = g(T,B) = 1$, The first curvature $\kappa_1(s) = 0$, if $\alpha(s)$ is straight line, or $\kappa_1(s) = 1$ in all other cases.

3. Application of the improved version of Ritt-Wu's decomposition algorithm to null Bertrand type curves

In this section, we characterize the null Bertrand curves by using the improved version of Ritt-Wu's decomposition algorithm given in [17].

Let us consider two null curves C_1 and C_2 in $\mathbb{E}_1^3$, and let us attach the moving frames $\{e_{11}, e_{12}, e_{13}\}$ and $\{e_{21}, e_{22}, e_{23}\}$ to C_1 and C_2 at the corresponding points of C_1 and C_2, respectively. For shortness, we denote the curves and their moving frames by $(C_1, e_{11}, e_{12}, e_{13})$ and $(C_2, e_{21}, e_{22}, e_{23})$. In addition, we denote the arcs, curvatures and torsions of C_1 and C_2 by s_1, k_1, t_1 and s_2, k_2, t_2, respectively. Here, the parameter s_2 can be considered as a function of s_1, and we put $r = \frac{ds_2}{ds_1}$. The vectorial relationship between C_1 and C_2 can be given as follows:

$$C_2 = C_1 + a_1 e_{11} + a_2 e_{12} + a_3 e_{13}, \tag{3.1}$$

$$\left.\begin{aligned} e_{21} &= u_{11} e_{11} + u_{12} e_{12} + u_{13} e_{13} \\ e_{22} &= u_{21} e_{11} + u_{22} e_{12} + u_{23} e_{13} \\ e_{23} &= u_{31} e_{11} + u_{32} e_{12} + u_{33} e_{13} \end{aligned}\right\} \tag{3.2}$$

where a_i, $(i = 1, 2, 3)$, are variables and $U = (u_{ij})$ is a matrix of variables satisfying certain relations which will be presented in the following sections.

In this paper, we mainly consider cases which are more general than the classical Bertrand curve for a given couple of null curves. These cases can be given (with indices i, j) in the following forms:

MI_{ij}: $(1 \le i \le j \le 3)$, means that e_{2j} is identical with e_{1i} in metrical structure.

MP_{ij}: $(1 \le i \le j \le 3)$, means that e_{2j} is parallel with e_{1i} in metrical structure.

We will consider Bertrand type null curves C_1 and C_2 in $\mathbb{E}_1^3$ satisfying the conditions MI_{ij} and MP_{ij}. Thus we will investigate 12 kinds of Bertrand type null curves in Minkowski 3-space $\mathbb{E}_1^3$.

3.1. Bertrand type null curves in Minkowski 3-space. As determined above, let $\{e_{11}, e_{12}, e_{13}\}$ and $\{e_{21}, e_{22}, e_{23}\}$ be the Frenet frames of C_1 and C_2, respectively. Differentiating these vectors with respect to s_1, we get the following Frenet formulae.

$$e'_{11} = k_1 e_{12},\ e'_{12} = t_1 e_{11} - k_1 e_{13},\ e'_{13} = -t_1 e_{12}, \tag{3.3}$$

$$e'_{21} = r k_2 e_{22},\ e'_{22} = r t_2 e_{21} - r k_2 e_{23},\ e'_{23} = -r t_2 e_{22}. \tag{3.4}$$

We know from [4, 5] that for null curves, having $k_1 = 0$ is equivalent to the curve being part of a straight line. This case will be excluded throughout this paper, that is, we assume that $k_1 = 1$ and $k_2 = 1$. With these assumptions (3.3) and (3.4) become,

$$\begin{aligned} &e'_{11} = e_{12},\ e'_{12} = t_1 e_{11} - e_{13},\ e'_{13} = -t_1 e_{12}, \\ &e'_{21} = r e_{22},\ e'_{22} = r t_2 e_{21} - r e_{23},\ e'_{23} = -r t_2 e_{22}. \end{aligned} \tag{3.5}$$

Differentiating (3.1) and (3.2); eliminating e'_{11}, e'_{12}, e'_{13}, e'_{21}, e'_{22} and e'_{23} using (3.5) and (3.6); eliminating e_{21}, e_{22} and e_{23} using (3.2); and finally comparing coefficients for the vectors e_{11}, e_{12} and e_{13}, we obtain

$$\left.\begin{aligned} &a'_1 + t_1 a_2 - r u_{11} + 1 = 0 \\ &a'_2 + a_1 - t_1 a_3 - r u_{12} = 0 \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} &u'_{11} + t_1 u_{12} - r u_{21} = 0 \\ &u'_{12} + u_{11} - t_1 u_{13} - r u_{22} = 0 \\ &u'_{13} - u_{12} - r u_{23} = 0, \end{aligned}\right\} \tag{3.7}$$

$$\left.\begin{aligned} &u'_{21}+t_1u_{22}-rt_2u_{11}+ru_{31}=0\\ &u'_{22}+u_{21}-t_1u_{23}-rt_2u_{12}+ru_{32}=0\\ &u'_{23}-u_{22}-rt_2u_{13}+ru_{33}=0,\end{aligned}\right\} \tag{3.8}$$

$$\left.\begin{aligned} &u'_{31}+t_1u_{32}+rt_2u_{21}=0\\ &u'_{32}+u_{31}-t_1u_{33}+rt_2u_{22}=0\\ &u'_{33}-u_{32}+rt_2u_{23}=0.\end{aligned}\right\} \tag{3.9}$$

In addition, from (3.2), (u_{ij}) must satisfy;

$$\left.\begin{aligned} &u_{12}^2+2u_{11}u_{13}=0,\\ &u_{22}^2+2u_{21}u_{23}=1,\\ &u_{32}^2+2u_{31}u_{33}=0,\\ &u_{11}u_{23}+u_{12}u_{22}+u_{13}u_{21}=0,\\ &u_{11}u_{33}+u_{12}u_{32}+u_{13}u_{31}=1,\\ &u_{21}u_{33}+u_{22}u_{32}+u_{23}u_{31}=0,\\ &(u_{11}u_{22}-u_{12}u_{21})u_{33}\\ &\quad-(u_{11}u_{23}-u_{13}u_{21})u_{32}\\ &\qquad+(u_{12}u_{23}-u_{13}u_{22})u_{31}=\mp 1.\end{aligned}\right\} \tag{3.10}$$

3.2. The identical case. For the case MI_{ij}, the variables a_i and u_{ij} must satisfy

$$a_m=0 \text{ for } m\neq i,\ u_{ji}=1,\ u_{jn}=0 \text{ for } n\neq i. \tag{3.11}$$

Throughout this paper we assume that $r\neq 0$. Otherwise, i.e. for $r=0$, C_2 will be a fixed point.

3.2.1. *Case MI_{11}:* $e_{21}=e_{11}$. From (3.5) - (3.12), we get $a_2=a_3=0$, $u_{22}=\mp 1$. In this case we obtain $a'_1-r+1,=0$. Since $a_1=0$, we get $r=1$. This means that the curves C_1 and C_2 are identical.

3.1. Corollary. *Let C_1 and C_2 be two null curves in $\mathbb{E}^3_1$, with Frenet frames $\{e_{11},e_{12},e_{13}\}$ and $\{e_{21},e_{22},e_{23}\}$. If the relation $e_{21}=e_{11}$ holds, then C_1 and C_2 are identical.*

3.2.2. *Case MI_{12}:* $e_{22}=e_{11}$. Under this condition, it is already seen that $u_{22}^2+2u_{21}u_{23}=0$. This is a contradiction with the second equality of (3.11).

3.2. Corollary. *Let C_1 and C_2 be two null curves in $\mathbb{E}^3_1$, with Frenet frames $\{e_{11},e_{12},e_{13}\}$ and $\{e_{21},e_{22},e_{23}\}$. There exist no null curves in $\mathbb{E}^3_1$ satisfying the relation $e_{22}=e_{11}$.*

3.2.3. *Case MI_{13}:* $e_{23}=e_{11}$. There exist no curves satisfying $e_{23}=e_{11}$ under the condition $r\neq 0$.

3.3. Corollary. *Let C_1 and C_2 be two null curves in $\mathbb{E}^3_1$, with Frenet frames $\{e_{11},e_{12},e_{13}\}$ and $\{e_{21},e_{22},e_{23}\}$. There are no null curves in $\mathbb{E}^3_1$ satisfying the relation $e_{23}=e_{11}$.*

3.2.4. *Case MI_{22}:* $e_{22}=e_{12}$. From (3.11) we obtain $u_{12}=u_{32}=0$. So, to be consistent with (3.11), these equalities must be satisfied:

$$u_{11}u_{13}=0 \text{ and } u_{31}u_{33}=0.$$

According to this we discuss the following four possible case:

(i) $u_{11}=0$ and $u_{33}=0$,
(ii) $u_{13}=0$ and $u_{31}=0$,
(iii) $u_{11}=0$ and $u_{31}=0$,

(iv) $u_{13} = 0$ and $u_{33} = 0$.

It is clear that in the cases (*iii*) and (*iv*), the transition matrix $U = (u_{ij})$ is singular. Thus we deal only with the cases (i) and (ii).

Case (i). $u_{11} = 0$ and $u_{33} = 0$. In this case, we easily obtain from (3.5)-(3.11):

$$u_{12} = u_{32} = u_{21} = u_{23} = 0,\ \frac{1}{u_{31}} = u_{13} = \lambda_1 \text{ (constant)}, a_2 = \lambda \text{ (constant)},$$
$$t_1 = \frac{-1}{\lambda},\ t_2 = -\frac{\lambda}{\lambda_1^2} \text{ and } r = \frac{\lambda_1}{\lambda}.$$

It is clear that $\det(u_{ij}) = -1$.

Case (ii). $u_{13} = 0$ and $u_{31} = 0$. In this case, from (3.5)-(3.11), we have:

$$u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = 0,\ u_{11}u_{33} = 1,\ a_2 = 0,$$
$$t_1 = t_2,\ r = \mp 1.$$

It is clear that $\det(u_{ij}) = 1$. In this case we obtain that $C_1 = C_2$.

Thus we have proved the following theorem:

3.4. Theorem. *Let C_1 and C_2 be two null curve in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11}, e_{12}, e_{13}, k_1 = 1, t_1\}$, $\{e_{21}, e_{22}, e_{23}, k_2 = 1, t_2\}$, respectively. If the relationship $e_{22} = e_{12}$ holds then C_1 and C_2 must satisfy one of the following conditions:*

(i) $C_2 = C_1$,
(ii) $C_2 = C_1 + \mu e_{12}$, *where* $\mu = \frac{-1}{t_1}$. *In this case the second curvatures t_1 and t_2 of the curves C_1 and C_2 are constant functions and $t_1 t_2 > 0$.*

3.5. Corollary. *Let C_1 be a null curve in $\mathbb{E}_1^3$ with Frenet frame e_{11}, e_{12}, e_{13} and curvatures $k_1 = 1$ and t_1. If C_1 is a Bertrand curve then there exist only two Bertrand mates of the curve C_1: one is $C_2 = C_1$, the other is $C_2 = C_1 + \mu e_{12}$, where $\mu = \frac{-1}{t_1}$.*

3.6. Example. We consider the null curve $C_1(s) = (\sinh s, s, \cosh s)$ in $\mathbb{E}_1^3$. We can easily obtain the Frenet vectors and the curvatures of the curve C_1 as follows:

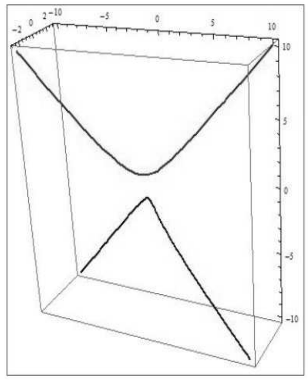
$$e_{11} = (\cosh s, 1, \sinh s),$$
$$e_{12} = (\sinh s, 0, \cosh s),$$
$$e_{13} = \Big(-\frac{1}{2}\cosh s, \frac{1}{2}, -\frac{1}{2}\sinh s\Big),$$
$$k_1 = 1,\ t_1 = \frac{1}{2}.$$

By using the above theorem, we can easily find one of its Bertrand mates as $C_2 = C_1 - \frac{1}{t_1} e_{12}$, and $C_2(s) = (-\sinh s, s, -\cosh s)$ (see Figure 1).

3.7. Corollary. *The null Bertrand curve C_1 and its Bertrand mate C_2 ($C_2 \neq C_1$) have opposite orientations.*

Proof. This is clear from the fact that the determinant of the transition matrix $U = (u_{ij})$ is $\det(u_{ij}) = -1$. □

Figure 1. Null curves satisfying the condition $e_{22} = e_{12}$ in $\mathbb{E}_1^3$



3.2.5. *Case MI_{23}:* $e_{23} = e_{12}$. There exist no curves satisfying $e_{23} = e_{11}$ under the condition $\det(u_{ij}) \neq 0$.

3.8. Remark. In Euclidean 3-space, if there exist a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of β, then α is called a Mannheim curve, and β a Mannheim partner curve of α. Mannheim partner curves in Euclidean 3-space and Minkowski 3-space (for non-null curves) have been studied by Liu and Wang [11].

Thus we can give the following interesting corollary as a result of case MI_{23}.

3.9. Corollary. *There are no null Mannheim partner curves in Minkowski* 3*-space.*

3.2.6. *Case MI_{33}:* $e_{23} = e_{13}$. By considering (3.5) - (3.11), and after some calculations, we obtain $u_{22} = \mp 1,\ u_{11} = 1,\ u_{21} = 0$ and the following cases,

(i) If $u_{22} = 1$, we get

$$u_{12} = u_{13} = u_{23} = 0,\ a_3 = \mu \text{ (const.)},\ r = 1 \text{ and } t_1 = t_2.$$

(ii) If $u_{22} = -1$, we obtain the following,

$$u_{12} = u_{13} = u_{23} = 0,\ a_3 = \mu \text{ (const.)},\ r = 1 \text{ and } t_1 = -t_2.$$

Thus we obtain the following theorem:

3.10. Theorem. *Let C_1 and C_2 be two null curves in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11}, e_{12}, e_{13}, k_1 = 1, t_1\}$, $\{e_{21}, e_{22}, e_{23}, k_2 = 1, t_2\}$, respectively. If the relationship $e_{23} = e_{13}$ holds then C_1 and C_2 must satisfy one of the following conditions:*

(i) *$C_2 = C_1$,*

(ii) *$C_2 = C_1 + \mu e_{12}$, where μ is non zero constant. In this case the second curvatures t_1 and t_2 of the curves C_1 and C_2 satisfy the condition $t_1 = t_2 = 0$.*

3.11. Remark. We note that the curves which satisfy the condition (ii) in the above theorem are null cubic curves with curvatures $k_1 = k_2 = 1, t_1 = t_2 = 0$.

3.12. Example. We consider the null cubic curves

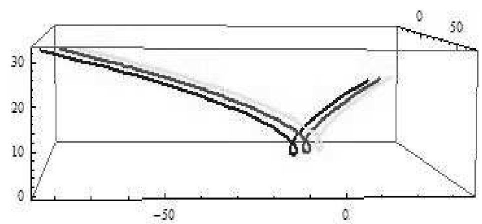
$$C_1(s) = \left(\frac{1}{\sqrt{2}} \left(\frac{\frac{s^3}{3} + s^2 + 3s + 1}{2} \right), \frac{1}{\sqrt{2}} \left(\frac{-\frac{s^3}{3} - s^2 + s - 1}{2} \right), \frac{1}{2}s^2 + s + 1 \right)$$

in $\mathbb{E}_1^3$. We can easily obtain the Frenet vectors and the curvatures of the curve C_1 as follows:

$$\begin{aligned}
e_{11} &= \left(\frac{1}{\sqrt{2}}\left(\frac{s^2+2s+3}{2}\right), \frac{1}{\sqrt{2}}\left(\frac{-s^2-2s+1}{2}\right), s+1\right),\\
e_{12} &= \left(\frac{1}{\sqrt{2}}(s+1), -\frac{1}{\sqrt{2}}(s+1), 1\right),\\
e_{13} &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right),\\
k_1 &= 1,\ t_1 = 0.
\end{aligned}$$

By using the above theorem, we can easily find some its Bertrand mates in the form $C_2 = C_1 + \mu e_{13}$ by taking $\mu = \sqrt{2},\ 5\sqrt{2},\ -5\sqrt{2}$ (see Figure 2).

Figure 2. Null curves satisfying the condition $e_{23} = e_{13}$ in $\mathbb{E}_1^3$



3.13. Corollary. *Let C_1 and C_2 be two null curves in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11}, e_{12}, e_{13}, k_1 = 1, t_1\}$, $\{e_{21}, e_{22}, e_{23}, k_2 = 1, t_2\}$, respectively. If the relationship $e_{23} = e_{13}$ holds then C_1 is congruent to C_2.*

3.3. The parallel case. For the case MP_{ij}, the variables u_{ij} must satisfy $u_{ik} = 0$ for $k \neq j$. Throughout this paper we assume that $r \neq 0$. Otherwise, i.e. for $r = 0$, C_2 will be a fixed point.

3.3.1. *Case MP_{11} :* $e_{21} = u_{11}e_{11}$. Considering (3.5)-(3.11), and after some calculations, we obtain $u_{22} = \mp 1$, and the following cases.

(i) If $u_{22} = 1$ we get,

$$u_{11} = \frac{1}{u_{33}} = r = \mu \text{ (const.} \neq 0),\ u_{21} = u_{23} = u_{31} = u_{32} = 0,$$
$$a_3a_1 + a_3a_2 + a_2a_1 = a_3(\mu^2 - 1) \text{ and } t_1 - \mu^2 t_2 = 0,$$

(ii) If $u_{22} = -1$, we get:

$$u_{11} = \frac{1}{u_{33}} = -r = \mu \text{ (const.} \neq 0),\ u_{21} = u_{23} = u_{31} = u_{32} = 0,$$
$$a_3a_1 + a_3a_2 + a_2a_1 = -a_3(\mu^2 + 1) \text{ and } t_1 - \mu^2 t_2 = 0.$$

3.14. Corollary. *Let C_1 and C_2 be two null curves in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11}, e_{12}, e_{13}, k_1 = 1, t_1\}$, $\{e_{21}, e_{22}, e_{23}, k_2 = 1, t_2\}$, respectively. If the Frenet vector e_{21} of C_2 is parallel to the Frenet vector e_{11} of C_1 then the components* *of the transition matrix* $U = (u_{ij})$ *and second curvatures of the curves must satisfy the following conditions:*

$$u_{22} = \mp 1,\ u_{21} = u_{23} = u_{31} = u_{32} = 0,$$
$$a_3a_1 + a_3a_2 + a_2a_1 = a_3\lambda,\ \lambda \in \mathbb{R}_0,$$
$$\frac{t_1}{t_2} = const.$$

3.3.2. *Case* MP_{12} : $e_{22} = u_{21}e_{11}$. There exist no curves satisfying $e_{22} = u_{11}e_{21}$ under the condition $\det(u_{ij}) \neq 0$.

3.3.3. *Case* MP_{13} : $e_{23} = u_{31}e_{11}$. Considering (3.5)-(3.11), and after some calculations, we obtain $u_{22} = \mp 1$, and the following conditions,

$$u_{11} = \mp\frac{1}{2}\frac{(\sigma')^2}{\sigma^3},\ u_{12} = \mp\frac{\sigma'}{\sigma^2},\ u_{13} = \pm\frac{1}{\sigma},$$
$$u_{21} = \mp\frac{\sigma'}{\sigma},\ u_{23} = 0,\ u_{31} = \pm\sigma,$$
$$t_1 = r\sigma - \frac{\sigma''}{\sigma} + \frac{3}{2}\Big(\frac{\sigma'}{\sigma}\Big)^2,$$

where $\sigma = rt_2$.

3.15. Corollary. *Let* C_1 *and* C_2 *be two null curves in* $\mathbb{E}_1^3$*, with Frenet vectors and non-zero curvature functions* $\{e_{11}, e_{12}, e_{13}, k_1 = 1, t_1\}$, $\{e_{21}, e_{22}, e_{23}, k_2 = 1, t_2\}$*, respectively. If the Frenet vector* e_{23} *of* C_2 *is parallel to the Frenet vector* e_{11} *of* C_1 *then the components of the transition matrix* $U = (u_{ij})$ *and second curvatures of the curves must satisfy the following conditions:*

$$u_{22} = \mp 1, u_{23} = u_{32} = u_{33} = 0,$$
$$t_1 = r\sigma - \frac{\sigma''}{\sigma} + \frac{3}{2}\Big(\frac{\sigma'}{\sigma}\Big)^2,\ \textit{where } \sigma = rt_2.$$

3.3.4. *Case* MP_{22}*:* $e_{22} = u_{22}e_{12}$. From (3.11), we obtain $u_{12} = u_{32} = 0$ and $u_{22} = \mp 1$. So, in order to be consistent with (3.11), these equalities must be satisfied:

$$u_{11}u_{13} = 0 \text{ and } u_{31}u_{33} = 0.$$

According to this we discuss the following four possible case:

(i) $u_{11} = 0$ and $u_{33} = 0$,
(ii) $u_{13} = 0$ and $u_{31} = 0$,
(iii) $u_{11} = 0$ and $u_{31} = 0$,
(iv) $u_{13} = 0$ and $u_{33} = 0$.

It is clear that in the cases (iii) and (iv), the transition matrix $U = (u_{ij})$ is singular. Thus we deal only with the cases (i) and (ii).

(i.1) If $u_{22} = 1$ then $u_{11} = u_{33} = 0$, and the following are obtained:

$$u_{12} = u_{32} = u_{21} = u_{23} = 0,\ \frac{1}{u_{31}} = u_{13} = \lambda \text{ (const.)}$$
$$t_1 = \frac{-r}{\lambda},\ t_2 = -\frac{1}{r\lambda},$$
$$a_3a_1' + a_2a_3' + a_1a_2 + a_3 = 0.$$

(i.2) If $u_{22}=-1$ then $u_{11}=u_{33}=0$, and the following are obtained:

$$u_{12}=u_{21}=u_{23}=u_{32}=0,\ \frac{1}{u_{31}}=u_{13}=\lambda,$$
$$t_1=\frac{r}{\lambda},\ t_2=\frac{1}{r\lambda},$$
$$a_2a_3-a_2^2-\lambda a_1=\lambda.$$

(ii.1) If $u_{22}=1$ then $u_{13}=u_{31}=0$, and the following are obtained:

$$u_{12}=u_{32}=u_{21}=u_{23}=0,\ u_{11}=\frac{1}{u_{33}}=r=\lambda\ (\text{const.}),$$
$$t_1-\lambda^2t_2=0,\ a_2=a_3',$$
$$a_3a_1+a_2+a_1+a_3(-\lambda^2+1)=0.$$

(ii.2) If $u_{22}=-1$ then $u_{13}=u_{31}=0$, and the following are obtained:

$$u_{12}=u_{21}=u_{23}=u_{32}=0,\ u_{11}=\frac{1}{u_{33}}=-r,$$
$$t_1-\lambda^2t_2=0,\ a_2=a_3',$$
$$a_3a_1+a_2+a_1+a_3(\lambda^2+1)=0.$$

Thus we have proved the following theorem:

3.16. Theorem. *Let C_1 and C_2 be two null curves in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11},e_{12},e_{13},k_1=1,t_1\}$, $\{e_{21},e_{22},e_{23},k_2=1,t_2\}$, respectively. If the Frenet vector e_{22} of C_2 is parallel to the Frenet vector e_{12} of C_1 then the components of the transition matrix $U=(u_{ij})$ and second curvatures of the curves must satisfy one of the following conditions:*

(i) $u_{22}=\mp1,\ u_{11}=u_{33}=0$ *and* $t_1t_2=\text{const.}>0$,

(ii) $u_{22}=\mp1,\ u_{13}=u_{31}=0$ *and* $\frac{t_1}{t_2}=\text{const.}>0$.

3.3.5. *Case MP_{23}: $e_{23}=u_{33}e_{12}$.* There exist no curves satisfying $e_{22}=u_{11}e_{21}$ under the condition $\det(u_{ij})\neq0$.

3.3.6. *Case MP_{33}: $e_{23}=u_{33}e_{13}$.* Considering (3.5)-(3.11), and after some calculations, we obtain $u_{31}=u_{32}=0$, and $u_{22}=\mp1$. Hence, we get:

$$\begin{aligned}&u_{11}=\mp\frac{t_1}{rt_2},\ u_{13}=\pm\frac{1}{2}\frac{(\sigma't_1-\sigma t_1')^2}{(t_1\sigma)^3},\\ &u_{12}=\mp\frac{\sigma't_1-\sigma t_1'}{t_1\sigma^2},\ u_{23}=\mp\frac{\sigma t_1'-\sigma' t_1}{\sigma t_1^2},\\ &u_{33}=\frac{1}{u_{11}},\ u_{21}=0.\end{aligned}\tag{3.12}$$

$$\begin{aligned}&a_1'+t_1a_2-ru_{11}+1=0\\ &a_2'+a_1-t_1a_3-ru_{12}=0\\ &a_3-a_2-ru_{13}=0.\end{aligned}\tag{3.13}$$

Also we find that $\det U=\mp1$. Thus we have proved the following theorem:

3.17. Theorem. *Let C_1 and C_2 be two null curves in $\mathbb{E}_1^3$, with Frenet vectors and non-zero curvature functions $\{e_{11},e_{12},e_{13},k_1=1,t_1\}$, $\{e_{21},e_{22},e_{23},k_2=1,t_2\}$, respectively. If the Frenet vector e_{23} of C_2 is parallel to the Frenet vector e_{13} of C_1 then the components of the transition matrix $U=(u_{ij})$ and the curvatures of the curves must satisfy the following conditions:*

(i) $u_{22} = \mp 1$, $u_{21} = u_{31} = u_{32} = 0$, *and the equalities in* (3.13),

(ii) *The equations given in* (3.14). □

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