Hacettepe Journal of Mathematics and Statistics
Volume 39 (4) (2010), 567 – 575

UNIVALENCE CONDITIONS FOR A NEW GENERAL INTEGRAL OPERATOR

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Received 31 : 01 : 2010 : Accepted 06 : 04 : 2010

Abstract

Using the Hadamard product, we define a new general integral operator. The aim of this paper is to obtain new sufficient conditions for the univalence of this general integral operator. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. For two functions, $f(z) \in \mathcal{A}$ and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

their *Hadamard product* (or *convolution*) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.2}$$

For a function $g \in \mathcal{A}$ defined by (1.1), where $b_n \geq 0, (n \geq 2)$, we define the family $\mathcal{B}(g, \mu)$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z^2 (f * g)'(z)}{[(f * g)(z)]^2} - 1 \right| < 1 - \mu \quad (z \in \mathcal{U};\ 0 \leq \mu < 1), \tag{1.3}$$

provided that $(f * g)(z) \neq 0$.

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Note that $\mathcal{B}\big(\frac{z}{1-z},\mu\big)=\mathcal{B}(\mu)$, where the class $\mathcal{B}(\mu)$ of analytic and univalent functions was introduced and studied by Frasin and Darus [10] (see also [8]).

Very recently, Frasin [7] and Frasin and Ahmad [9] defined the following new general integral operators:

1.1. Definition. [7] Given $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i=1,\dots,n$, $n\in\mathbb{N}$, we define the integral operator $I(f_1,\dots,f_n;g_1,\dots,g_n):\mathcal{A}^n\to\mathcal{A}$ by

$$I(f_1,\dots,f_n;g_1,\dots,g_n)(z)=\int_0^z\left(\frac{(f_1*g_1)(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{(f_n*g_n)(t)}{t}\right)^{\alpha_n}dt, \tag{1.4}$$

where $(f*g)(z)/z\neq 0$, $z\in\mathcal{U}$.

1.2. Definition. [9] Given $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ for all $i=1,\dots,n$, $n\in\mathbb{N}$, $\beta\in\mathbb{C}$ with $\Re(\beta)>0$. We define the integral operator $I_\beta(f_1,\dots,f_n;g_1,\dots,g_n):\mathcal{A}^n\to\mathcal{A}$ by

$$I_\beta(f_1,\dots,f_n;g_1,\dots,g_n)(z)=\left\{\int_0^z\beta t^{\beta-1}\left(\frac{(f_1*g_1)(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{(f_n*g_n)(t)}{t}\right)^{\alpha_n}dt\right\}^{\frac{1}{\beta}}, \tag{1.5}$$

where $(f*g)(z)/z\neq 0$, $z\in\mathcal{U}$.

Using the Hadamard product defined by (1.2), we introduce the following new general integral operator:

1.3. Definition. Given $f_i, g_i \in \mathcal{A}$ for all $i=1,\dots,n$, $n\in\mathbb{N}$, $\gamma\in\mathbb{C}$. We define the integral operator $I_\gamma(f_1,\dots,f_n;g_1,\dots,g_n):\mathcal{A}^n\to\mathcal{A}$ by

$$\begin{aligned}&I_\gamma(f_1,\dots,f_n;g_1,\dots,g_n)(z)\\&\quad=\left((1+n(\gamma-1))\int_0^z((f_1*g_1)(t))^{\gamma-1}\cdots((f_n*g_n)(t))^{\gamma-1}\,dt\right)^{\frac{1}{1+n(\gamma-1)}},\end{aligned} \tag{1.6}$$

where $(f*g)(z)/z\neq 0$, $z\in\mathcal{U}$.

1.4. Remark. Note that the integral operator $I_\gamma(f_1,\dots,f_n;g_1,\dots,g_n)(z)$ generalizes many operators introduced and studied by several authors, for example:

(1) For $g_1=\dots=g_n=\dfrac{z}{1-z}$, we obtain the integral operator

$$F_\gamma(z)=\left((1+n(\gamma-1))\int_0^z(f_1(t))^{\gamma-1}\cdots(f_n(t))^{\gamma-1}\,dt\right)^{\frac{1}{1+n(\gamma-1)}} \tag{1.7}$$

introduced and studied by Breaz and Breaz [3].

(2) For $g_1=\dots=g_n=\dfrac{z}{(1-z)^2}$, we obtain the integral operator

$$G_\gamma(z)=\left((1+n(\gamma-1))\int_0^z t^{n(\gamma-1)}(f_1'(t))^{\gamma-1}\cdots(f_n'(t))^{\gamma-1}\,dt\right)^{\frac{1}{1+n(\gamma-1)}}$$

introduced by Selvaraj and Karthikeyan [17].

(3) For $g_1=\dots=g_n=z+\sum\limits_{n=2}^{\infty}[1+(n-1))\lambda]^m z^n$, we obtain the following integral operator introduced and studied by Bulut [5].

$$G_{n,m,\gamma}(z)=\left((1+n(\gamma-1))\int_0^z(D^m f_1(t))^{\gamma-1}\cdots\big(D^m f_n(t)\big)^{\gamma-1}\,dt\right)^{\frac{1}{1+n(\gamma-1)}}, \tag{1.8}$$

where $D^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1))\lambda]^m a_n z^n,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the Al-Oboudi differential operator [1].

(4) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}(n-1)!} z^n$, we obtain the following integral operator introduced and studied by Selvaraj and Karthikeyan [17].

$$F_\gamma(\alpha_1, \beta_1; z) = \left((1 + n(\gamma - 1)) \int_0^z (H_s^q(\alpha_1, \beta_1) f_1(t))^{\gamma-1} \cdots (H_s^q(\alpha_1, \beta_1) f_n(t))^{\gamma-1} dt\right)^{\frac{1}{1+n(\gamma-1)}}, \tag{1.9}$$

where $H_s^q(\alpha_1, \beta_1) f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}(n-1)!} a_n z^n$ is the Dziok-Srivastava operator [6].

(5) For $g_1 = \cdots = g_n = \frac{z}{1-z}$ and $f_1 = \cdots = f_n = f \in \mathcal{A}$, we obtain the integral operator

$$G_\gamma^n(z) = \left((1 + n(\gamma - 1)) \int_0^z (f(t))^{n(\gamma-1)} dt\right)^{\frac{1}{1+n(\gamma-1)}},\ n \in \mathbb{N} \tag{1.10}$$

introduced and studied by Breaz and Breaz [3].

(6) For $g_1 = \cdots = g_n = \frac{z}{1-z}$ and $f_1 = \cdots = f_n = f \in \mathcal{A},\ n = 1$, we obtain the integral operator

$$G_\gamma(z) = \left(\gamma \int_0^z (f(t))^{(\gamma-1)}\, dt\right)^{\frac{1}{\gamma}},\ n \in \mathbb{N} \tag{1.11}$$

introduced and studied by Pescar [14].

(7) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} C_{k+n-1}^k z^n$, we obtain the following integral operator

$$I_\gamma(f_1, \ldots, f_n)(z) = \left((1 + n(\gamma - 1)) \int_0^z (R^k f_1(t))^{\gamma-1} \cdots (R^k f_n(t))^{\gamma-1}\, dt\right)^{\frac{1}{1+n(\gamma-1)}}, \tag{1.12}$$

where $R^k f(z) = z + \sum_{n=2}^{\infty} C_{k+n-1}^k a_n z^n,\ k \in \mathbb{N}_0$ is the Ruscheweyh differential operator [15].

(8) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} n^k z^n$, we obtain the following integral operator

$$D_\gamma^k F(z) = \left((1 + n(\gamma - 1)) \int_0^z (D^k f_1(t))^{\gamma-1} \cdots (D^k f_n(t))^{\gamma-1} dt\right)^{\frac{1}{1+n(\gamma-1)}}, \tag{1.13}$$

where $D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ k \in \mathbb{N}_0$ is the Sălăgean differential operator [16].

In order to derive our main results, we have to recall here the following univalence criteria.

1.5. Lemma. [12, 13] *Let* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zf''(z)}{f'(z)}\right| \le 1,$$

for all $z \in \mathcal{U}$, *then the integral operator*

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)\,dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$. □

1.6. Lemma. [14] *Let* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) > 0$, $c \in \mathbb{C}$ *with* $|c| \le 1$, $c \ne -1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right| \le 1,$$

for all $z \in \mathcal{U}$ *then the integral operator*

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Also, we need the following Schwarz Lemma

1.7. Lemma. [11] *Let the function* f *be regular in the disk* $\mathcal{U}$ *with* $f(0) = 0$. *If* $|f(z)| < 1$, $(z \in \mathcal{U})$, *then*

$$|f(z)| \le |z|, (z \in \mathcal{U}).$$

Equality can hold only if

$$f(z) = Kz,\ (z \in \mathcal{U}),$$

where $|K| = 1$.

2. Univalence conditions for $I_\gamma(f_1, \ldots, f_n; g_1, \ldots, g_n)$

2.1. Theorem. *Let* $f_i \in \mathcal{A}$ *for all* $i = 1, \ldots, n$, $\gamma \in \mathbb{C}$, *and* $M \ge 1$ *with*

$$|\gamma - 1| \le \frac{\mathrm{Re}(\gamma)}{\sum_{i=1}^{n}[(2-\mu_i)M+1]}. \tag{2.1}$$

If for all $i = 1, \ldots, n$, $f_i \in \mathcal{B}(g_i, \mu_i)$, $0 \le \mu_i < 1$, *and*

$$|(f_i * g_i)(z)| \le M,\ (z \in \mathcal{U}),$$

then the integral operator $I_\gamma(f_1, \ldots, f_n; g_1, \ldots, g_n)$ *defined by* (1.6) *is analytic and univalent in* $\mathcal{U}$.

Proof. Define

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{(f_i * g_i)(t)}{t}\right)^{\gamma-1} dt$$

we observe that $h(0) = h'(0) - 1 = 0$. On the other hand, it is easy to see that

$$h'(z) = \prod_{i=1}^{n}\left(\frac{(f_i * g_i)(z)}{z}\right)^{\gamma-1}. \tag{2.2}$$

Now if we differentiate logarithmically and multiply with z on both sides of (2.2), we obtain

$$\frac{zh''(z)}{h'(z)} = (\gamma - 1)\sum_{i=1}^{n}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right).$$

Thus we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma - 1|\sum_{i=1}^{n}\left|\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1\right|, \tag{2.3}$$

so

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left[|\gamma - 1|\sum_{i=1}^{n}\left(\left|\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)}\right| + 1\right)\right] \\ &\leq \frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left[|\gamma - 1|\sum_{i=1}^{n}\left(\left|\frac{z^2(f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2}\right|\right.\right. \\ &\qquad \left.\left.\times\left|\frac{(f_i * g_i)(z)}{z}\right| + 1\right)\right].\end{aligned}$$

Since $|(f_i * g_i)(z)| \leq M$, $(z \in \mathcal{U},\ i = 1, \ldots, n)$, and $f_i \in \mathcal{B}(g_i, \mu_i)$ for all $i = 1, \ldots, n$, then from Schwarz Lemma and (1.3), we obtain

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left[|\gamma - 1|\sum_{i=1}^{n}\left(\left|\frac{z^2(f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2} - 1\right|M\right.\right. \\ &\qquad \left.\left. + M + 1\right)\right] \\ &\leq \frac{|\gamma - 1|}{\mathrm{Re}(\gamma)}\left(\sum_{i=1}^{n}[(2-\mu_i)M + 1]\right),\ (z \in \mathcal{U}),\end{aligned}$$

which, in the light of the hypothesis (2.1), yields

$$\frac{1-|z|^{2\mathrm{Re}(\gamma)}}{\mathrm{Re}(\gamma)}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1,\ (z \in \mathcal{U}).$$

Applying Lemma 1.5 for the function $h(z)$ we obtain that $I_\gamma(f_1, \ldots, f_n; g_1, \ldots, g_n) \in \mathcal{S}$. □

Letting $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_s)_{n-1}(n-1)!}z^n$, $\mu_i = 0$ (for all $i = 1, \ldots, n$) and $M = 1$ in Theorem 2.1, we have

2.2. Corollary. [17] *Let $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$, $n \in \mathbb{N}$ and $\gamma \in \mathbb{C}$ with*

$$|\gamma - 1| \leq \frac{\mathrm{Re}(\gamma)}{3n}.$$

If

$$\left|\frac{z^2(H_s^q(\alpha_1, \beta_1)f_i(z))'}{(H_s^q(\alpha_1, \beta_1)f_i(z))^2} - 1\right| < 1,\ (z \in \mathcal{U})$$

and

$$|H_s^q(\alpha_1, \beta_1)f_i(z)| \leq 1,\ (z \in \mathcal{U})$$

for all $i = 1, \ldots, n$, then the integral operator $F_\gamma(\alpha_1, \beta_1; z)$ defined by (1.9) is analytic and univalent in $\mathcal{U}$. □

Letting $g_1 = \cdots = g_n = \dfrac{z}{1-z}$, $\mu_i = 0$ (for all $i = 1, \ldots, n$) and $M = 1$ in Theorem 2.1, we have

2.3. Corollary. [2] *Let $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n,\ n \in \mathbb{N}$ and $\gamma \in \mathbb{C}$ with*

$$|\gamma - 1| \leq \frac{\operatorname{Re}(\gamma)}{3n}.$$

If

$$\left| \frac{z^2 f_i'(z)}{(f_i(z))^2} - 1 \right| < 1$$

and

$$|f_i(z)| \leq 1,\ \ (z \in \mathcal{U})$$

for all $i = 1, \ldots, n$, then the integral operator $F_\gamma(z)$ defined by (1.7) is analytic and univalent in $\mathcal{U}$. □

Making use of Lemma 1.6, we prove

2.4. Theorem. *Let $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$; $c \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $M \geq 1$ with*

$$|c| \leq 1 + \left(\frac{1-\gamma}{n(\gamma - 1) + 1} \right) \sum_{i=1}^{n} [(2 - \mu_i)M + 1] \tag{2.4}$$

and

$$\gamma \in \left[1, \frac{\sum_{i=1}^{n} [(2 - \mu_i)M + 1] - n + 1}{\sum_{i=1}^{n} [(2 - \mu_i)M + 1] - n} \right]. \tag{2.5}$$

If for all $i = 1, \ldots, n,\ f_i \in \mathcal{B}(g_i, \mu_i),\ 0 \leq \mu_i < 1$, and

$$|(f_i * g_i)(z)| \leq M,\ \ (z \in \mathcal{U}),$$

then the integral operator $I_\gamma(f_1, \ldots, f_n; g_1, \ldots, g_n)$ defined by (1.6) is analytic and univalent in $\mathcal{U}$.

Proof. From the proof of Theorem 2.1, we have

$$\frac{zh''(z)}{h'(z)} = (\gamma - 1) \sum_{i=1}^{n} \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right).$$

Thus, we have

$$\begin{aligned}
&\left| c\,|z|^{2[n(\gamma-1)+1]} + (1 - |z|^{2[n(\gamma-1)+1]}) \frac{zh''(z)}{[n(\gamma - 1) + 1]h'(z)} \right| \\
&\quad = \Bigg| c\,|z|^{2[n(\gamma-1)+1]} + (1 - |z|^{2[n(\gamma-1)+1]}) \left(\frac{\gamma - 1}{n(\gamma - 1) + 1} \right) \\
&\qquad\qquad \times \sum_{i=1}^{n} \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right) \Bigg| \\
&\quad \leq |c| + \left(\frac{\gamma - 1}{n(\gamma - 1) + 1} \right) \sum_{i=1}^{n} \left(\left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} \right| + 1 \right) \\
&\quad \leq |c| + \left(\frac{\gamma - 1}{n(\gamma - 1) + 1} \right) \sum_{i=1}^{n} \left(\left| \frac{z^2(f_i * g_i)'(z)}{[(f_i * g_i)(z)]^2} \right| \left| \frac{(f_i * g_i)(z)}{z} \right| + 1 \right)
\end{aligned}$$

Since $|(f_i * g_i)(z)| \leq M,\ (z \in \mathcal{U},\, i = 1, \ldots, n)$ and $f_i \in \mathcal{B}(g_i, \mu_i)$ for all $i = 1, \ldots, n$, then once again from Schwarz' Lemma and (1.3), we obtain

$$\left| c\,|z|^{2[n(\gamma-1)+1]} + \left(1 - |z|^{2[n(\gamma-1)+1]}\right) \frac{zh''(z)}{[n(\gamma-1)+1]h'(z)} \right| \leq |c| + \left(\frac{\gamma-1}{n(\gamma-1)+1}\right) \sum_{i=1}^{n} [(2-\mu_i)M + 1]$$

since $|c| \leq 1 + \left(\frac{1-\gamma}{n(\gamma-1)+1}\right) \sum\limits_{i=1}^{n} [(2-\mu_i)M + 1]$, then we have

$$\left| c\,|z|^{2[n(\gamma-1)+1]} + \left(1 - |z|^{2[n(\gamma-1)+1]}\right) \frac{zh''(z)}{[n(\gamma-1)+1]h'(z)} \right| \leq 1,\ (z \in \mathcal{U}).$$

Applying Lemma 1.6 for the function $h(z)$, we obtain that $I_\gamma(f_1, \ldots, f_n; g_1, \ldots, g_n) \in \mathcal{S}$. □

Letting $g_1 = \cdots = g_n = z + \sum\limits_{n=2}^{\infty} [1 + (n-1))\lambda]^m z^n$ and $\mu_i = 0$ (for all $i = 1, \ldots, n$) in Theorem 2.4, we have

2.5. Corollary. *Let $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$; $c \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $M \geq 1$ with*

$$|c| \leq 1 + \left(\frac{1-\gamma}{n(\gamma-1)+1}\right)(2M+1)n$$

and

$$\gamma \in \left[1, \frac{2Mn+1}{2Mn}\right].$$

If for all $i = 1, \ldots, n$

$$\left| \frac{z^2 (D^m f_i(z))'}{(D^m f_i(z))^2} - 1 \right| < 1,\ (z \in \mathcal{U};\ m \in \mathbb{N}_0)$$

and

$$|(D^m f_i(z))| \leq M,\ (z \in \mathcal{U};\ i = 1, \ldots, n),$$

then the integral operator $G_{n,m,\gamma}(z)$ defined by (1.8) *is analytic and univalent in $\mathcal{U}$.* □

Letting $\mu_i = 0$ (for all $i = 1, \ldots, n$) and $m = 0$ in Corollary 2.5, we have

2.6. Corollary. *Let $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$; $c \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $M \geq 1$ with*

$$|c| \leq 1 + \left(\frac{1-\gamma}{n(\gamma-1)+1}\right)(2M+1)n$$

and

$$\gamma \in \left[1, \frac{2Mn+1}{2Mn}\right].$$

If for all $i = 1, \ldots, n$

$$\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| < 1,\ (z \in \mathcal{U})$$

and

$$|f_i(z)| \leq M,\ (z \in \mathcal{U};\ i = 1, \ldots, n),$$

then the integral operator $F_\gamma(z)$ defined by (1.7) *is analytic and univalent in $\mathcal{U}$.* □

Letting $n = 1,\ M = 1,\ f_1 = f$ in Corollary 2.6, we have

2.7. Corollary. [14] *Let* $f \in \mathcal{A}$*,* $c \in \mathbb{C}$ *and* $\gamma \in \mathbb{R}$ *with*

$$|c| \leq \frac{3-2\gamma}{\gamma}, \ (c \neq -1)$$

and

$$\gamma \in \left[1, \frac{3}{2}\right].$$

If

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, \ (z \in \mathcal{U})$$

and

$$|f(z)| \leq 1, \ (z \in \mathcal{U}),$$

then the integral operator $G_\gamma(z)$ *defined by* (1.11) *is analytic and univalent in* $\mathcal{U}$. □

2.8. Remark. Taking different choices of $g_1, \ldots, g_n$ as stated in Section 1, Theorems 2.1 and 2.4 lead to new sufficient conditions for univalency of the other integral operators defined in Section 1.

Acknowledgement. The authors would like to thank the referee for his valuable comments and suggestions.

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