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ON THE WEAK CONVERGENCE OF THE ERGODIC DISTRIBUTION FOR AN INVENTORY MODEL OF TYPE (s,S)

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Abstract

In this study, a renewal - reward process with a discrete interference of chance is constructed. This process describes in particular a semi-Markovian inventory model of type (s, S). The ergodic distribution of this process is expressed by a renewal function, and a second-order approximation for the ergodic distribution of the process is obtained as $S - s \to \infty$ when the interference has a triangular distribution. Then, the weak convergence theorem is proved for the ergodic distribution and the limit distribution is derived. Finally, the accuracy of the approximation formula is tested by the Monte Carlo simulation method.

Keywords: Renewal-reward process, Discrete interference of chance, Asymptotic expansion, Triangular distribution, Weak convergence, Renewal function.

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1. Introduction

Many interesting problems that are related to the theories of inventory, stock control, queuing, reliability, mathematical biology, stochastic finance, mathematical insurance, etc., can be expressed by renewal-reward or random walk processes. There are many interesting studies on these topics in the literature, see for example [1, 2, 4, 6, 9, 15, 16, 17, 18, 19, 20, 21]. But most of these studies are generally theoretical and are not helpful enough in solving concrete problems in practice due to the complexity of their mathematical structure. In addition to these theoretical studies, there are also some

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investigations devoted to the approximation methods for these kinds of the problems, including [1, 2, 6, 9, 14, 15, 16, 19, 21, 24]. The approximation results are generally simpler and easier for application. On the other hand, it is desirable that approximation results should be reasonably close to exact expressions. One of the most effective methods to obtain this kind of approximation is the asymptotic expansion method. In many cases, it is possible to obtain such approximations, which are closer to the exact expressions, by increasing the number of terms in the asymptotic expansions. However, when the number of terms in the asymptotic expansions is increased, the approximations start to lose their simplicity and meaning. For this reason, two- or three-term asymptotic expansions are sufficient to obtain convenient approximation formulas. Because of that, in this study, an inventory model of type (s, S) is considered, and a two-term asymptotic expansion is obtained for the ergodic distribution of this model.

In recent years, the inventory models of type (s, S) have been extensively considered and some of their characteristics investigated in the literature [3, 6, 7, 10, 12, 13, 20, 22, 23, and 26]. In most of these studies, analytical solutions could not be obtained. Instead of analytical methods, heuristic approaches, dynamic programming, etc., are used in these studies. In addition, in most of these studies both the demand quantity (η_n) and the inter-arrival time (ξ_n) are assumed to have exponential distributions. However, in our study, η_n and ξ_n are assumed to be arbitrary positive random variables.

Let us consider the following inventory model before expressing the problem mathematically. Assume that the stock level in a depot at the initial time $(t = 0)$ is equal to $X(0) \equiv X_0 \equiv z \in (s, S)$, $0 < s < S < \infty$, where s represents the stock control level and S the maximum stock level. In addition, it is assumed that at random times $T_1, T_2, \ldots, T_n, \ldots$ the stock level $(X(t))$ in the depot at time t decreases by $\eta_1, \eta_2, \ldots, \eta_n, \ldots$, respectively, until the inventory level falls below the predetermined control level s. In other words, the stock level in the depot changes as follows:

$$X(T_1) \equiv X_1 = z - \eta_1, X(T_2) \equiv X_2 = z - (\eta_1 + \eta_2), \ldots, X(T_n) \equiv X_n = z - \sum_{i=1}^{n} \eta_i,$$

where η_n represents the quantity of the nth demand, $n = 1, 2, 3, \ldots$.

In other words, demands are inserted into the system at random times $T_n = \sum_{i=1}^{n} \xi_i$, where ξ_n represents the time between the $(n-1)$st and nth demands, $n = 1, 2, 3, \ldots$. The system passes from one state to another by jumping at time T_n, according to quantities of demand $\{\eta_n\}$, $n \geq 1$. This natural variation of the system continues up to a certain random time τ_1, where τ_1 is the first time that the inventory level drops below the control level $s > 0$. When this occurs, the system is brought to a random level $\zeta_1 \in (s, S)$ immediately. That completes the first period and starts the second. Afterwards, the system will carry on its "natural variation" from a new initial state ζ_1 similar to the first period. When the stock level drops below s for the second time, by an interference to the system the stock level is brought to a random level $\zeta_2 \in (s, S)$ similar to the preceding period. Here, $\zeta_1, \zeta_2, \ldots$ are independent and identically distributed random variables. The important criteria on choosing the distribution for the random variable ζ_1 can be expressed as follows:

1) The re-starting state ζ_1 of the system is desired not to be too close to the control level s. In other words, it is desired that ζ_1 takes values close to s with very low probability. Note that if ζ_1 takes values close to s, then the system may re-start again in very short time intervals. This is not desirable in practice as it increases the number of orders, which causes an increase in the total ordering cost.
2) The re-starting state ζ_1 of the system is also desired not to be too close to the maximum stock level S. In other words, it is desired that ζ_1 takes values close

to S with a very low probability. Having ζ_1 very close to S increases the average inventory level, which, in turn, increases the holding cost.

The triangular distributions defined on the interval (s, S) satisfy the criteria above. For this reason, in this study it is assumed that the random variable ζ_1 has a triangular distribution on the interval (s, S).

The aim of this study is to construct a renewal-reward process with a discrete interference of chance which expresses the above mentioned semi-Markovian inventory model, and to obtain the second-order approximation for the ergodic distribution of the process while the interference has a triangular distribution. The final aim is to prove the weak convergence theorem for the ergodic distribution of the process and to derive the exact form of the limit distribution.

2. Mathematical construction of the process $X(t)$

Let $\{(\xi_n, \eta_n, \zeta_n)\},\ n \geq 1$, be a sequence of independent and identically distributed vectors of random variables defined on the probability space $(\Omega, \Im, \mathbb{P})$, such that the ξ_n and the η_n are positive-valued random variables, the random variable ζ_n takes values within the interval (s, S). Suppose that the random variables ξ_n, η_n and ζ_n are independent from each other and their distribution functions are known. Let these distribution functions be denoted by $\Phi(t)$, $F(x)$ and $\pi(z)$, respectively. So,

$$\Phi(t) = P\{\xi_1 \leq t\},\ F(x) = P\{\eta_1 \leq x\},\ \pi(z) = P\{\zeta_1 \leq z\}.$$

Define the renewal sequences $\{T_n\}$ and $\{S_n\}$ as follows, using the initial sequences of the random variables $\{(\xi_n, \eta_n)\}$, as:

$$T_0 = S_0 = 0,\ T_n = \sum_{i=1}^{n} \xi_i,\ S_n = \sum_{i=1}^{n} \eta_i,\ n \geq 1,$$

and define a sequence of integer-valued random variables $\{N_n\}$, $n \geq 0$, as:

$$N_0 = 0;\ N_1 = N(z-s) = \inf\{k \geq 1 : z - S_k \leq s\},\ z \in (s, S),$$

$$N_{n+1} = \inf\{k \geq N_n + 1 : \zeta_n - S_k + S_{N_n} \leq s\},\ n \geq 1.$$

Here $\inf(\oslash) = +\infty$ is stipulated. Put $\tau_0 = 0$, $\tau_1 \equiv T_{N_1} = \sum_{i=1}^{N(z-s)} \xi_i$, $\tau_n \equiv T_{N_n} = \sum_{i=1}^{N_n} \xi_i$, $n \geq 2$, and define $\nu(t)$ as $\nu(t) = \max\{n \geq 0 : T_n \leq t\},\ t > 0$. Here the random variable τ_1 represents the first time of the stock level drops below the control level s. We can now construct the desired stochastic process $X(t)$ as follows:

$$\begin{aligned} X(t) &= \zeta_n - (\eta_{N_n+1} + \eta_{N_n+2} + \ldots + \eta_{\nu(t)}) \\ &= \zeta_n - \left(\sum_{i=1}^{\nu(t)} \eta_i - \sum_{i=1}^{N_n} \eta_i \right) = \zeta_n - (S_{\nu(t)} - S_{N_n}), \end{aligned}$$

where $\tau_n \leq t < \tau_{n+1},\ n \geq 0$, $\zeta_0 \equiv z \in (s, S)$ are given.

The process $X(t)$ can be also rewritten as follows:

$$X(t) = \sum_{n=0}^{\infty} \big(\zeta_n - (S_{\nu(t)} - S_{N_n})\big) I_{[\tau_n;\tau_{n+1})}(t)$$

Here $I_A(t)$ represents the indicator function of the set A, so

$$I_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}$$

In this study, the process $X(t)$ is called "a renewal- reward process with a triangular interference of chance", because in this case it is assumed that the random variable ζ_1, which expresses a discrete interference of chance, has a triangular distribution.

3. Ergodicity of the process $X(t)$ and the exact formulas for the ergodic distribution

To investigate the stationary characteristics of the considered process, it is necessary to prove that $X(t)$ is ergodic under some assumptions. This property can be given by the following proposition.

3.1. Proposition. *Let the initial sequence of the random variables $\{(\xi_n, \eta_n, \zeta_n)\}$ satisfy the following supplementary conditions:*

i) $0 < \mathrm{E}(\xi_1) < \infty$,
ii) $\mathrm{E}(\eta_1) > 0$,
iii) η_1 *is a non-arithmetic random variable,*
iv) *The random variable ζ_1 has a continuous distribution in the interval (s, S) such that $P\{\zeta_1 = s\} = P\{\zeta_1 = S\} = 0$.*

Then, the process $X(t)$ is ergodic and the following expression is correct with probability 1 for each measurable bounded function $f(x)$, $(f : (s, S) \to \mathbb{R})$:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(u))du = \frac{\int_s^S \int_s^S f(x)[U_\eta(z-s) - U_\eta(z-x)]\, d\pi(z)\, dx}{\int_s^S U_\eta(z-s)\, d\pi(z)}. \tag{3.1}$$

Here $U_\eta(x)$ is a renewal function generated by the sequence $\{\eta_n\}$. So,

$$U_\eta(x) = \sum_{n=0}^{\infty} F^{*n}(x),$$

where the notation $F^{*n}(x)$ represents the nth convolution of the distribution function $F(x)$.

Proof. As mentioned in the Introduction, the process $X(t)$ belongs to a wide class of processes which is known in the literature as the class of semi-Markov processes with a discrete interference of chance. Furthermore, for this class the general ergodic theorem of type Smith's 'key renewal theorem' exists in the literature [11, p.243]. According to the general ergodic theorem, the following conditions must be satisfied to make the process $X(t)$ ergodic.

1) The monotone increasing random times sequence (γ_n) must exist such that the values of the process $X(t)$ at these times, meaning the values of $\aleph_n \equiv X(\gamma_n + 0)$, must form an ergodic Markov chain.
2) Additionally, the expected values of the differences $\gamma_{n+1} - \gamma_n$, $n = 1, 2, \ldots$, must be finite, that is $\mathrm{E}(\gamma_{n+1} - \gamma_n) < \infty$.

Let us show that these two assumptions are satisfied under the conditions of Proposition 3.1. Firstly, let us prove that the first assumption is satisfied. For this purpose, let us consider the sequence $\{\tau_n\}$, $n = 1, 2, \ldots$, that is defined in Section 2 instead of $\{\gamma_n\}$. Here, τ_n is the nth time that the process $X(t)$ crosses the control level s and $0 < \tau_1 < \tau_2 < \ldots$. Due to definition we have $X(\tau_n + 0) \equiv \zeta_n$. Furthermore, according to the conditions of Proposition 3.1, the random variables ζ_n form a sequence of independent random variables which have a continuous distribution in the interval $[s, S]$. So the sequence $\{\zeta_n\}$ can be considered as an embedded Markov chain. Since the ζ_n are independent and identically distributed random variables with a common distribution function

$$\pi(z) = P\left\{\zeta_1 \le z\right\},\ s \le z \le S,$$

then the sequence $\{\zeta_n\}$ forms an ergodic Markov chain. So, this shows the first assumption of the general ergodic theorem is satisfied.

Now, let us prove that the second assumption is satisfied. It is enough to show only $E(\tau_1) < \infty$ because the differences $\tau_1; \tau_2 - \tau_1; \tau_3 - \tau_2; \ldots; \tau_n - \tau_{n-1}; \ldots$ have identical distribution. According to the Wald identity, $\mathrm{E}(\tau_1)$ can be represented as follows:

$$\mathrm{E}(\tau_1) = \mathrm{E}(\sum_{i=1}^{N_1} \xi_i) = \mathrm{E}(\xi_1)\mathrm{E}(N_1).$$

Here $\mathrm{E}(N_1) = \int_s^S E(N(z-s))\, d\pi(z)$. Additionally, according to Proposition 3.1, $\mathrm{E}(\xi_1) < \infty$ is satisfied. Having $\mathrm{E}(N_1)$ finite is enough for $E(\tau_1)$ to be finite. Let us show this.

Let us recall that,

$$\mathrm{E}(N(z-s)) \equiv U_\eta(z-s) = 1 + \sum_{n=1}^{\infty} F_\eta^{*n}(z-s).$$

Here the renewal function, which is formed by the random variables $\{\eta_n\}$, is shown as $U_\eta(x)$. The function $U_\eta(x)$ is finite for each finite x [9]. On the other hand, the renewal function $U_\eta(x)$, is a positive-valued and non-decreasing function. Therefore, for each $z \in [s, S]$, $U_\eta(z-s) \le U_\eta(S-s)$ is satisfied. So, we have

$$\begin{aligned}
\mathrm{E}(N_1) &= \int_s^S \mathrm{E}(N(z-s))\, d\pi(z) \\
&= \int_s^S U_\eta(z-s)\, d\pi(z) \le \int_s^S U_\eta(S-s)\, d\pi(z) \\
&= U_\eta(S-s) \int_s^S d\pi(z) = U_\eta(S-s) < \infty.
\end{aligned}$$

Briefly, for each $0 < s < S < \infty$, $E(\tau_1) < \infty$ is true, and this shows the second assumption is satisfied. Therefore, under the conditions of Proposition 3.1, both assumptions of the general ergodic theorem [11, p.243] are satisfied. That means that the process $X(t)$ is ergodic when the conditions of Proposition 3.1 are satisfied. This completes the proof of Proposition 3.1. □

Let us denote the ergodic distribution function of the process $X(t)$ by $Q_X(x)$:

$$Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \le x\},\ x \in [s, S].$$

We can obtain the following exact result for the ergodic distribution function $Q_X(x)$ by replacing the function $f(x)$ with indicator function in Proposition 3.1:

$$Q_X(x) = 1 - \frac{\int_x^S U_\eta(z-x) d\pi(z)}{\int_s^S U_\eta(z-s) d\pi(z)},\ x \in [s, S]. \tag{3.2}$$

In this case, Equation (3.2) yields the following exact expression for $Q_X(x)$. For each $x \in [s, \frac{S+s}{2}]$:

$$Q_X(x) = 1 - \frac{\int_x^{(S+s)/2} (z-s) U_\eta(z-x)\, dz + \int_{(S+s)/2}^S (S-z) U_\eta(z-x)\, dz}{\int_s^{(S+s)/2} (z-s) U_\eta(z-s)\, dz + \int_{(S+s)/2}^S (S-z) U_\eta(z-s)\, dz}; \tag{3.3}$$

and for each $x \in [\frac{S+s}{2}, S]$:

$$(3.4) \qquad Q_X(x) = 1 - \frac{\int_x^S (S-z)U_\eta(z-x)\,dz}{\int_s^{(S+s)/2}(z-s)U_\eta(z-s)\,dz + \int_{(S+s)/2}^S (S-z)U_\eta(z-s)\,dz}.$$

3.2. Remark. It is possible to find the exact form of the renewal function $U_\eta(x)$ when the random variable η_1 has a suitable simple distribution (for example exponential, Erlang, etc). But, for most of distributions, it is very difficult to find an exact and closed form for the renewal function $U_\eta(x)$. Moreover the obtained expressions will be very complex. Let us give the following two examples to show this complexity.

3.3. Example. Let the random variable η_1 have an exponential distribution with parameter $\lambda > 0$, and the random variable ζ_1 a triangular distribution in the interval (s, S) with mode $(S + s)/2$. Then an exact expression of the ergodic distribution function $Q_X(x)$ of the process $X(t)$ can be given as follows:

$$Q_X(x) = 1 - \frac{3\lambda(S-s)^2(S+s-2x) + 4(x-s)^2[\lambda(x-s)-3] + 6(S-s)^2}{3(S-s)^2[\lambda(S-s)+2]},$$

when $x \in \left[s, \frac{S+s}{2}\right]$, and

$$Q_X(x) = 1 - \frac{4(S-x)^2[\lambda(S-x)+3]}{3(S-s)^2[\lambda(S-s)+2]},$$

when $x \in \left[\frac{S+s}{2}, S\right]$.

3.4. Example. Let the random variable η_1 have the second order Erlang distribution with parameter $\lambda > 0$, and the random variable ζ_1 a triangular distribution in the interval (s, S) with mode $(S+s)/2$. Then an exact expression of the ergodic distribution function $Q_X(x)$ of the process $X(t)$ can be given as follows:

$$Q_X(x) = \begin{cases} 1 - \dfrac{J_1(x)}{J_1(s)}, & x \in \left[s, \dfrac{S+s}{2}\right], \\ 1 - \dfrac{J_2(x)}{J_1(s)}, & x \in \left[\dfrac{S+s}{2}, S\right], \end{cases}$$

where

$$\begin{aligned} J_1(x) &= \lambda^3(S+s-2x)(S-s)^2 + 3\lambda^2(S-s)^2 + \lambda^2(x-s)^2[(4\lambda/3)(x-s)-6] \\ &\quad + 2\lambda(x-s) + 1 + \exp(-2\lambda(S-x)) - 2\exp(-\lambda(S+s-2x)), \\ J_2(x) &= 2\lambda^2(S-x)^2[(2\lambda/3)(S-x)+3] - 2\lambda(S-x)[1-\exp(-2\lambda(S-x))]. \end{aligned}$$

3.5. Remark. As is seen from Example 3.1 and Example 3.2, the exact expressions for the ergodic distribution of the process can be extremely complex even in very simple cases. Besides, when the distribution of the random variable η_1 is not the exponential or Erlang distribution, it becomes a very complex problem to find the exact form of the renewal function $U_\eta(x)$. Thus it is advisable to derive an approximation formula for the ergodic distribution which is simpler, instead of the exact but complex formula.

4. Exact and asymptotic results for the ergodic distribution of the process $Y(t)$

As can be seen from the above examples, the exact expression of the ergodic distribution function $Q_X(x)$ of the process $X(t)$ has a very complex mathematical structure. The most effective way to remove this complexity is to obtain an asymptotic expansion for $Q_X(x)$ as $S - s \to \infty$, by using certain asymptotic methods. But before using these asymptotic methods, it will be useful to "standardize" the process $X(t)$. For this purpose, let us define the process $Y(t)$, which is a linear transformation of the process $X(t)$, as follows:

$$Y(t) = \frac{X(t) - s}{a} \in [0, 2], \text{where} \quad a \equiv (S - s)/2.$$

Additionally, set

$$Q_Y(v) = \lim_{t\to\infty} P\{Y(t) \le v\}, \quad v \in [0, 2].$$

By considering these notations we can give the following proposition.

4.1. Proposition. *In addition to the assumptions of Proposition 3.1, let the following condition be satisfied:*

> *The random variables ζ_n have a triangular distribution in the interval (s, S) with mode $(S + s)/2$.*

Then, the ergodic distribution function $Q_Y(v)$ of the process $Y(t)$ has the following exact form:

$$Q_Y(v) = 1 - \frac{\int_{av}^{2a} U_\eta(x - av)\widetilde{\rho}_a(x)dx}{\int_0^{2a} U_\eta(x)\widetilde{\rho}_a(x)\,dx}, \ v \in [0, 2], \tag{4.1}$$

where $\widetilde{\rho}_a(x) = x/a^2$ if $0 \le x \le a$, and $\widetilde{\rho}_a(x) = (2a - x)/a^2$ if $a < x \le 2a$.

Proof. For each $v \in [0, 2]$

$$P\{Y(t) \le v\} = P\left\{\frac{X(t) - s}{a} \le v\right\} = P\{X(t) \le s + av\}.$$

Therefore,

$$Q_Y(v) = Q_X(s + av) = 1 - \frac{\int_{s+av}^{s+2a} U_\eta(z - s - av)d\pi(z)}{\int_s^{s+2a} U_\eta(z - s)d\pi(z)}.$$

Since the random variable ζ_1 has a triangular distribution in the interval (s, S) with mode $(S + s)/2$, then the random variable $\widetilde{\zeta_1} = \zeta_1 - s$ will have a triangular distribution in the interval $[0, 2a]$, with mode $a \equiv (S - s)/2$. So we have,

$$Q_Y(v) = 1 - \frac{\int_{av}^{2a} U_\eta(x - av)\widetilde{\rho}_a(x)\,dx}{\int_0^{2a} U_\eta(x)\widetilde{\rho}_a(x)\,dx}, \ v \in [0, 2].$$

This completes the proof of Proposition 4.1. □

Now, let us try to obtain the asymptotic expansion for the ergodic distribution function of the process $Y(t)$, when $a \to \infty$. For this purpose, let us give some asymptotic results about the renewal function $U_\eta(x)$. Note that there are many valuable studies about the asymptotic behaviour of the renewal function $U_\eta(x)$ in the literature (see, for example, [9,21, 24, 25]).

4.2. Lemma. *[9, p.366]. Assume that η_1 is a non-arithmetic random variable, and that the condition $\mathrm{E}(\eta_1^2) < \infty$ is satisfied. Then, the asymptotic expansion for the renewal function $U_\eta(x)$ can be written as follows, when $x \to \infty$:*

$$U_\eta(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} + o(1),$$

where $m_k = E(\eta_1^k)$, $k = 1, 2, \ldots$. □

4.3. Remark. If we know the asymptotic behaviour of the renewal function $U_\eta(x)$, it is not difficult to obtain the asymptotic expansion for the integrals depending on $U_\eta(x)$ (see, for example, [8,9]).

4.4. Lemma. *Under the conditions of Lemma 4.1, as $x \to \infty$ we have:*

$$I_0(x) \equiv \int_0^x U_\eta(t)dt = \frac{x^2}{2m_1} + cx + o(x), \ c = \frac{m_2}{2m_1^2}. \qquad \square$$

4.5. Lemma. *Under the conditions of Lemma 4.1, as $x \to \infty$ we have:*

$$I_1(x) \equiv \int_0^x tU_\eta(t)\,dt = \frac{x^3}{3m_1} + \frac{cx^2}{2} + o(x^2), \ c = \frac{m_2}{2m_1^2}. \qquad \square$$

Let us now state the theorem which is the basic aim of this study.

4.6. Theorem. *Suppose that, in addition to the assumptions of Proposition 4.1, the condition $\mathrm{E}(\eta_1^2) < \infty$ is satisfied. Then for each $v \in [0,2]$ we have the following asymptotic expansion for the ergodic distribution function $Q_Y(v)$ of the process $Y(t)$ when $a \equiv (S-s)/2 \to \infty$:*

$$Q_Y(v) = G_0(v) - \frac{m_2}{12m_1}\frac{G_1(v)}{a} + o\Big(\frac{1}{a}\Big), \tag{4.2}$$

where

$$G_0(v) = \begin{cases} 1-(2-v)^3/6+(1-v)^3/3, & v \in [0,1], \\ 1-(2-v)^3/6, & v \in (1,2], \end{cases} \tag{4.3}$$

$$G_1(v) = \begin{cases} [(2-v)^2(1-v) - 2(1-v)^2(2+v), & v \in [0,1], \\ (2-v)^2(1+v), & v \in (1,2]. \end{cases} \tag{4.4}$$

Proof. To simplify the notation, let us denote the numerator and denominator in Equation (4.1) by $J_{1a}(v) = \int_{av}^{2a} U_\eta(x-av)\widetilde{\rho}_a(x)\,dx$ and $J_{1a}(0) = \int_0^{2a} U_\eta(x)\widetilde{\rho}_a(x)\,dx$, respectively. Then, the denominator can be written as follows:

$$J_{1a}(0) = \frac{1}{a^2}[2I_1(a) - I_1(2a)] + \frac{2}{a}[I_0(2a) - I_0(a)], \tag{4.5}$$

where $I_0(x) = \int_0^x U_\eta(t)\,dt$; $I_1(x) = \int_0^x tU_\eta(t)\,dt$. By substituting the results of Lemma 4.1 and Lemma 4.2 in Equation (4.5), we can obtain the following asymptotic expansion for $a \to \infty$:

$$J_{1a}(0) = \frac{a}{m_1} + \frac{m_2}{2m_1^2} + o(1). \tag{4.6}$$

Moreover, the integral $J_{1a}(v)$ can be written in the following exact form:

$$J_{1a}(v) = \frac{1}{a^2}[2I_1(a(1-v)) - I_1(a(2-v))] + \frac{1}{a}[2(v-1)I_0(a(1-v)) + (2-v)I_0(a(2-v))], \tag{4.7}$$

when $v \in [0,1]$, and

$$J_{1a}(v) = \frac{1}{a^2}[(a(2-v))I_0(a(2-v)) - I_1(a(2-v))], \tag{4.8}$$

when $v \in (1,2]$. Taking into account Lemma 4.1 and Lemma 4.2 in Equation (4.7) and Equation (4.8), we can write the following asymptotic expansion for $J_{1a}(v)$ when $a \to \infty$:

$$J_{1a}(v) = \begin{cases} \dfrac{a[(2-v)^3 - 2(1-v)^3]}{6m_1} + \dfrac{c[(2-v)^2 - 2(1-v)^2]}{2} + o(1), & v \in [0,1], \\ \dfrac{a(2-v)^3}{6m_1} + \dfrac{c(2-v)^2}{2} + o(1), & v \in (1,2]. \end{cases} \tag{4.9}$$

Substituting the asymptotic expansions (4.7), (4.8) and (4.9) in Equation (4.1), we have the following asymptotic expansion for $Q_Y(v)$ when $v \in [0,1]$ and $a \to \infty$:

$$
\begin{aligned}
Q_Y(v) &= 1 - \frac{J_{1a}(v)}{J_{1a}(0)} \\
&= 1 - \frac{\dfrac{a\big[(2-v)^3 - 2(1-v)^3\big]}{6m_1} + \dfrac{c\big[(2-v)^2 - 2(1-v)^2\big]}{2} + o(1)}{\dfrac{a}{m_1} + c + o(1)} \\
&= 1 - \frac{(2-v)^3 - 2(1-v)^3}{6} - \frac{m_2}{12m_1 a}\big[(2-v)^2(1+v) - 2(1-v)^2(2+v)\big] \\
&\qquad + o\Big(\frac{1}{a}\Big).
\end{aligned}
$$

In a similar manner, it is not difficult to obtain the following asymptotic expansion for $Q_Y(v)$ when $v \in (1,2]$ and $a \to \infty$:

$$
\begin{aligned}
Q_Y(v) &= 1 - \frac{J_{1a}(v)}{J_{1a}(0)} = 1 - \left\{\left[\frac{a(2-v)^3}{6m_1} + \frac{c(2-v)^2}{2} + o(1)\right]\Big/\left[\frac{a}{m_1} + c + o(1)\right]\right\} \\
&= 1 - \frac{(2-v)^3}{6} - \frac{m_2}{12m_1 a}(2-v)^2(1-v) + o\Big(\frac{1}{a}\Big).
\end{aligned}
$$

Therefore, for each $v \in [0,2]$, we have:

$$
Q_Y(v) = G_0(v) - \frac{m_2}{12m_1}\frac{G_1(v)}{a} + o\Big(\frac{1}{a}\Big), \text{ as } a \to \infty.
$$

Thus we have proved Theorem 4.1, which is the basic aim of this study. □

Now, we can obtain the weak convergence theorem for the ergodic distribution function $(Q_Y(v))$ of the process $Y(t)$, as $a \to \infty$.

4.7. Proposition. *Assume that the conditions of Theorem 4.1 are satisfied. Then the ergodic distribution of the process $Y(t)$ weakly converges to the limit distribution $G_0(v)$, i.e.,*

$$
Q_Y(v) \to G_0(v)
$$

when $a \to \infty$ and $v \in [0,2]$. The function $G_0(v)$ is defined in Equation (4.3).

Proof. From Equation (4.4), it is not difficult to see that the value $v^* = \sqrt{3}-1$ maximizes the function $G_1(v)$. At this point the value of the function $G_1(v)$ is equal to $2K$. Here the constant K is equal to $K = 3\sqrt{3} - 4 \cong 1.196152423\ldots$. Therefore, for each $v \in [0,2]$, since $|G_1(v)| \le 2K < \infty$, then $G_1(v)/a \to 0$ while $a \to \infty$.

In other words, the ergodic distribution of the process $Y(t)$ weakly converges to the limit distribution $G_0(v)$ when $a \to \infty$, i.e., for each $v \in [0,2]$, $Q_Y(v) \to G_0(v)$. This completes the proof of Proposition 4.2. □

4.8. Remark. It can be observed that the form of the limit distribution $G_0(v)$, which is obtained by applying asymptotic methods, is simpler than the exact formulas (see, Example 3.1 and Example 3.2). By using this simple form of the limit distribution, it is possible to derive many probability characteristics of the process very easily and quickly, which are very important for inventory models. But besides simplicity, it is very important to show that the approximated formulas are very close to the exact formulas. For this purpose, in the following section, we compare the approximated values of the ergodic distribution function of the process to those obtained by the Monte Carlo simulation method.

5. Simulation Results

Suppose that the random variable ζ_1 has a triangular distribution in the interval $(1, 1+2a)$ and the random variable η_1 has exponential distribution with a parameter $\lambda = 1$. Besides, assume that $\hat{Q}_Y(v)$ denotes the value of the ergodic distribution function of the process $X(t)$, which is calculated using the Monte Carlo simulation method, and $\widetilde{Q}_Y(v)$ denotes the value of the first two terms of the asymptotic expansion given by Theorem 4.1, i.e.,

$$\widetilde{Q}_Y(v) \equiv G_0(v) - (m_2 G_1(v)/12 m_1 a)\,, \; v \in (0,2], \; a \equiv (S-s)/2.$$

Furthermore let $\Delta = \left|\hat{Q}_Y(v) - \widetilde{Q}_Y(v)\right|$; $\delta = \frac{\Delta}{\hat{Q}_Y(v)} 100\%$ and $AP = 100 - \delta$. In other words, the numbers Δ, δ and AP denote the absolute error, relative error and accuracy percentage between the simulation and asymptotic results for the ergodic distribution function of the process $Y(t)$, respectively. So we can generate Table 1, Table 2, and Table 3.

For the calculation of each value of $\hat{Q}_Y(v)$ in the tables, we simulated 2×10^6 trajectories of the process $X(t)$. As seen from the presented tables, the approximating formulas are of high accuracy even for small values of the parameter $a \equiv (S-s)/2$. For example, as seen from Table 2 and Table 3, the accuracy percentage (AP) is greater than %99, for each value of the parameters $a \geq 10$ and $v \in (0,2]$. This indicates that the asymptotic expansion obtained can be safely applied to different problems of inventory or queuing models, even for values of the parameter $a \equiv (S-s)/2$ that are not large.

Table 1. Comparison of the Asymptotic and the Simulation Values of the Ergodic Distribution for the case $a = 5$, $(a \equiv (S-s)/2)$.

v	$\hat{Q}_Y(v)$	$\widetilde{Q}_Y(v)$	Δ	$\delta(\%)$	$AP(\%)$
0,1	0,0840	0,0809	0,0031	3,6905	96,3095
0,2	0,1688	0,1629	0,0059	3,4953	96,5047
0,3	0,2532	0,2454	0,0078	3,0806	96,9194
0,4	0,3376	0,3275	0,0101	2,9917	97,0083
0,5	0,4202	0,4083	0,0119	2,8320	97,1680
0,6	0,5005	0,4872	0,0133	2,6573	97,3427
0,7	0,5760	0,5633	0,0127	2,2049	97,7951
0,8	0,6488	0,6357	0,0131	2,0191	97,9809
0,9	0,7161	0,7038	0,0123	1,7176	98,2824
1,0	0,7776	0,7667	0,0109	1,4017	98,5983
1,1	0,8310	0,8218	0,0092	1,1071	98,8929
1,2	0,8756	0,8677	0,0079	0,9022	99,0978
1,3	0,9118	0,9053	0,0065	0,7129	99,2871
1,4	0,9401	0,9352	0,0049	0,5212	99,4788
1,5	0,9619	0,9583	0,0035	0,3639	99,6361
1,6	0,9779	0,9755	0,0024	0,2454	99,7546
1,7	0,9886	0,9874	0,0012	0,1214	99,8786
1,8	0,9955	0,9949	0,0006	0,0603	99,9397
1,9	0,9990	0,9989	0,0001	0,0100	99,9900
2,0	1,0000	1,0000	0,0000	0,0000	100,0000

Table 2. Comparison of the Asymptotic and the Simulation Values of the Ergodic Distribution for the case $a = 10$, $(a \equiv (S - s)/2)$.

v	$\hat{Q}_Y(v)$	$\widetilde{Q}_Y(v)$	Δ	$\delta(\%)$	$AP(\%)$
0,1	0,0912	0,0904	0,0008	0,8772	99,1228
0,2	0,1824	0,1808	0,0016	0,8772	99,1228
0,3	0,2731	0,2705	0,0026	0,9520	99,0480
0,4	0,3612	0,3584	0,0030	0,8301	99,1699
0,5	0,4472	0,4438	0,0034	0,7603	99,2397
0,6	0,5296	0,5256	0,0040	0,7553	99,2447
0,7	0,6063	0,6031	0,0032	0,5278	99,4722
0,8	0,6783	0,6752	0,0031	0,4570	99,5430
0,9	0,7444	0,7412	0,0032	0,4299	99,5701
1,0	0,8032	0,8000	0,0032	0,3984	99,6016
1,1	0,8526	0,8502	0,0024	0,2815	99,7185
1,2	0,8936	0,8912	0,0024	0,2686	99,7314
1,3	0,9258	0,9241	0,0017	0,1836	99,8164
1,4	0,9508	0,9496	0,0012	0,1262	99,8738
1,5	0,9696	0,9688	0,0008	0,0825	99,9175
1,6	0,9829	0,9824	0,0005	0,0509	99,9490
1,7	0,9918	0,9915	0,0003	0,0302	99,9698
1,8	0,9970	0,9968	0,0002	0,0201	99,9799
1,9	0,9994	0,9994	0,0000	0,0000	100,0000
2,0	1,0000	1,0000	0,0000	0,0000	100,0000

Table 3. Comparison of the Asymptotic and the Simulation Values of the Ergodic Distribution for the case $a = 20$, $(a \equiv (S - s)/2)$.

v	$\hat{Q}_Y(v)$	$\widetilde{Q}_Y(v)$	Δ	$\delta(\%)$	$AP(\%)$
0,1	0,0955	0,0951	0,0004	0,4188	99,5812
0,2	0,1909	0,1897	0,0012	0,6286	99,3714
0,3	0,2850	0,2830	0,0020	0,7018	99,2982
0,4	0,3767	0,3739	0,0028	0,7433	99,2567
0,5	0,4649	0,4615	0,0034	0,7313	99,2687
0,6	0,5486	0,5448	0,0038	0,5839	99,3073
0,7	0,6269	0,6229	0,0040	0,6381	99,3619
0,8	0,6984	0,6949	0,0035	0,5011	99,4989
0,9	0,7633	0,7598	0,0035	0,4585	99,5415
1,0	0,8195	0,8167	0,0028	0,3417	99,6583
1,1	0,8661	0,8643	0,0018	0,2078	99,7922
1,2	0,9044	0,9029	0,0015	0,1659	99,8341
1,3	0,9348	0,9334	0,0014	0,1498	99,8502
1,4	0,9575	0,9568	0,0007	0,0731	99,9269
1,5	0,9744	0,9740	0,0004	0,0411	99,9589
1,6	0,9862	0,9859	0,0003	0,0304	99,9696
1,7	0,9937	0,9935	0,0002	0,0201	99,9799
1,8	0,9978	0,9977	0,0001	0,0100	99,9900
1,9	0,9996	0,9996	0,0000	0,0000	100,0000
2,0	1,0000	1,0000	0,0000	0,0000	100,0000

6. Summary and Conclusion

In this study, a semi-Markovian model of type (s, S) is considered. This model is expressed by using a renewal-reward process with a discrete interference of chance. An exact expression of the ergodic distribution function of the process is derived when the random variable ζ_1, which describes a discrete interference of chance, has a triangular distribution. However, it is very difficult to use this exact expression in solving concrete problems of inventory or queuing theory, because this exact expression has a very complex mathematical structure. In this study, an asymptotic method is used to overcome this mathematical difficulty. At the same time, the two-term asymptotic expansion for the ergodic distribution is obtained, when $S - s \to \infty$. Using this expansion, a weak convergence theorem for the ergodic distribution is proved, and the exact expression of the limit distribution $(G_0(v))$ is derived. The accuracy of the approximated result is tested by using the Monte Carlo simulation method. The approximation is satisfactory even for small values of the parameter $S - s$. It is easy to see that this approximation formula is not only simple and clear, but also accurate.

Note that it is important to obtain a similar asymptotic result for the delayed (s, S) models by using the methods and approaches introduced in this paper. Such investigation can also be applied to the semi-Markovian random walk process. Applying this approach to distribution other than triangular distribution is another promising direction for future research.

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