Hacettepe Journal of Mathematics and Statistics
Volume 39 (3) (2010), 305 – 312

ON PRIME AND MAXIMAL k-SUBSEMIMODULES OF SEMIMODULES

Gürsel Yeşilot*

Received 05 : 02 : 2009 : Accepted 29 : 01 : 2010

Abstract

In this paper, we present some characterizations of prime k-ideals and maximal k-ideals of a semiring. Then we extend these properties to prime k-subsemimodules and maximal k-subsemimodules of a semimodule. After that, the correspondence between prime k-ideals and prime k-subsemimodules, and between maximal k-ideals and maximal k-subsemimodules are given.

Keywords: Prime k-ideal, Maximal k-ideal, Prime k-subsemimodule, Maximal k-subsemimodule.

2000 AMS Classification: 16 Y 60.

1. Introduction

In this work we investigate k-ideals of semirings and k-subsemimodules of semimodules. In Section 2 the definitions of prime k-ideals and maximal k-ideals are given. Then we give some characterizations of prime k-ideals and maximal k-ideals, and give the prime avoidance theorem for k-ideals. In Section 3 we extend these properties to prime k-subsemimodules and maximal k-subsemimodules. Then, relations between k-ideals and k-subsemimodules are given.

First of all we recall some known definitions.

A set R together with two associative binary operations called addition $(+)$ and multiplication $(\cdot)$ is called a semiring provided:

i) Addition is a commutative operation, and
ii) Multiplication distributes over addition both from the left and from the right.

*Yıldız Technical University, Department of Mathematics, Davutpaşa, 34210 Esenler, Istanbul, Turkey. E-mail: `gyesilot@yildiz.edu.tr`

An element $0 \in R$ such that $x + 0 = 0 + x = x$ and $x0 = 0x = 0$ for each $x \in R$ is called an absorbing zero element.

A subset I of a semiring R is called an ideal of R if for $a, b \in I$ and $r \in R$ we have $a + b \in I$, $ra \in I$ and $ar \in I$. An ideal I of a semiring R is called trivial iff $I = R$ or $I = \{0\}$. For each ideal I of a semiring R the k-closure $\overline{I}$ of I is defined by

$$\overline{I} = \{\bar{a} \in R \mid \bar{a} + a_1 = a_2 \text{ for some } a_1, a_2 \in I\},$$

and is an ideal of R satisfying $I \subseteq \overline{I}$ and $\overline{\overline{I}} = \overline{I}$. An ideal I is called a k-ideal of R if and only if $I = \overline{I}$ holds.

Given two semirings R and R', a mapping η from R to R' is called a homomorphism if $\eta(a + b) = \eta(a) + \eta(b)$ and $\eta(ab) = \eta(a)\eta(b)$ for each $a, b \in R$. An isomorphism is a one-to-one homomorphism (see [1] for more details). In this situation the semirings R and R' are called isomorphic.

Each ideal I of a semiring R defines a congruence κ_I on $(R, +, \cdot)$ by

$$r\kappa_I r' \iff r + a_1 = r' + a_2 \text{ for some } a_1, a_2 \in I.$$

The corresponding congruence class semiring R/κ_I consisting of the classes $[r]_{\kappa_I} = [r]$, contains the k-closure $\overline{I}$ of I as one of its classes, and $\overline{I}$ is a multiplication absorbing zero of R/κ_I. (see [6]).

An ideal I is called prime if $ab \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in R$. For details of prime ideals the reader is refereed to [3,6] and [7]. Throughout this paper we let the semiring R be commutative.

If R is a semiring, an additively written commutative semigroup M with neutral element θ is called an R-semimodule if

i) $r \cdot (m + m') = r \cdot m + r \cdot m'$,
ii) $(r + r') \cdot m = r \cdot m + r' \cdot m$,
iii) $(rr') \cdot m = r \cdot (r' \cdot m)$,
iv) $1 \cdot m = m$,
v) $r \cdot \theta = 0 \cdot m = \theta$,

for all $m, m' \in M$ and $r, r' \in R$. A subset N of the R-semimodule M will be called a subsemimodule of M if $a, b \in N$ and $r \in R$ implies $a + b \in N$ and $ra \in N$.

The k-closure of a subsemimodule of the R-semimodule M is defined by

$$\overline{N} = \{\bar{a} \in M \mid \bar{a} + a_1 = a_2 \text{ for some } a_1, a_2 \in N\}.$$

Note that $\overline{N}$ is a subsemimodule, satisfying $N \subset \overline{N}$ and $\overline{N} = \overline{\overline{N}}$. We say that a subsemimodule N of M is a k-subsemimodule if $N = \overline{N}$.

The annihilator of a subsemimodule M/N is defined by

$$A_N(M) = \{a \in R : aM \subseteq N\}.$$

Clearly, $A_N(M)$ is an ideal of the semiring R. A subsemimodule N is called prime if $rm \in N$ implies $r \in A_N(M)$ or $m \in N$ for all $r \in R$ and $m \in M$.

Each subsemimodule N of a semimodule M defines a congruence κ_N on $(M, +)$ by

$$m\kappa_N m' \iff m + a_1 = m' + a_2 \text{ for some } a_1, a_2 \in N.$$

The corresponding congruence class semimodule M/κ_N, consisting of the classes $[m]_{\kappa_N} = [m]$, contains the k-closure $\overline{N}$ of N as one of its classes, and $\overline{N}$ is a multiplication absorbing zero of M/κ_N.

2. Prime k-ideals and maximal k-ideals

Recall that the natural homomorphism $\Psi : R \to R/\kappa_I$ of a semiring R for some ideal I of R is defined by $r \mapsto [r]_{\kappa_I} = [r]$.

2.1. Lemma. *Let R be a semiring and I an ideal of the semiring R. Consider the natural homomorphism $\Psi : R \longrightarrow R/\kappa_I$. Then*

(i) *If $I \subseteq J$ is a k-ideal of R, then $\Psi(J)$ is a k-ideal of R/κ_I,*
(ii) *If J is a k-ideal of R/κ_I then $\Psi^{-1}(J)$ is a k-ideal of R.*

Proof. Let $\Psi : R \to R/\kappa_I$ be the natural homomorphism, i.e $\Psi(r) = [r]$.
(i) Let $I \subseteq J$ be a k-ideal of R. Suppose that $[x] + [r_1] = [r_2]$ for some $r_1, r_2 \in J$ and $x \in R$. Then we have for some $a_1, a_2 \in I$, $x + r_1 + a_1 = r_2 + a_2$. Since $r_1 + a_1, r_2 + a_2 \in J$ and J is a k-ideal we get $x \in J$, so $\Psi(x) = [x] \in \Psi(J)$.

(ii) Let J be a k-ideal of R/κ_I, and suppose that $x + r_1 = r_2$ for some $r_1, r_2 \in \Psi^{-1}(J)$ and $x \in R$. Therefore we get $[r_2] = [x + r_1] = [x] + [r_1]$, and since J is a k-ideal we get $[x] \in J$. So $\Psi(x) = [x] \in J$, consequently $x \in \Psi^{-1}(J)$. □

2.2. Definition. Let $(R, +, \cdot)$ be a semiring and I a prime ideal of R. If I is a k-ideal, then we call I a *prime k-ideal*.

2.3. Theorem. *Let R be a semiring and I an ideal of the semiring R. Consider the natural homomorphism $\Psi : R \to R/\kappa_I$ for some ideal I of R. Then an ideal $I \subseteq J$ of R is a prime k-ideal if and only if $\Psi(J)$ is prime k-ideal of R/κ_I.*

Proof. Let $I \subseteq J$ be a prime k-ideal of R and for some $r_1, r_2 \in R$ let $\Psi(r_1)\Psi(r_2) \in \Psi(J)$. Then $\Psi(r_1 r_2) = \Psi(x) \in \Psi(J)$ for some $x \in J$. Then for some $a_1, a_2 \in I$, $r_1 r_2 + a_1 = x + a_2$ and since J is a k-ideal we get $r_1 r_2 \in J$. Therefore $r_1 \in J$ or $r_2 \in J$. Thus $\Psi(r_1) \in \Psi(J)$ or $\Psi(r_2) \in \Psi(J)$.

Conversely, assume that $\Psi(J)$ is a prime k-ideal for an ideal J containing I. If $r_1 r_2 \in J$ for some $r_1, r_2 \in R$, then $\Psi(r_1 r_2) \in \Psi(J)$. So $\Psi(r_1)\Psi(r_2) \in \Psi(J)$. Therefore $\Psi(r_1) \in \Psi(J)$ or $\Psi(r_2) \in \Psi(J)$. Thus $r_1 \in J$ or $r_2 \in J$. □

2.4. Lemma. *Let P be a prime k-ideal of the commutative semiring R, and let $I_1, \dots, I_n$ be ideals of R. Then the following statements are equivalent:*

(i) *$P \supseteq I_j$ for some j with $1 \leq j \leq n$,*
(ii) *$P \supseteq \bigcap_{i=1}^n I_i$,*
(iii) *$P \supseteq \prod_{i=1}^n I_i$.*

Proof. (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are trivial.

(iii) $\Longrightarrow$ (i) Let $P \supseteq \prod_{i=1}^n I_i$ for some ideals $I_1, \dots, I_n$ of R and prime k-ideal P. Assume that $P \not\supseteq I_j$ for all j with $1 \leq j \leq n$. Then for all j there exist elements $a_j \in I_j \setminus P$, where $1 \leq j \leq n$. Therefore $a_1 \cdots a_n \in \prod_{i=1}^n I_i \subseteq P$. Since P is prime we get for some j with $1 \leq j \leq n$, $a_j \in P$ which is a contradiction. □

Let R be a ring with identity. One of the fundamental theorems of commutative ring theory is the "prime avoidance" theorem, which states that if $P_1, P_2, \dots, P_n$ are prime ideals of R and I is an ideal of R such that $I \subseteq \bigcup_{i=1}^n P_i$, then $I \subseteq P_i$ for some $1 \leq j \leq n$. In the following Theorem we will prove the "prime avoidance" theorem for semirings. Now let us give a lemma for the theorem.

2.5. Lemma. *Let P_1, P_2 be k-ideals of a commutative semiring R and I an ideal of R such that $I \subseteq P_1 \cup P_2$. Then $I \subseteq P_1$ or $I \subseteq P_2$.*

Proof. Let $I \subseteq P_1 \cup P_2$ and we assume that P_1, P_2 are k-ideals. Suppose that $I \nsubseteq P_1$ and $I \nsubseteq P_2$. Thus there exist elements $a_1 \in I \setminus P_1$ and $a_2 \in I \setminus P_2$ so that $a_1 \in P_2$ and $a_2 \in P_1$. Since I is an ideal we get $a_1 + a_2 \in I \subseteq P_1 \cup P_2$, and thus $a_1 + a_2 \in P_1$ or $a_1 + a_2 \in P_2$. Since P_1, P_2 are k-ideals, we have $a_1 \in P_1$ or $a_2 \in P_2$, which is a contradiction. Therefore $I \subseteq P_1$ or $I \subseteq P_2$. □

2.6. Theorem. *Let $P_1, \ldots, P_n$, $n \geq 2$, be k-ideals of the commutative semiring R such that at most two of $P_1, \ldots, P_n$ are not prime. Let I be an ideal of R such that $I \subseteq \bigcup_{i=1}^{n} P_i$. Then $I \subseteq P_j$ for some j with $1 \leq j \leq n$.*

Proof. We will prove this using induction on n. Consider first the case $n = 2$. This is given by Lemma 2.5. Now assume that it is true for $n = k$ and let $n = k + 1$. Let $I \subseteq \bigcup_{i=1}^{k+1} P_i$ and, since at most 2 of the P_i are not prime, we can assume that they have been indexed in such a way that P_{k+1} is prime.

Suppose that for each $j = 1, \ldots, k+1$, it is the case that $I \nsubseteq \bigcup_{i=1,\, i\neq j}^{k+1} P_i$. We will obtain a contradiction, so by the induction hypothesis $I \subseteq P_t$ for some $t = 1, \ldots, k+1$ where $t \neq j$. Now we have elements $a_j \in I \setminus \bigcup_{i=1,\, i\neq j}^{k+1} P_i$ with $a_j \in P_j$. Also, since P_{k+1} is prime, we have $a_1 \cdots a_k \notin P_{k+1}$. Thus $a_1 \cdots a_k \in \bigcap_{i=1}^{k} P_i \setminus P_{k+1}$ and $a_{k+1} \in P_{k+1} \setminus \bigcup_{i=1}^{k} P_i$. Now, consider the element $b = a_1 \cdots a_k + a_{k+1}$. Since $a_1 \cdots a_k \in I$ and $a_{k+1} \in I$ we get $b \in I \subseteq \bigcup_{i=1}^{k+1} P_i$. Thus we have $b \in P_{k+1}$ or $b \in P_j$ for some $1 \leq j \leq k$. If $b \in P_{k+1}$ then $a_1 \cdots a_k \in P_{k+1}$, which is a contradiction. If $b \in P_j$ for some $1 \leq j \leq k$ then $a_{k+1} \in P_j$, which is again a contradiction. Hence we get $I \subseteq \bigcup_{i=1,\, i\neq j}^{k+1} P_i$ for some j and by the induction hypothesis we get $I \subseteq P_i$ for some $1 \leq i \leq k+1$, $i \neq j$. □

2.7. Definition. Let R be a semiring with an absorbing zero element 0. Then we call R an *integral semidomain* if $ab = 0$ implies $a = 0$ or $b = 0$ for any $a, b \in R$.

2.8. Example. 1) $\mathbb{N}$, the set of all non-negative integers, is an integral semidomain.

2) $\mathbb{N}$ is a semiring with the binary operations $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$. Then 0 is an absorbing zero of $\mathbb{N}$, and $\mathbb{N}$ is an integral semidomain.

Note that, if R is a semiring and I is an ideal of R then R/κ_I has $\{\overline{I}\} = 0_{K_I}$ as an absorbing zero.

2.9. Theorem. *Let R be a semiring. Then a k-ideal P of R is a prime k-ideal if and only if R/κ_P is an integral semidomain.*

Proof. Let P be a prime k-ideal of R and $[r_1][r_2] = \{P\} = 0_{\kappa_P}$ for some $r_1, r_2 \in R$. Then $[r_1 r_2] = 0_{\kappa_P}$, so there exist $a_1, a_2 \in P$ such that $r_1 r_2 + a_1 = a_2$. Since P is a k-ideal we get $r_1 r_2 \in P$. Therefore, $r_1 \in P$ or $r_2 \in P$. Hence $[r_1] = 0_{\kappa_P}$ or $[r_2] = 0_{\kappa_P}$.

For the converse, suppose that R/κ_P is an integral semidomain. Assume that $r_1 r_2 \in P$ and $r_2 \notin P$ for some $r_1, r_2 \in R$. Then $[r_1 r_2] = [r_1][r_2] = 0_{\kappa_P}$. Since R/κ_P is an integral semidomain and $r_2 \notin P$ we obtain $[r_1] = 0_{\kappa_P}$. Thus, $r_1 \in P$. □

2.10. Definition. Let R be a semiring. A k-ideal $\hbar \subset R$ is called a maximal k-ideal of R if there is no k-ideal I of R satisfying $\hbar \subset I \subset R$.

We note that a maximal k-ideal of R need not to be a maximal ideal of R. For example, by [5, Sen 4.2], let $\mathbb{N}$ be the set of all non-negative integers. Then the maximal k-ideals of $\mathbb{N}$ are of the form $(p) = \{pn : n \in \mathbb{N}\}$, where p is prime, but none of these are maximal ideals. Indeed, for a prime element p, (p) is properly contained in the ideal $B = \{b \in \mathbb{N} \mid b \geq p\}$.

2.11. Theorem. *Let R be a semiring. Then every maximal k-ideal is a prime k-ideal.*

Proof. See [5, Proposition 3.1]. □

2.12. Definition. A semiring R is called a *k-semifield* if it has only the trivial k-ideals.

2.13. Theorem. *Let R be a semiring. Then $\hbar$ is a maximal k-ideal of R if and only if $R/\kappa_\hbar$ is a k-semifield.*

Proof. Let $\hbar$ be a maximal k-ideal of R. Assume that J is an k-ideal of $R/\kappa_\hbar$ such that $J \neq \{\hbar\}$. By Lemma 2.1, $\Psi^{-1}(J)$ is an k-ideal and we get $\hbar \subseteq \Psi^{-1}(J)$. Since $\hbar$ is a maximal k-ideal we obtain $\Psi^{-1}(J) = R$. Thus $J = R/\kappa_\hbar$.

Conversely, let $\hbar \subsetneq J$ be a k-ideals of R. Then $\Psi(J)$ is a k-ideal of $R/\kappa_\hbar$. Since $R/\kappa_\hbar$ is a k-semifield we get $\Psi(J) = R/\kappa_\hbar$. Hence, $J = R$. □

3. Prime k-subsemimodules and maximal k-subsemimodules

3.1. Lemma. *Let M be an R-semimodule, N a proper subsemimodule of M and consider the natural homomorphism $\Psi : M \to M/\kappa_N$ given by $\Psi(m) = [m]$. Then:*

(i) *If $N \subset K$ is a k-subsemimodule of M then $\Psi(K)$ is a k-subsemimodule of M/κ_N.*
(ii) *If K is a k-subsemimodule of M/κ_N then $\Psi^{-1}(K)$ is a k-subsemimodule of M.*

Proof. Similar to the proof of Lemma 2.1. □

3.2. Theorem. *Let M be a semimodule over a commutative semiring R. If N is a k-subsemimodule of M, then $A_N(M)$ is a k-ideal of R.*

Proof. We know that $A_N(M) \subseteq \overline{A_N(M)}$ by the property of k-closure. Now, let $x \in \overline{A_N(M)}$. Then there exist $r_1, r_2 \in A_N(M)$ such that $x + r_1 = r_2$. Therefore we obtain, for all $m \in M$, that $xm + r_1m = r_2m$. Since $r_1m, r_2m \in N$ and N is a k-subsemimodule we get $xm \in N$. Hence, $x \in A_N(M)$. □

The converse of this theorem is not true in general. To show this we give the following example.

3.3. Example. Consider the semigroup $(\mathbb{Z}, +)$ as an $\mathbb{N}$-semimodule, where $(\mathbb{N}, +, .)$ is regarded as a semiring. The subset $\mathbb{N}$ of $\mathbb{Z}$ is a subsemimodule of $\mathbb{Z}$ which is not a k-subsemimodule. Indeed, $-2+2 = 0$ for $2, 0 \in \mathbb{N}$ and $-2 \in \mathbb{Z}$, but $-2 \notin \mathbb{N}$. But on the other hand, $A_{\mathbb{N}}(\mathbb{Z}) = \{0\}$, is a k-ideal of $\mathbb{N}$.

Recall that a subsemimodule N of a semimodule M is called prime if $rm \in N$ implies $r \in A_N(M)$ or $m \in N$ for $r \in R$ and $m \in M$. The reader is refereed to [7] for details.

3.4. Definition. Let M be an R-semimodule and N a prime subsemimodule of M. Then N is called a *prime k-subsemimodule* if it is k-subsemimodule.

3.5. Theorem. *Let M be an R-semimodule and N be subsemimodule of M. Then $A_N(M)$ is a prime k-ideal of R if N is a prime k-subsemimodule.*

Proof. Let N be a prime k-subsemimodule of M. By Theorem 3.2 above, $A_N(M)$ is a k-ideal. Now we will show that $A_N(M)$ is a prime ideal of R. Let $a, b \in R$ such that $ab \in A_N(M)$ but $b \notin A_N(M)$. Then there exists an element $m \in M$ such that $bm \notin N$ but $a(bm) \in N$. Since N is prime we get $a \in A_N(M)$. □

Now we give an example showing that the converse of this theorem is not true in general.

3.6. Example. $(\mathbb{N}\times\mathbb{N}, +)$ is an $\mathbb{N}$-semimodule. It is clear that (0) is a prime k-ideal of $\mathbb{N}$. If we consider the subsemimodule $K = 0 \times 6\mathbb{N}$ of $\mathbb{N}\times\mathbb{N}$, then K is a k-subsemimodule but not prime even though the annihilator (0) is prime.

3.7. Definition. Let M be an R-semimodule over a semiring R. We call M a *k-multiplication semimodule* if for all subsemimodules N of M there exists a k-ideal I of R such that $N = IM$.

3.8. Theorem. *Let M be a k-multiplication semimodule of the semiring R. Then a k-subsemimodule N is prime if and only if $A_N(M)$ is a prime k-ideal.*

Proof. Let N be a k-subsemimodule of M such that $A_N(M)$ is a prime k-ideal. Assume that $rm \in N$ for some $r \in R$ and $m \in M$. Then $(r)(m) \subseteq N$. Since M is a k-multiplication semimodule, there exist a k-ideal of R such that $(m) = IM$. Thus $N \supseteq (r)(m) = (r)(IM) = (rI)M$. So we get $(rI) \subseteq A_N(M)$. Hence, $(r) \subseteq A_N(M)$ or $I \subseteq A_N(M)$. If $(r) \subseteq A_N(M)$, then $r \in A_N(M)$ and if $I \subseteq A_N(M)$, then $IM \subseteq N$. So $(m) \subseteq N$ and $m \in N$.

For the converse we may use the above theorem. □

3.9. Definition. Let M be an R-semimodule. A k-subsemimodule $N \subset M$ is called a *maximal k-subsemimodule* of M if there is no k-subsemimodule K of M satisfying $N \subset K \subset M$.

3.10. Theorem. *A proper k-subsemimodule N of an R-semimodule M is maximal if and only if M/κ_N has only trivial k-subsemimodules.*

Proof. Let N be maximal k-subsemimodule of M. Assume that C is a k-subsemimodule of M/κ_N such that $\{N\} \subset C \subseteq M/\kappa_N$. Since $\Psi^{-1}(C)$ is a k-subsemimodule we get $\Psi^{-1}(C) = M$. Thus, $C = M/\kappa_N$.

Conversely, assume that M/κ_N has only trivial k-subsemimodules. Consider a k-subsemimodule B of M such that $N \subset B \subseteq M$. Then $\Psi(B)$ is a k-subsemimodule of M/κ_N, so $\{N\} \subsetneq \Psi(B) \subset M/\kappa_N$. Therefore we obtain $\Psi(B) = M/\kappa_N$, which gives us $B = M$. □

3.11. Theorem. *Let M be a finitely generated semimodule over a semiring R. Then each proper k-subsemimodule N of M is contained in a maximal k-subsemimodule of M.*

Proof. Let $M = (m_1, \dots, m_n)$ be a finitely generated semimodule over a semiring R, N a k-subsemimodule of M and Σ the set of all k-subsemimodules K of M satisfying $N \subseteq K \subset M$. This set is partially ordered by inclusion. Consider a chain $\{K_i \mid i \in I\}$ in Σ, where I is a index set. Then the subsemimodule $K = \bigcup_{i \in I} K_i$ is a k-subsemimodule of M and $K \neq M$ since $M = (m_1, \dots, m_n)$. Hence $K \in \Sigma$ is an upper bound of the chain. So, by Zorn's Lemma, Σ has a maximal element, as required. □

3.12. Example. If the semimodule M is cyclic then each proper k-subsemimodule of M is contained in a maximal k-subsemimodule of M.

3.13. Definition. A semimodule M is said to *satisfy condition* (C) if and only if for all $a \in M' = M \setminus \{0\}$ and all $m \in M$ there are $r_1, r_2 \in R$ such that $m + r_1 a = r_2 a$.

3.14. Example. Consider the semigroup $(\mathbb{N}, +)$. Then since every semigroup is a $(\mathbb{N}, +, \cdot)$-semimodule, so is $(\mathbb{N}, +)$. Thus the $\mathbb{N}$-semimodule $(\mathbb{N}, +)$ satisfies condition (C).

3.15. Example. The set $U = \{0\} \cup \{u \in \mathbb{N} \mid u \geq c\}$ is a semigroup under usual addition and is a $\mathbb{N}$-semimodule. Then U satisfies condition (C) since $u + 0 \cdot u = 1 \cdot u$ for all $u \in U$.

3.16. Lemma. *If a semimodule M satisfies condition (C), then $rm = \theta$ for $r \in R$, $m \in M$, implies $m = \theta$ or $r \in A_N(M)$.*

Proof. Assume that $rm = \theta$ and $r \neq 0,\ m \neq \theta$ for some $r \in R,\ m \in M$. Then for all $s \in M$ there exist $r_1, r_2 \in R$ such that $s + r_1 m = r_2 m$. So, $rs + rr_1 m = rr_2 m$, which gives us $rs = \theta$ for all $s \in M$. There for $x \in M$, if we use condition (C), $x + r_3 s = r_4 s$ for some $r_3, r_4 \in R$. Hence, $x = \theta$. □

3.17. Theorem. *Let M be a semimodule. Then condition (C) implies that M contains only trivial k-subsemimodules. The converse is true if $Rm = \{rm \mid r \in R\} \neq \{\theta\}$ holds for all $m \in M'$.*

Proof. Assume that M satisfies condition (C). Let $\{\theta\} \neq N$ be a k-subsemimodule of M, i.e N contains at least one element $a \in M'$. Then by condition (C), for all $m \in M$ there exist $r_1, r_2 \in R$ such that $m + r_1 a = r_2 a$, since $r_1 a, r_2 a \in N$, and N is a k-subsemimodule. This gives $m \in N$. Thus, $N = M$.

For the converse, let $\theta \neq m \in M$. Then $Rm \neq \{\theta\}$ is a subsemimodule of M. By our assumption, the k-subsemimodule $\overline{Rm}$ is equal to M, i.e $\overline{Rm} = M$. This gives us

$$\{\theta\} \neq Rm \subseteq \overline{Rm} = \{\bar{m} \in M \mid \bar{m} + r_1 m = r_2 m \text{ for some } r_i \in R\} = M,$$

so the condition (C) is satisfied. □

3.18. Theorem. *Let M be a semimodule over a commutative semiring R, and N a k-subsemimodule of M. Then N is maximal if and only if M/N satisfies condition (C).*

Proof. Suppose that N is a maximal k-subsemimodule of M. Let $c\rho_N \in (M/N)'$. Then $c \notin N$. Now let K be the smallest subsemimodule which contains N and c. Since $N \subset K$ we get

$$\overline{K} = M\overline{K} = \{\bar{m} \in M \mid \bar{m} + \bar{s_1} = \overline{s_2},\ \overline{s_i} \in K\},$$

where $\overline{s_i} = s_i c + n_i$ for $s_i \in R$, $n_i \in N$. So $\bar{m} + s_1 c + n_1 = s_2 c + n_2$, thus $\bar{m}\rho_N + s_1(c\rho_N) = s_2(c\rho_N)$. This shows that M/N satisfies condition (c).

For the converse suppose that M/N satisfies condition (c), and let K be a k-subsemimodule of M satisfying $N \subset K$. Then there is an element $c \in K/N$, so that $c\rho_N \in (M/N)'$. By condition (c), for all $m\rho_N \in M/N$ there exist $r_1, r_2 \in R$ such that $m\rho_N + r_1(c\rho_N) = r_2(c\rho_N)$. Hence there exist $n_1, n_2 \in N$ such that $m + r_1 c + n_1 = r_2 c + n_2$. Since K is a k-subsemimodule, $m \in K$. Thus $K = M$. □

3.19. Theorem. *Let M be a semimodule over a commutative semiring R. Then each maximal k-subsemimodule N of M is prime.*

Proof. By Theorem 3.18, M/N satisfies condition (C). Then for $r \notin A_N(M),\ m \notin N$ we have $r \neq 0$ and $m\rho_N \neq \theta\rho_N$. Now if we use Lemma 3.16, we get $r(m\rho_N) \neq \theta\rho_N$, thus $rm \notin N$. □

References

[1] Allen, P. J. *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21**, 412–416, 1969.

[2] Allen, P. J. Neggers, J. and Kim, H. S. *Ideal theory in commutative A-semirings*, Kyungpook Math. J. **46**, 261–271, 2006.

[3] Iseki, K. *Ideal theory of semirings*, Proceedings Japan Academy **32**, 554–559, 1956.

[4] LaTorre, D. R. *A note on quotient semirings*, Proc. Amer. Math. Soc. **24**, 463–465, 1970.

[5] Chin-Pi Lu, C.-P. *Unions of prime submodules*, Houston J. Math. **23** (2), 203–213, 1997.

[6] Sen, M. K. and Adhikari, M. R. *On maximal k-ideals of semirings*, Proc. Amer. Math. Soc. **118** (3), 699–703, 1993.

[7] Shabir, M. and Iqbal, M. S. *One-sided prime ideals in semirings*, Kyungpook Math. J. **47**, 473–480, 2007.

[8] Weinert, H. J., Sen, M. K. and Adhikari, M. R. *One-sided k-ideals and h-ideals in semirings*, Math. Pannon. **7** (1), 147–162, 1996.