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# IMPROVED BOUNDS FOR THE SPECTRAL RADIUS OF DIGRAPHS

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**Abstract**

Let $G = (V, E)$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs. The spectral radius $\rho(G)$ of $G$ is the largest eigenvalue of its adjacency matrix. In this note, we obtain two sharp upper and lower bounds on $\rho(G)$. These bounds improve those obtained by G. H. Xu and C.-Q Xu (*Sharp bounds for the spectral radius of digraphs*, Linear Algebra Appl. **430**, 1607–1612, 2009).

**Keywords:** Digraph, Spectral radius, Bound.

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## 1. Introduction

Let $G$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs on the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. If $(u, v)$ be an arc of $G$, then $u$ is called the *initial vertex* and $v$ the *terminal vertex* of this arc. The outdegree $d_i^+$ of a vertex $v_i$ in the digraph $G$ is defined to be the number of arcs in $G$ with initial vertex $v_i$. Let $d_1^+, d_2^+, \dots, d_n^+$ be the outdegree sequence and $\delta^+(G)$ the minimum outdegree of $G$. For convenience, we sometimes abbreviate $\delta^+(G)$ to $\delta^+$.

Let $t_i^+$ be the sum of the outdegrees of all vertices in $N_i^+(v_i) = \{v_j : (v_i, v_j) \in E\}$, and call it the 2-*outdegree*. Moreover, call $m_i^+ = \frac{t_i^+}{d_i^+}$ the *average* 2-*outdegree*, $1 \le i \le n$. If the average 2-outdegrees of the vertices in $V$ are the same, we call $G$ an *average* 2-*outdegree regular* digraph. If $V = U \cup W$, and the average 2-outdegrees of the vertices in $U$ and

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$W$ are $m_1^+$ and $m_2^+$, respectively, we call $G$ an *average* 2-*outdegree semiregular* digraph. Now we define:

$$(\alpha_{t^+})_i = \sum_{(v_i,v_j)\in E} \left(d_j^+\right)^\alpha \text{ and } (\alpha_{m^+})_i = \frac{\sum\limits_{(v_i,v_j)\in E} \left(d_j^+\right)^\alpha}{\left(d_i^+\right)^\alpha},$$

where $\alpha$ is a real number. Note that $d_i^+ = (0_{t^+})_i = (0_{m^+})_i$, $t_i^+ = (1_{t^+})_i$ and $m_i^+ = (1_{m^+})_i$.

The spectral radius $\rho(G)$ of $G$ is defined to be largest eigenvalue of its adjacency matrix $A(G)$. Recently, the spectral radius of a digraph has been well studied in [2,3,5,6].

In this note, we present two sharp upper and lower bounds on the spectral radius of a digraph $G$, and obtain some known results from it. In fact, for undirected graphs, the following result has been obtained in [4].

**1.1. Lemma.** [4] *Let $G$ be a connected undirected graph. Then*

$$\rho(G) \le \min_{\alpha} \max_{(v_i,v_j)\in E} \left\{ \sqrt{(\alpha_m)_i\,(\alpha_m)_j} \right\},$$

*where $d_i$ is the degree of $v_i$ and $(\alpha_m)_i = \frac{\sum_{(v_i,v_j)\in E} (d_j)^\alpha}{(d_i)^\alpha}$. Moreover, the equality holds for a particular value of $\alpha$ if and only if $(\alpha_m)_1 = (\alpha_m)_2 = \cdots = (\alpha_m)_n$, or $G$ is a bipartite graph with the partition $\{v_1,\ldots,v_{n_1}\} \cup \left\{v_{n_1+1},\ldots,v_n\right\}$ and $(\alpha_m)_1 = \cdots = (\alpha_m)_{n_1}$, $(\alpha_m)_{n_1+1} = \cdots = (\alpha_m)_n$.* □

Now, we will give a generation of this result on the spectral radius for digraphs.

## 2. Upper bound on the spectral radius of digraphs

Throughout this section, let $G$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs. Let $(d_1^+, d_2^+, \ldots, d_n^+)$ be the outdegree sequence and $A(G)$ the adjacency matrix of $G$. Let

$$\bar{D} = \operatorname{diag}\left((d_1^+)^\alpha, \ldots, (d_n^+)^\alpha\right).$$

**2.1. Lemma.** [1] *Let $A$ be a nonnegative matrix of order $n$. Let $R_i$ be the sum of the $i$th row of $A$. Then*

$$\min\{R_i : 1 \le i \le n\} \le \rho(A) \le \max\{R_i : 1 \le i \le n\}.$$

*If $A$ is irreducible, then equality holds in both cases if and only if $R_1 = R_2 = \cdots = R_n$.* □

Now, we give our main result of this section.

**2.2. Theorem.** *Let $G$ be a digraph with $n$ vertices, and $\delta^+$ the minimum outdegree of $G$, $\delta^+ \ge 1$. Then*

$$\rho(G) \le \min_{\alpha} \max_{(v_i,v)\in E_j} \left\{ \sqrt{(\alpha_{m^+})_i\,(\alpha_{m^+})_j} \right\}. \tag{1}$$

*Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m^+})_1 = (\alpha_{m^+})_2 = \cdots = (\alpha_{m^+})_n$, or $G$ is a bipartite graph with the partition $\{v_1,\ldots,v_{n_1}\} \cup \{v_{n_1+1},\ldots,v_n\}$ and $(\alpha_{m^+})_1 = \cdots = (\alpha_{m^+})_{n_1}$, $(\alpha_{m^+})_{n_1+1} = \cdots = (\alpha_{m^+})_n$.*

*Proof.* Note that $\rho(G)=\rho(\bar{D}^{-1}A(G)\bar{D})$. Now the $(i,j)$th element of $\bar{D}^{-1}A(G)\bar{D}$ is

$$\begin{cases} \dfrac{\left(d_j^+\right)^\alpha}{\left(d_i^+\right)^\alpha} & \text{if } (v_i,v_j)\in E,\\ 0 & \text{otherwise.}\end{cases}$$

Let $X=(x_1,x_2,\ldots,x_n)^T$ be an eigenvector of $\bar{D}^{-1}A(G)\bar{D}$ corresponding to the eigenvalue $\rho(G)$. We can assume that one eigen-component, say $x_i$, is equal to 1 and the other eigen-components are less than or equal to 1, that is, $x_i=1$ and $0<x_k\leq 1$, for all $k$. Let

$$x_j=\max\left\{x_k:(v_i,v_k)\in E\right\}.$$

Since

$$\bar{D}^{-1}A(G)\bar{D}X=\rho(G)X,$$

we have

$$\rho(G)x_i=\sum_k\left\{\frac{\left(d_k^+\right)^\alpha}{\left(d_i^+\right)^\alpha}x_k:(v_i,v_k)\in E\right\}\leq\left(\alpha_{m^+}\right)_i x_j, \tag{2}$$

$$\rho(G)x_j=\sum_k\left\{\frac{\left(d_k^+\right)^\alpha}{\left(d_j^+\right)^\alpha}x_k:(v_j,v_k)\in E\right\}\leq\left(\alpha_{m^+}\right)_j. \tag{3}$$

From (2) and (3), we get

$$\rho(G)\leq\sqrt{\left(\alpha_{m^+}\right)_i\left(\alpha_{m^+}\right)_j}.$$

Now we assume that in (1) equality holds for a particular value of $\alpha$. Then all the inequalities in the above argument must be equalities. In particular, we have from (2) that $x_k=x_j$ for all $k$ such that $(v_i,v_k)\in E$. Also, from (3) we have that $x_k=x_i=1$ for all $k$ such that $(v_j,v_k)\in E$. Let $U=\{v_k\in V(G):x_k=1\}$. Then $v_i\in U$.

If $x_j=1$, then we will show that $U=V(G)$. Otherwise, if $U\neq V(G)$, there exist vertices $v_a,v_b\in U$, $v_c\notin U$, such that $(v_a,v_b)\in E$ and $(v_b,v_c)\in E$ since $G$ is strongly connected. Therefore, from

$$\rho(G)x_a=\sum_k\left\{\frac{\left(d_k^+\right)^\alpha}{\left(d_a^+\right)^\alpha}x_k:(v_a,v_k)\in E\right\}\leq\left(\alpha_{m^+}\right)_a$$

and

$$\rho(G)x_b=\sum_k\left\{\frac{\left(d_k^+\right)^\alpha}{\left(d_b^+\right)^\alpha}x_k:(v_b,v_k)\in E\right\}<\left(\alpha_{m^+}\right)_b,$$

we have

$$\rho(G)<\sqrt{\left(\alpha_{m^+}\right)_a\left(\alpha_{m^+}\right)_b},$$

which contradicts that equality holds in (1). Thus $U=V(G)$ and

$$\left(\alpha_{m^+}\right)_1=\left(\alpha_{m^+}\right)_2=\cdots=\left(\alpha_{m^+}\right)_n=\rho(G).$$

Suppose that $x_j<1$, and let $W=\{v_k\in V(G):x_k=x_j\}$. Then, $N_G(v_j)\subseteq U$ and $N_G(v_i)\subseteq W$. Now we show that $N_G\left(N_G\left(v_i\right)\right)\subseteq U$. Let $v_r\in N_G\left(N_G\left(v_i\right)\right)$, there exists a vertex $v_p$ such that $(v_i,v_p)\in E$ and $(v_r,v_p)\in E$. Therefore,

$$x_p=x_j \text{ and } \rho(G)x_p=\sum_w\left\{\frac{\left(d_w^+\right)^\alpha}{\left(d_p^+\right)^\alpha}x_k:(v_p,v_w)\in E\right\}\leq\left(\alpha_{m^+}\right)_p.$$

Using (2), we get $\rho(G)^2 \le (\alpha_{m^+})_i (\alpha_{m^+})_p$. We have $\rho(G)^2 \ge (\alpha_{m^+})_i (\alpha_{m^+})_p$, therefore

$$\rho(G)^2 = (\alpha_{m^+})_i (\alpha_{m^+})_p ,$$

which shows that $x_r = 1$. Hence $N_G(N_G(v_i)) \subseteq U$. By a similar argument, we can show that $N_G(N_G(v_j)) \subseteq W$. Continuing the procedure, since $G$ is strongly connected it is easy to see that $V = U \cup W$, and that the directed subgraphs induced by $U$ and $W$, respectively, are empty digraphs. Hence $G$ is bipartite. Moreover, $(\alpha_{m^+})_p$ are the same for all $v_p \in U$ and $(\alpha_{m^+})_q$ are the same for all $v_q \in W$.

Conversely, If $G$ is a graph with $(\alpha_{m^+})_1 = (\alpha_{m^+})_2 = \cdots = (\alpha_{m^+})_n$, then the equality in (1) is satisfied. Let $G$ be a bipartite graph with bipartition $V = U \cup W$ and $(\alpha_{m^+})_i = a$ for $v_i \in U$, $(\alpha_{m^+})_i = b$ for $v_i \in W$. Let $M = \bar{K}^{-1}\left(\bar{D}^{-1}A(G)\bar{D}\right)\bar{K}$, where $\bar{K} = \operatorname{diag}\{\sqrt{(\alpha_{m^+})_1}, \dots, \sqrt{(\alpha_{m^+})_n}\}$.

Note that the $(i,j)$th element of $M$ is

$$\begin{cases} \sqrt{\frac{b}{a}}\frac{(d_j^+)^\alpha}{(d_i^+)^\alpha} & \text{if } (v_i,v_j) \in E \text{ and } v_i \in U, \\ \sqrt{\frac{a}{b}}\frac{(d_j^+)^\alpha}{(d_i^+)^\alpha} & \text{if } (v_i,v_j) \in E \text{ and } v_i \in W, \\ 0, & \text{otherwise.} \end{cases}$$

So each row sum of the matrix $M$ is equal to $\sqrt{ab}$. Thus, by Lemma 2.1, we have $\rho(G) = \rho(M) = \sqrt{ab}$. □

**2.3. Corollary.** *Let $G$ be a graph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \ge 1$. Then*

$$\rho(G) \le \min_{\alpha} \max_{1\le i\le n} \left\{(\alpha_{m^+})_i\right\}. \tag{4}$$

*Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m^+})_1 = (\alpha_{m^+})_2 = \cdots = (\alpha_{m^+})_n$.* □

If $\alpha = 1$ in (1), then we get the following result.

**2.4. Corollary.** [5] *Let $G$ be a digraph on $n$ vertices and $\delta^+$ the minimum outdegree of $G$, $\delta^+ \ge 1$. Then*

$$\rho(G) \le \max\left\{\sqrt{m_i^+ m_j^+} : (v_i,v_j) \in E\right\}. \tag{5}$$

*Moreover, If $G$ is a strongly connected digraph, equality holds if and only if $G$ is average 2-outdegree regular or average 2-outdegree semiregular.*

## 3. Lower bound on the spectral radius of digraphs

**3.1. Theorem.** *Let $G$ be a digraph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \ge 1$. Then*

$$\rho(G) \ge \max_{\alpha} \min_{(v_i,v_j)\in E} \left\{\sqrt{(\alpha_{m^+})_i (\alpha_{m^+})_j}\right\}. \tag{6}$$

*Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m^+})_1 = (\alpha_{m^+})_2 = \cdots = (\alpha_{m^+})_n$, or $G$ is a bipartite graph with the partition $\{v_1,\dots,v_{n_1}\} \cup \{v_{n_1+1},\dots,v_n\}$ and $(\alpha_{m^+})_1 = \cdots = (\alpha_{m^+})_{n_1}$, $(\alpha_{m^+})_{n_1+1} = \cdots = (\alpha_{m^+})_n$.*

*Proof.* Let $X = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of $\bar{D}^{-1}A(G)\bar{D}$ corresponding to the eigenvalue $\rho(G)$. We can assume that one eigen-component, say $x_i$, is equal to 1 and the other eigen-components are greater than or equal to 1, that is, $x_i = 1$ and $x_k \geq 1$ for all $k \neq i$. Let $x_j = \min\{x_k : (v_i, v_k) \in E\}$.

Since

$$\bar{D}^{-1}A(G)\bar{D}X = \rho(G)X,$$

we have

$$\rho(G)x_i = \sum_k \left\{ \frac{\left(d_k^+\right)^\alpha}{\left(d_i^+\right)^\alpha} x_k : (v_i, v_k) \in E \right\} \geq \left(\alpha_{m^+}\right)_i x_j, \tag{7}$$

$$\rho(G)x_j = \sum_k \left\{ \frac{\left(d_k^+\right)^\alpha}{\left(d_j^+\right)^\alpha} x_k : (v_j, v_k) \in E \right\} \geq \left(\alpha_{m^+}\right)_j. \tag{8}$$

From (7) and (8), we get

$$\rho(G) \geq \sqrt{\left(\alpha_{m^+}\right)_i \left(\alpha_{m^+}\right)_j}.$$

Similarly as in the proof of the Theorem 2.2, we can show that equality holds for a particular value of $\alpha$ if and only if $\left(\alpha_{m^+}\right)_1 = \left(\alpha_{m^+}\right)_2 = \dots = \left(\alpha_{m^+}\right)_n$, or $G$ is a bipartite graph with the partition $\{v_1, \dots, v_{n_1}\} \cup \{v_{n_1+1}, \dots, v_n\}$ and $\left(\alpha_{m^+}\right)_1 = \dots = \left(\alpha_{m^+}\right)_{n_1}$, $\left(\alpha_{m^+}\right)_{n_1+1} = \dots = \left(\alpha_{m^+}\right)_n$. □

**3.2. Corollary.** *Let $G$ be a digraph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \geq 1$. Then*

$$\rho(G) \geq \max_\alpha \min_{1 \leq i \leq n} \left\{ \left(\alpha_{m^+}\right)_i \right\}. \tag{9}$$

*Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $\left(\alpha_{m^+}\right)_1 = \left(\alpha_{m^+}\right)_2 = \dots = \left(\alpha_{m^+}\right)_n$.* □

If $\alpha = 1$ in (6), the we get the following result.

**3.3. Corollary.** [5] *Let $G$ be a strongly connected digraph. Then*

$$\rho(G) \geq \min \left\{ \sqrt{m_i^+ m_j^+} : (v_i, v_j) \in E \right\}. \tag{10}$$

*Moreover, equality holds if and only if $G$ is average 2-outdegree regular or average 2-outdegree semiregular.* □

**3.4. Example.** Let $G$ be a digraph with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then the bound (1) is 1.847 when $\alpha = 0.5$, and the bound (5) from [5] is 2. For the same graph, the bound (6) is 1.414 when $\alpha = 0.5$, and the bound (10) from [5] is 1.154. Thus in both cases, the results obtained in this paper for $\alpha = 0.5$ are better than the bounds obtained in [5].

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