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A FORMULA FOR COMPUTING INTEGER POWERS FOR ONE TYPE OF TRIDIAGONAL MATRIX

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Abstract

In this paper, we derive the general expression of the r^{th} power ($r \in \mathbb{N}$) for one type of tridiagonal matrix.

Keywords: Tridiagonal matrix, Matrix power, Eigenvalues, Chebyshev polynomial.

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1. Introduction

In the present paper, we derive a general expression for the rth power for one type of tridiagonal matrix, where $r \in \mathbb{N}$ and $\mathbb{N}$ denotes the set of natural numbers. In [4], Rimas derived a general expression for the lth power for one type of symmetric tridiagonal matrices of even order.

General expressions for the rth power of a matrix are obtained by using the equality $A^r = PJ^rP^{-1}$ [2], where J is the Jordan form of A and P the transforming matrix. We need the eigenvalues and eigenvectors of the matrix A to compute the matrices J and P. The eigenvalues of A are the roots

$$x_{nk} = \cos\frac{k\pi}{n+1},\ k = 1, 2, \dots, n$$

of the nth degree Chebyshev polynomial

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))},\ -1 \le x \le 1,$$

of the second kind.

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2. Main results

Let A be the following n-square tridiagonal matrix

$$A = \begin{bmatrix} 0 & -1 & & & & & \\ -1 & 0 & 1 & & & & \\ & 1 & 0 & -1 & & & \\ & & -1 & 0 & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & 0 & (-1)^{n-1} \\ & & & & & (-1)^{n-1} & 0 \end{bmatrix}. \tag{2.1}$$

The eigenvalues of A are the roots of the characteristic equation

$$|A - \lambda E| = 0,$$

and also the roots of $U_n(x)$, the nth degree Chebyshev polynomial of the second kind [3].

In the present paper, we investigate integer powers of this matrix. Initially we will scrutinize the integer powers for even orders. Let us write

$$D_n(\alpha) = |A - \lambda E| = \begin{vmatrix} \alpha & -1 & & & & & \\ -1 & \alpha & 1 & & & & \\ & 1 & \alpha & -1 & & & \\ & & -1 & \alpha & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & \alpha & -1 \\ & & & & & -1 & \alpha \end{vmatrix} \tag{2.2}$$

and

$$\Delta_n(\alpha) = \begin{vmatrix} \alpha & 1 & & & & & \\ 1 & \alpha & -1 & & & & \\ & -1 & \alpha & 1 & & & \\ & & 1 & \alpha & \ddots & & \\ & & & \ddots & \ddots & -1 & \\ & & & & -1 & \alpha & 1 \\ & & & & & 1 & \alpha \end{vmatrix} \tag{2.3}$$

where $\alpha = -\lambda \in \mathbb{R}$.

By (2.2) and (2.3), we obtain the following two results

$$D_n(\alpha) = \Delta_n(\alpha) \tag{2.4}$$

$$\Delta_n(\alpha) = \alpha\Delta_{n-1}(\alpha) - \Delta_{n-2}(\alpha), \tag{2.5}$$

where $\Delta_2(\alpha) = \alpha^2 - 1$, $\Delta_1(\alpha) = \alpha$ and $\Delta_0(\alpha) = 1$.

We obtain (2.4) by the induction principle. For $n = 2$,

$$D_2(\alpha) = \Delta_2(\alpha) = \alpha^2 - 1.$$

Let $D_{2k}(\alpha) = \Delta_{2k}(\alpha)$ for $n = 2k$. We have to show that $D_{2k+2}(\alpha) = \Delta_{2k+2}(\alpha)$ for $n = 2k+2$. Let

$$D_{2k+2}(\alpha) = \begin{vmatrix} \alpha & -1 & & & & \\ -1 & \alpha & 1 & & & \\ & 1 & \alpha & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \alpha & -1 \\ & & & & -1 & \alpha \end{vmatrix}_{(2k+2)\times(2k+2)}. \tag{2.6}$$

If we expand the determinant (2.6) according to first row, then we have

$$D_{2k+2}(\alpha) = \alpha \begin{vmatrix} \alpha & 1 & & & & \\ 1 & \alpha & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & \alpha & 1 & \\ & & & 1 & \alpha & -1 \\ & & & & -1 & \alpha \end{vmatrix} + \begin{vmatrix} -1 & 1 & & & & \\ 0 & \alpha & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & \alpha & 1 & \\ & & & 1 & \alpha & -1 \\ & & & & -1 & \alpha \end{vmatrix} \tag{2.7}$$

and if we expand the first determinant in (2.7) according to last row and the second determinant according to first column, then we obtain

$$D_{2k+2}(\alpha) = \alpha \left\{ \alpha \underbrace{\begin{vmatrix} \alpha & 1 & & & \\ 1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & 1 \\ & & & 1 & \alpha \end{vmatrix}}_{\Delta_{2k}(\alpha)} + \begin{vmatrix} \alpha & 1 & & & \\ 1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & 0 \\ & & & 1 & -1 \end{vmatrix} \right\}$$
$$- \underbrace{\begin{vmatrix} \alpha & -1 & & & \\ -1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & -1 \\ & & & -1 & \alpha \end{vmatrix}}_{D_{2k}(\alpha)=\Delta_{2k}(\alpha)}.$$

Then

$$D_{2k+2}(\alpha) = \alpha\left\{\alpha\Delta_{2k}(\alpha) - \underbrace{\begin{vmatrix} \alpha & 1 & & & \\ 1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & -1 \\ & & & -1 & \alpha \end{vmatrix}}_{\Delta_{2k-1}(\alpha)}\right\} - \Delta_{2k}(\alpha)$$

$$= \alpha\underbrace{[\alpha\Delta_{2k}(\alpha) - \Delta_{2k-1}(\alpha)]}_{\Delta_{2k+1}(\alpha)} - \Delta_{2k}(\alpha)$$

$$= \alpha\Delta_{2k+1}(\alpha) - \Delta_{2k}(\alpha)$$

$$= \Delta_{2k+2}(\alpha).$$

By solving the difference equation (2.5), we obtain [1]

$$\Delta_n(\alpha) = U_n\left(\frac{\alpha}{2}\right). \tag{2.8}$$

It is known that the roots of the polynomial $U_n(x)$ are [3]

$$x_{nk} = \cos\frac{k\pi}{n+1},\ k = 1, 2, \ldots, n, \tag{2.9}$$

which are included in the interval $[-1, 1]$.

Taking (2.6) and (2.7) into account, we find the eigenvalues of the matrix A to be

$$\lambda_k = -2\cos\frac{k\pi}{n+1},\ k = 1, 2, \ldots, n. \tag{2.10}$$

Since the multiplicity of each of the eigenvalues λ_k is 1, and applying the relation

$$\lambda_k = -\lambda_{n-k+1},\ (k = 1, 2, \ldots, \frac{n}{2})$$

we write down the Jordan form of the matrix A as

$$J = \operatorname{diag}(-\lambda_n, -\lambda_{n-1}, -\lambda_{n-2}, \ldots, -\lambda_{\frac{n}{2}+1}, \lambda_{\frac{n}{2}+1}, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n). \tag{2.11}$$

Let us solve the homogeneous linear equations

$$(\lambda_i E - A)x = 0,$$

where λ_i is the ith eigenvalue of the matrix A $(1 \le i \le n)$. By elementary row operations, the coefficient matrix of the system is

$$\begin{bmatrix} 1 & \lambda_i & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda_i & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda_i & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\lambda_i & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \lambda_i \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & D_n(-\lambda_i) \end{bmatrix}.$$

Since $D_n(-\lambda_i) = 0$, $\operatorname{rank}(\lambda_i E - A) = n - 1$. Then

$$\begin{aligned} x_1 + \lambda_i x_2 - x_3 &= 0, \\ x_2 - \lambda_i x_3 - x_4 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ x_{n-2} - \lambda_i x_{n-1} - x_n &= 0, \\ x_{n-1} + \lambda_i x_n &= 0. \end{aligned}$$

Now we will scrutinize the solutions of the above system of linear equations according to n and k.

Let $x_n = -1 = -U_0\Big(\dfrac{\lambda_i}{2}\Big)$, and for any n and k define

$$a = \begin{cases} 1 & n-k \equiv 1 \text{ or } 2 \mod 4, \\ -1 & n-k \equiv 0 \text{ or } 3 \mod 4, \end{cases}$$

$$e = \begin{cases} 1 & n-1 \equiv 1 \text{ or } 2 \mod 4, \\ -1 & n-1 \equiv 0 \text{ or } 3 \mod 4. \end{cases}$$

We consider the following two cases:

Case 1. Let n be an even number.

i) If k is an odd number, then

$$\begin{aligned} x_{n-1} &= \lambda_i = U_1\Big(\frac{\lambda_i}{2}\Big) \\ x_{n-2} &= \lambda_i^2 - 1 = U_2\Big(\frac{\lambda_i}{2}\Big) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{k+2} &= -a\lambda_i^{n-k-2} + c\lambda_i^{n-k-4} + \cdots + \frac{n-k-1}{2}\lambda_i = -aU_{n-k-2}\Big(\frac{\lambda_i}{2}\Big) \\ x_{k+1} &= -a\lambda_i^{n-k-1} + b\lambda_i^{n-k-3} + d\lambda_i^{n-k-5} + \cdots - 1 = -aU_{n-k-1}\Big(\frac{\lambda_i}{2}\Big) \\ x_k &= a\lambda_i^{n-k} + (-a-b)\lambda_i^{n-k-2} + (c-d)\lambda_i^{n-k-4} + \cdots \\ &\qquad\qquad \cdots + \frac{n-k+1}{2}\lambda_i = aU_{n-k}\Big(\frac{\lambda_i}{2}\Big), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1 &= e\lambda_i^{n-1} - e(n-2)\lambda_i^{n-3} + \cdots + \frac{n}{2}\lambda_i = eU_{n-1}\Big(\frac{\lambda_i}{2}\Big). \end{aligned}$$

where

$$\begin{aligned} b &= a(n-k-2), \\ c &= a(n-k-3), \end{aligned}$$

and d is the sum of the coefficients of the terms λ_i^{n-k-5} in x_{k-3} and λ_i^{n-k-6} in x_{k-2}.

ii) If k is an even number, then

$$
\begin{aligned}
x_{n-1} &= \lambda_i = U_1\Big(\frac{\lambda_i}{2}\Big),\\
x_{n-2} &= \lambda_i^2 - 1 = U_2\Big(\frac{\lambda_i}{2}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{k+3} &= -a\lambda_i^{n-k-3} + \cdots + \frac{n-k-2}{2}\lambda_i = -aU_{n-k-3}\Big(\frac{\lambda_i}{2}\Big),\\
x_{k+2} &= -a\lambda_i^{n-k-2} + c\lambda_i^{n-k-4} + \cdots - 1 = -aU_{n-k-2}\Big(\frac{\lambda_i}{2}\Big),\\
x_{k+1} &= a\lambda_i^{n-k-1} + b\lambda_i^{n-k-3} + d\lambda_i^{n-k-5} + \cdots + \frac{n-k}{2}\lambda_i = aU_{n-k-1}\Big(\frac{\lambda_i}{2}\Big),\\
x_k &= a\lambda_i^{n-k} + (b-a)\lambda_i^{n-k-2} + (c+d)\lambda_i^{n-k-4} + \cdots - 1 = aU_{n-k}\Big(\frac{\lambda_i}{2}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_1 &= e\lambda_i^{n-1} - e(n-2)\lambda_i^{n-3} + \cdots + \frac{n}{2}\lambda_i = eU_{n-1}\Big(\frac{\lambda_i}{2}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
b &= -a(n-k-2),\\
c &= a(n-k-3),
\end{aligned}
$$

and d is the difference of the coefficients of the terms λ_i^{n-k-5} in x_{k+3} and λ_i^{n-k-6} in x_{k+2}.

Case 2 Let n be an odd number.

i) If k is an odd number

$$
\begin{aligned}
x_{n-1} &= \lambda_i = U_1\Big(\frac{\lambda_i}{2}\Big),\\
x_{n-2} &= \lambda_i^2 - 1 = U_2\Big(\frac{\lambda_i}{2}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{k+3} &= -a\lambda_i^{n-k-3} + \cdots + \frac{n-k-2}{2}\lambda_i = -aU_{n-k-3}\Big(\frac{\lambda_i}{2}\Big),\\
x_{k+2} &= -a\lambda_i^{n-k-2} + c\lambda_i^{n-k-4} + \cdots - 1 = -aU_{n-k-2}\Big(\frac{\lambda_i}{2}\Big),\\
x_{k+1} &= a\lambda_i^{n-k-1} + b\lambda_i^{n-k-3} + d\lambda_i^{n-k-5} + \cdots + \frac{n-k}{2}\lambda_i = aU_{n-k-1}\Big(\frac{\lambda_i}{2}\Big),\\
x_k &= a\lambda_i^{n-k} + (b-a)\lambda_i^{n-k-2} + (c+d)\lambda_i^{n-k-4} + \cdots - 1 = aU_{n-k}\Big(\frac{\lambda_i}{2}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_1 &= e\lambda_i^{n-1} - e(n-2)\lambda_i^{n-3} + \cdots - 1 = eU_{n-1}\Big(\frac{\lambda_i}{2}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
b &= -a(n-k-2),\\
c &= a(n-k-3),
\end{aligned}
$$

and d is the difference of the coefficients of the terms λ_i^{n-k-5} in x_{k+3} and λ_i^{n-k-6} in x_{k+2}.

ii) If k is an even number, then

$$\begin{aligned}
x_{n-1} &= \lambda_i = U_1\Big(\frac{\lambda_i}{2}\Big)\\
x_{n-2} &= \lambda_i^2 - 1 = U_2\Big(\frac{\lambda_i}{2}\Big)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{k+2} &= -a\lambda_i^{n-k-2} + c\lambda_i^{n-k-4} + \cdots + \frac{n-k-1}{2}\lambda_i = -aU_{n-k-2}\Big(\frac{\lambda_i}{2}\Big)\\
x_{k+1} &= -a\lambda_i^{n-k-1} + b\lambda_i^{n-k-3} + d\lambda_i^{n-k-5} + \cdots - 1 = -aU_{n-k-1}\Big(\frac{\lambda_i}{2}\Big)\\
x_k &= a\lambda_i^{n-k} + (-a-b)\lambda_i^{n-k-2} + (c-d)\lambda_i^{n-k-4} + \cdots\\
&\qquad \cdots + \frac{n-k+1}{2}\lambda_i = aU_{n-k}\Big(\frac{\lambda_i}{2}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_1 &= e\lambda_i^{n-1} - e(n-2)\lambda_i^{n-3} + \cdots - 1 = eU_{n-1}\Big(\frac{\lambda_i}{2}\Big).
\end{aligned}$$

where

$$\begin{aligned}
b &= a(n-k-2),\\
c &= a(n-k-3),
\end{aligned}$$

and d is the sum of the coefficients of the terms λ_i^{n-k-5} in x_{k-3} and λ_i^{n-k-6} in x_{k-2}.

Taking

$$P_i = x_{\lambda_i}$$

into account,

$$P = \begin{bmatrix} P_1 & P_2 & \dots & P_n \end{bmatrix}.$$

Using this expression, we can write down the transforming matrix P as

$$P = \begin{bmatrix}
eU_{n-1}(\frac{\lambda_1}{2}) & eU_{n-1}(\frac{\lambda_2}{2}) & \dots & eU_{n-1}(\frac{\lambda_n}{2})\\
\vdots & \vdots & & \vdots\\
aU_{n-k}(\frac{\lambda_1}{2}) & aU_{n-k}(\frac{\lambda_2}{2}) & \dots & aU_{n-k}(\frac{\lambda_n}{2})\\
\vdots & \vdots & & \vdots\\
-U_3(\frac{\lambda_1}{2}) & -U_3(\frac{\lambda_2}{2}) & \dots & -U_3(\frac{\lambda_n}{2})\\
U_2(\frac{\lambda_1}{2}) & U_2(\frac{\lambda_2}{2}) & \dots & U_2(\frac{\lambda_n}{2})\\
U_1(\frac{\lambda_1}{2}) & U_1(\frac{\lambda_2}{2}) & \dots & U_1(\frac{\lambda_n}{2})\\
-U_0(\frac{\lambda_1}{2}) & -U_0(\frac{\lambda_2}{2}) & \dots & -U_0(\frac{\lambda_n}{2})
\end{bmatrix} \tag{2.12}$$

Denoting the ith row of P by $\Re_i$ $(i = \overline{1,n})$, from (2.12) we have

$$\Re_i = \begin{bmatrix} t_iU_{n-i}(\frac{\lambda_1}{2}) & t_iU_{n-i}(\frac{\lambda_2}{2}) & \dots & t_iU_{n-i}(\frac{\lambda_n}{2}) \end{bmatrix}, \tag{2.13}$$

where

$$t_i = \begin{cases} 1 & n-i \equiv 1 \text{ or } 2 \mod 4,\\ -1 & n-i \equiv 0 \text{ or } 3 \mod 4. \end{cases}$$

Denoting jth column of the inverse matrix P^{-1} by ρ_j^{-1}, we obtain

$$\rho_j^{-1} = \begin{bmatrix} m_j\left(\frac{4-\lambda_1^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_1}{2}\right) \\ m_j\left(\frac{4-\lambda_2^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_2}{2}\right) \\ m_j\left(\frac{4-\lambda_3^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_3}{2}\right) \\ \vdots \\ m_j\left(\frac{4-\lambda_n^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_n}{2}\right) \end{bmatrix}, \tag{2.14}$$

where

$$m_j = \begin{cases} 1 & n-j \equiv 1 \text{ or } 2 \mod 4, \\ -1 & n-j \equiv 0 \text{ or } 3 \mod 4. \end{cases}$$

Taking expression (2.14) into account, we obtain the matrix P^{-1} as follows:

$$P^{-1} = \begin{bmatrix} e\left(\frac{4-\lambda_1^2}{2n+2}\right)U_{n-1}\left(\frac{\lambda_1}{2}\right) & \dots & -\left(\frac{4-\lambda_1^2}{2n+2}\right)U_3\left(\frac{\lambda_1}{2}\right) & \left(\frac{4-\lambda_1^2}{2n+2}\right)U_2\left(\frac{\lambda_1}{2}\right) & \left(\frac{4-\lambda_1^2}{2n+2}\right)U_1\left(\frac{\lambda_1}{2}\right) & -\left(\frac{4-\lambda_1^2}{2n+2}\right)U_0\left(\frac{\lambda_1}{2}\right) \\ e\left(\frac{4-\lambda_2^2}{2n+2}\right)U_{n-1}\left(\frac{\lambda_2}{2}\right) & \dots & -\left(\frac{4-\lambda_2^2}{2n+2}\right)U_3\left(\frac{\lambda_2}{2}\right) & \left(\frac{4-\lambda_2^2}{2n+2}\right)U_2\left(\frac{\lambda_2}{2}\right) & \left(\frac{4-\lambda_2^2}{2n+2}\right)U_1\left(\frac{\lambda_2}{2}\right) & -\left(\frac{4-\lambda_2^2}{2n+2}\right)U_0\left(\frac{\lambda_2}{2}\right) \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ e\left(\frac{4-\lambda_n^2}{2n+2}\right)U_{n-1}\left(\frac{\lambda_n}{2}\right) & \dots & -\left(\frac{4-\lambda_n^2}{2n+2}\right)U_3\left(\frac{\lambda_n}{2}\right) & \left(\frac{4-\lambda_n^2}{2n+2}\right)U_2\left(\frac{\lambda_n}{2}\right) & \left(\frac{4-\lambda_n^2}{2n+2}\right)U_1\left(\frac{\lambda_n}{2}\right) & -\left(\frac{4-\lambda_n^2}{2n+2}\right)U_0\left(\frac{\lambda_n}{2}\right) \end{bmatrix}$$

Using the equality $A^r = PJ^rP^{-1}$, we derive

$$\begin{aligned} \left\{A^r\right\}_{ij} = \left\{PJ^rP^{-1}\right\}_{ij} = \Re_i J^r \rho_j^{-1} &= \Re_i \begin{bmatrix} \lambda_1^r m_j\left(\frac{4-\lambda_1^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_1}{2}\right) \\ \lambda_2^r m_j\left(\frac{4-\lambda_2^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_2}{2}\right) \\ \lambda_3^r m_j\left(\frac{4-\lambda_3^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_3}{2}\right) \\ \vdots \\ \lambda_n^r m_j\left(\frac{4-\lambda_n^2}{2n+2}\right)U_{n-j}\left(\frac{\lambda_n}{2}\right) \end{bmatrix} \\ &= \frac{t_i m_j}{2n+2}\sum_{k=1}^{n} \lambda_k^r(4-\lambda_k^2)U_{n-i}\left(\frac{\lambda_k}{2}\right)U_{n-j}\left(\frac{\lambda_k}{2}\right), \end{aligned}$$

where λ_i are the eigenvalues of the matrix A, $U_k(x)$ is the kth degree Chebyshev polynomial of the second kind $(i, j = \overline{1, n})$, and we equate this 1×1 matrix with its unique element as usual.

Now we consider the case when the order n is odd. Working as before, we obtain

$$(2.15)\qquad D_n(\alpha)=|A-\lambda E|=\begin{vmatrix}\alpha & -1 & & & & & \\ -1 & \alpha & 1 & & & & \\ & 1 & \alpha & -1 & & & \\ & & -1 & \alpha & \ddots & & \\ & & & \ddots & \ddots & -1 & \\ & & & & -1 & \alpha & 1 \\ & & & & & 1 & \alpha\end{vmatrix}$$

and

$$(2.16)\qquad \Delta_n(\alpha)=\begin{vmatrix}\alpha & 1 & & & & & \\ 1 & \alpha & -1 & & & & \\ & -1 & \alpha & 1 & & & \\ & & 1 & \alpha & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & \alpha & -1 \\ & & & & & -1 & \alpha\end{vmatrix}$$

for odd orders, where $\alpha=-\lambda\in\mathbb{R}$. Also, the equalities (2.4), (2.5), (2.8), (2.10), (2.12) and (2.13) are valid for odd orders.

Since the multiplicity of all the eigenvalues λ_k is 1, and applying the relation

$$\lambda_k=-\lambda_{n-k+1},\ \left(k=1,2,\dots,\frac{n-1}{2}\right),$$

and $\lambda_{\frac{n+1}{2}}=0$ we can write down the Jordan form of the matrix A as:

$$(2.17)\qquad J=\operatorname{diag}(-\lambda_n,-\lambda_{n-1},-\lambda_{n-2},\dots,-\lambda_{\frac{n+3}{2}},0,\lambda_{\frac{n+3}{2}},\dots,\lambda_{n-2},\lambda_{n-1},\lambda_n).$$

Denoting jth column of the inverse matrix P^{-1} by ρ_j^{-1}, we obtain

$$(2.18)\qquad \rho_j^{-1}=\begin{bmatrix} m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_1}{2}\right)\\ m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_2}{2}\right)\\ m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+3}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_3}{2}\right)\\ \vdots\\ m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_n}{2}\right)\end{bmatrix},$$

where

$$m_j=\begin{cases}1 & n-j\equiv 1 \text{ or } 2 \mod 4,\\ -1 & n-j\equiv 0 \text{ or } 3 \mod 4.\end{cases}$$

Taking the expression (2.18) into account, we obtain the matrix P^{-1} as follows:

$$P^{-1} = \begin{bmatrix} e\left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_{n-1}\left(\frac{\lambda_1}{2}\right) & \dots & -\left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_3\left(\frac{\lambda_1}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_2\left(\frac{\lambda_1}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_1\left(\frac{\lambda_1}{2}\right) & -\left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_0\left(\frac{\lambda_1}{2}\right) \\ e\left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_{n-1}\left(\frac{\lambda_2}{2}\right) & \dots & -\left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_3\left(\frac{\lambda_2}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_2\left(\frac{\lambda_2}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_1\left(\frac{\lambda_2}{2}\right) & -\left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_0\left(\frac{\lambda_2}{2}\right) \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ e\left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_{n-1}\left(\frac{\lambda_n}{2}\right) & \dots & -\left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_3\left(\frac{\lambda_n}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_2\left(\frac{\lambda_n}{2}\right) & \left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_1\left(\frac{\lambda_n}{2}\right) & -\left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_0\left(\frac{\lambda_n}{2}\right) \end{bmatrix}$$

Using the equality $A^r = PJ^rP^{-1}$, we derive

$$\left\{A^r\right\}_{ij} = \left\{PJ^rP^{-1}\right\}_{ij} = \Re_i J^r \rho_j^{-1} = \Re_i \begin{bmatrix} \lambda_1^r m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+1}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_1}{2}\right) \\ \lambda_2^r m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+2}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_2}{2}\right) \\ \lambda_3^r m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+3}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_3}{2}\right) \\ \vdots \\ \lambda_n^r m_j\left(\frac{\lambda^2_{\frac{n+1}{2}+n}}{2n+2}\right)U_{n-j}\left(\frac{\lambda_n}{2}\right) \end{bmatrix}$$

$$= (-1)^r \frac{t_i m_j}{2n+2} \sum_{k=1}^{n} \lambda_k^r \lambda^2_{\frac{n+1}{2}+k} U_{n-i}\left(\frac{\lambda_k}{2}\right) U_{n-j}\left(\frac{\lambda_k}{2}\right),$$

where λ_i are the eigenvalues of the matrix A, $U_k(x)$ is the kth degree Chebyshev polynomial of the second kind $(i, j = \overline{1, n})$, and we equate the 1×1 matrix with its unique element.

3. Numerical examples

Taking into account the derived expressions for n even, one can see the effectiveness of the formula. For $n = 4$, we obtain;

$$J = \operatorname{diag}(-\lambda_4, -\lambda_3, \lambda_3, \lambda_4) = \operatorname{diag}(-a, -b, b, a),$$
$$a = 2\cos\frac{\pi}{5}, \; b = 2\cos\frac{2\pi}{5}.$$

If r is even,

$$\begin{aligned}
\{A^r\}_{11} &= \frac{1}{10}\big(2a^r(4-a^2)(a^3-2a)^2 + 2b^r(4-b^2)(b^3-2b)^2\big),\\
\{A^r\}_{12} &= \{A^r\}_{21} = 0,\\
\{A^r\}_{13} &= \{A^r\}_{31} = -\frac{1}{10}\big(2a^{r+1}(4-a^2)(a^3-2a) + 2b^{r+1}(4-b^2)(b^3-2b)\big),\\
\{A^r\}_{14} &= \{A^r\}_{41} = 0,\\
\{A^r\}_{22} &= \frac{1}{10}\big(2a^r(4-a^2)(a^2-1)^2 + 2b^r(4-b^2)(b^2-1)^2\big),\\
\{A^r\}_{23} &= \{A^r\}_{32} = 0,\\
\{A^r\}_{24} &= \{A^r\}_{42} = -\frac{1}{10}\big(2a^r(4-a^2)(a^2-1) + 2b^r(4-b^2)(b^2-1)\big),\\
\{A^r\}_{33} &= \frac{1}{10}\big(2a^{r+2}(4-a^2) + 2b^{r+2}(4-b^2)\big),\\
\{A^r\}_{34} &= \{A^r\}_{43} = 0,\\
\{A^r\}_{44} &= \frac{1}{10}\big(2a^r(4-a^2) + 2b^r(4-b^2)\big).
\end{aligned}$$

For example for $r = 4$;

$$A^4 = \begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & 5 & 0 & -3 \\ -3 & 0 & 5 & 0 \\ 0 & -3 & 0 & 2 \end{bmatrix}.$$

If r is odd,

$$\begin{aligned}
\{A^r\}_{11} &= 0,\\
\{A^r\}_{12} &= \{A^r\}_{21} = -\frac{1}{10}\big[2a^r(4-a^2)(a^3-2a)(a^2-1).\\
&\qquad + 2b^r(4-b^2)(b^3-2b)(b^2-1)\big]\\
\{A^r\}_{13} &= \{A^r\}_{31} = 0,\\
\{A^r\}_{14} &= \{A^r\}_{41} = \frac{1}{10}\big(2a^r(4-a^2)(a^3-2a) + 2b^r(4-b^2)(b^3-2b)\big),\\
\{A^r\}_{22} &= 0,\\
\{A^r\}_{23} &= \{A^r\}_{32} = \frac{1}{10}\big(2a^{r+1}(4-a^2)(a^2-1) + 2b^{r+1}(4-b^2)(b^2-1)\big),\\
\{A^r\}_{24} &= \{A^r\}_{42} = 0,\\
\{A^r\}_{33} &= 0,\\
\{A^r\}_{34} &= \{A^r\}_{43} = -\frac{1}{10}\big(2a^{r+1}(4-a^2) + 2b^{r+1}(4-b^2)\big),\\
\{A^r\}_{44} &= 0.
\end{aligned}$$

For example, for $r = 5$,

$$A^5 = \begin{bmatrix} 0 & -5 & 0 & 3 \\ -5 & 0 & 8 & 0 \\ 0 & 8 & 0 & -5 \\ 3 & 0 & -5 & 0 \end{bmatrix}.$$

We will now see the effectiveness of the formula for odd orders. For example if $n = 5$ we obtain,

$$J = \mathrm{diag}(-\lambda_5,\, -\lambda_4,\, 0,\, \lambda_4,\, \lambda_5) = \mathrm{diag}(-c,\, -d,\, 0,\, d,\, c),\ c = \sqrt{3},\ d = 1.$$

If r is even,

$$\begin{aligned}
\{A^r\}_{11} &= \frac{1}{6}\big(c^r d^2 + c^2 d^r\big) = \frac{1}{6}\big((\sqrt{3})^r + 3\big),\\
\{A^r\}_{12} &= \{A^r\}_{21} = 0,\\
\{A^r\}_{13} &= \{A^r\}_{31} = -\frac{1}{3}c^r d^2 = -(\sqrt{3})^{r-2},\\
\{A^r\}_{14} &= \{A^r\}_{41} = 0,\\
\{A^r\}_{15} &= \{A^r\}_{51} = \frac{1}{6}\big(c^r d^2 - c^2 d^r\big) = \frac{1}{6}\big((\sqrt{3})^r - 3\big),\\
\{A^r\}_{22} &= \frac{1}{6}(c^{r+2} d^2 + c^2 d^r) = \frac{1}{6}((\sqrt{3})^{r+2} + 3),\\
\{A^r\}_{23} &= \{A^r\}_{32} = 0,\\
\{A^r\}_{24} &= \{A^r\}_{42} = \frac{1}{6}\big(c^2 d^r - c^{r+2} d^2\big) = \frac{1}{6}\big(3 - (\sqrt{3})^{r+2}\big),\\
\{A^r\}_{25} &= \{A^r\}_{52} = 0,\\
\{A^r\}_{33} &= \frac{2}{3}a^r b^2 = \frac{2}{3}c^r = 2c^{r-2} = 2(\sqrt{3})^{r-2},\\
\{A^r\}_{34} &= \{A^r\}_{43} = 0,\\
\{A^r\}_{35} &= \{A^r\}_{53} = -\frac{1}{3}c^r d^2 = -(c)^{r-2} = -(\sqrt{3})^{r-2},\\
\{A^r\}_{44} &= \frac{1}{6}\big(c^{r+2} d^2 + c^2 d^r\big) = \frac{1}{6}\big((\sqrt{3})^{r+2} + 3\big),\\
\{A^r\}_{45} &= \{A^r\}_{54} = 0,\\
\{A^r\}_{55} &= \frac{1}{6}\big(c^r d^2 + c^2 d^r\big) = \frac{1}{6}\big((\sqrt{3})^r + 3\big).
\end{aligned}$$

For example, for $r = 4$ we have,

$$A^4 = \begin{bmatrix} 2 & 0 & -3 & 0 & 1 \\ 0 & 5 & 0 & -4 & 0 \\ -3 & 0 & 6 & 0 & -3 \\ 0 & -4 & 0 & 5 & 0 \\ 1 & 0 & -3 & 0 & 2 \end{bmatrix}.$$

If r is odd,

$$\begin{aligned}
\{A^r\}_{11} &= 0,\\
\{A^r\}_{12} &= \{A^r\}_{21} = -\frac{1}{6}\big(c^{r+1} d^2 + c^2 d^r\big) = -\frac{1}{6}\big((\sqrt{3})^{r+1} + 3\big),\\
\{A^r\}_{13} &= \{A^r\}_{31} = 0,\\
\{A^r\}_{14} &= \{A^r\}_{41} = \frac{1}{6}\big(c^{r+1} d^2 - c^2 d^r\big) = \frac{1}{6}\big((\sqrt{3})^{r+1} - 3\big),\\
\{A^r\}_{15} &= \{A^r\}_{51} = 0,\\
\{A^r\}_{22} &= 0,
\end{aligned}$$

$$\{A^r\}_{23} = \{A^r\}_{32} = \frac{1}{3}c^{r+1}d^2 = \left(\sqrt{3}\right)^{r-1},$$
$$\{A^r\}_{24} = \{A^r\}_{42} = 0,$$
$$\{A^r\}_{25} = \{A^r\}_{52} = \frac{1}{6}\left(c^2d^r - c^{r+1}d^2\right) = \frac{1}{6}\left(3 - (\sqrt{3})^{r+1}\right),$$
$$\{A^r\}_{33} = 0,$$
$$\{A^r\}_{34} = \{A^r\}_{43} = -\frac{1}{3}c^{r+1}d^2 = -c^{r-1} = -\left(\sqrt{3}\right)^{r-1},$$
$$\{A^r\}_{35} = \{A^r\}_{53} = 0,$$
$$\{A^r\}_{44} = 0,$$
$$\{A^r\}_{45} = \{A^r\}_{54} = \frac{1}{6}\left(c^{r+1}d^2 + c^2d^r\right) = \frac{1}{6}\left((\sqrt{3})^{r+1} + 3\right),$$
$$\{A^r\}_{55} = 0.$$

For example, for $r = 3$ we have:

$$A^3 = \begin{bmatrix} 0 & -2 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & -3 & 0 \\ 1 & 0 & -3 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 \end{bmatrix}.$$

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