

LIMIT POINT AND LIMIT CIRCLE CASES FOR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract

In this study, we show that analogues of the classical concepts of Weyl limit point and limit circle cases can be introduced and investigated for second order linear dynamic equations on time scales. Since dynamical equations on time scales unify and extend continuous and discrete dynamical equations (i.e., differential and difference equations), in this way we establish a more general theory of the limit point and limit circle cases.

Keywords: Time scale, Delta and nabla derivatives, Delta and nabla integrals, Limit point case, Limit circle case.

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1. Introduction

During the past twenty years a lot of effort has been put into the study of dynamic equations on time scales (see [7, 8] and references given therein). However, there is only a small body of work concerning eigenvalue problems (spectral problems) for operators on time scales [1, 3, 4, 9, 12, 13, 16, 21], and moreover these papers are concerned only with spectral problems on a bounded time scale interval (the regular case). Spectral problems on unbounded time scale intervals (the singular case) have started to be considered only quite recently [17]. The present paper deals with second order linear dynamic equations (differential equations on time scales are called the dynamic equations) on semi-unbounded time scale intervals of the form

$$(1.1) \quad -[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = \lambda y(t), \quad t \in (a, \infty)_{\mathbb{T}},$$

as well as of the form

$$(1.2) \quad -[p(t)y^\Delta(t)]^\Delta + q(t)y^\sigma(t) = \lambda y^\sigma(t), \quad t \in [a, \infty)_{\mathbb{T}},$$

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and develops for such equations an analogue of the classical Weyl limit point and limit circle theory given by him for the usual Sturm-Liouville equation

$$(1.3) \quad -[p(t)y'(t)]' + q(t)y(t) = \lambda y(t), \quad t \in (a, \infty)_{\mathbb{R}},$$

in the first decade of the twentieth century, [24]. The limit point and limit circle theory plays an important role in the spectral theory of equations on unbounded intervals (see [10, 23, 25]). For discrete analogues of Equations (1.1) and (1.2) (for infinite Jacobi matrices), the concepts of the limit point and limit circle cases were introduced and investigated by Hellinger [19] (see also [2, Chapter 1]).

Our consideration of the problem in this paper for Equations (1.1) and (1.2) on time scales allows to unify the known continuous and discrete cases (i.e., differential and difference equation cases), and extend them to the more general context of time scales. Note that the left-hand side of Equation (1.1) is formally selfadjoint with respect to an appropriate inner product, whereas the left-hand side of equation (1.2) is not (see [11, 15]).

For a general introduction to the calculus and dynamic equations on time scales, we refer the reader to [7, 8, 18, 20].

2. Limit point and limit circle for second order dynamic equations with mixed derivatives

Firstly, following [5] we present some needed facts about second order linear dynamic equations on time scales with mixed derivatives.

Let \mathbb{T} be a time scale (an arbitrary nonempty closed subset of the real numbers). Consider the following second order linear homogeneous dynamic equation

$$(2.1) \quad -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = 0, \quad t \in \mathbb{T}^* = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa},$$

where $q : \mathbb{T} \rightarrow \mathbb{C}$ is a piecewise continuous function, $p : \mathbb{T} \rightarrow \mathbb{C}$ is ∇ -differentiable on \mathbb{T}_{κ} , $p(t) \neq 0$, for all $t \in \mathbb{T}$, and $p^{\nabla} : \mathbb{T}_{\kappa} \rightarrow \mathbb{C}$ is piecewise continuous.

We define the *quasi Δ -derivative* $y^{[\Delta]}(t)$ of y at t by

$$y^{[\Delta]}(t) = p(t)y^{\Delta}(t).$$

For any point $t_0 \in \mathbb{T}^*$ and any complex constants c_0, c_1 Equation (2.1) has a unique solution y satisfying the initial conditions

$$y(t_0) = c_0, \quad y^{[\Delta]}(t_0) = c_1.$$

If $y_1, y_2 : \mathbb{T} \rightarrow \mathbb{C}$ are two Δ -differentiable functions on \mathbb{T}^{κ} , then their Wronskian is defined for $t \in \mathbb{T}^{\kappa}$ by

$$(2.2) \quad W_t(y_1, y_2) = y_1(t)y_2^{[\Delta]}(t) - y_1^{[\Delta]}(t)y_2(t) = p(t)[y_1(t)y_2^{\Delta}(t) - y_1^{\Delta}(t)y_2(t)].$$

The Wronskian of any two solutions of Equation (2.1) is independent of t . Two solutions of Equation (2.1) are linearly independent if and only if their Wronskian is nonzero. Equation (2.1) has two linearly independent solutions, and every solution of Equation (2.1) is a linear combination of these solutions.

We say that y_1 and y_2 form a *fundamental set* (or a *fundamental system*) of solutions for Equation (2.1) provided their Wronskian is nonzero.

Let us consider the nonhomogeneous equation

$$(2.3) \quad -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = h(t), \quad t \in \mathbb{T}^*,$$

where $h : \mathbb{T} \rightarrow \mathbb{C}$ is a piecewise continuous function. If y_1 and y_2 form a fundamental set of solutions of the homogeneous equation (2.1), and $\omega = W_t(y_1, y_2)$, then the general solution of the corresponding nonhomogeneous equation (2.3) is given by

$$(2.4) \quad y(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{1}{\omega} \int_{t_0}^t [y_1(t)y_2(s) - y_1(s)y_2(t)]h(s)\nabla s,$$

where t_0 is a fixed point in \mathbb{T} , and c_1, c_2 are arbitrary constants. Formula (2.4) is called the *variation of constants formula*.

Let now \mathbb{T} be a time scale which is bounded from below and unbounded from above, so that

$$\inf \mathbb{T} = a > -\infty \text{ and } \sup \mathbb{T} = \infty.$$

By the closedness of \mathbb{T} in \mathbb{R} we have $a \in \mathbb{T}$. We will denote such a \mathbb{T} also as $[a, \infty)_{\mathbb{T}}$, and call it a *semi-infinite* (or *semi-unbounded*) *time scale interval*.

Consider the equation

$$(2.5) \quad -[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = \lambda y(t), \quad t \in (a, \infty)_{\mathbb{T}},$$

where $p(t)$ is a real-valued continuous function on $[a, \infty)_{\mathbb{T}}$ and ∇ -differentiable on $(a, \infty)_{\mathbb{T}}$ with piecewise continuous ∇ -derivative $p^\nabla(t)$, $p(t) \neq 0$ for all t , $q(t)$ is a real-valued piecewise continuous function on $[a, \infty)_{\mathbb{T}}$, and λ is a complex parameter (spectral parameter).

A function $y : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{C}$ is called a *solution* of Equation (2.5), if y is Δ -differentiable on $[a, \infty)_{\mathbb{T}}$, y^Δ is ∇ -differentiable on $(a, \infty)_{\mathbb{T}}$, and Equation (2.5) is satisfied.

For any fixed point $t_0 \in [a, \infty)_{\mathbb{T}}$ and any complex numbers c_0, c_1 Equation (2.5) has a unique solution $y(t, \lambda)$ satisfying the initial conditions

$$y(t_0, \lambda) = c_0, \quad y^{[\Delta]}(t_0, \lambda) = c_1,$$

where $y^{[\Delta]}(t, \lambda) = p(t)y^\Delta(t, \lambda)$. The function $y(t, \lambda)$, together with its quasi Δ -derivative $y^{[\Delta]}(t, \lambda)$, is regular in the complex plane \mathbb{C} as a function of λ , in other words, it is an entire function of λ . This follows from the fact that $y(t, \lambda)$ can be obtained by solving the integral equation

$$y(t, \lambda) = c_0 + c_1 \int_{t_0}^t \frac{\Delta\tau}{p(\tau)} + \int_{t_0}^t \frac{1}{p(\tau)} \left\{ \int_{t_0}^\tau [q(s) - \lambda]y(s, \lambda)\nabla s \right\} \Delta\tau$$

by the method of successive approximations, and that for $y^{[\Delta]}(t, \lambda)$ we have the equation

$$y^{[\Delta]}(t, \lambda) = c_1 + \int_{t_0}^t [q(s) - \lambda]y(s, \lambda)\nabla s.$$

Moreover, if the initial values c_0 and c_1 are real, then

$$\overline{y(t, \lambda)} = y(t, \overline{\lambda}),$$

where the bar over a function (or over a number) denotes the complex conjugate.

Throughout the paper L will denote the formally selfadjoint operator defined by

$$Lx = -(px^\Delta)^\nabla + qx.$$

2.1. Lemma. *If $Lx(t, \lambda) = \lambda x(t, \lambda)$ and $Ly(t, \lambda') = \lambda' y(t, \lambda')$, then for any $b \in (a, \infty)_{\mathbb{T}}$,*

$$(2.6) \quad (\lambda' - \lambda) \int_a^b xy\nabla t = W_a(x, y) - W_b(x, y),$$

where $W_t(x, y)$ is the Wronskian of x and y defined by (2.2).

Proof. We have, using the integration by parts formula

$$\int_a^b f^\nabla(t)g(t)\nabla t = f(t)g(t) \Big|_a^b - \int_a^b f(t)g^\Delta(t)\Delta t$$

established in [15], that

$$\begin{aligned} (\lambda' - \lambda) \int_a^b xy\nabla t &= \int_a^b (xLy - yLx)\nabla t \\ &= - \int_a^b [x(py^\Delta)^\nabla - y(px^\Delta)^\nabla]\nabla t \\ &= -xpy^\Delta \Big|_a^b + \int_a^b x^\Delta py^\Delta \Delta t + ypx^\Delta \Big|_a^b - \int_a^b y^\Delta px^\Delta \Delta t \\ &= -p(xy^\Delta - x^\Delta y) \Big|_a^b = W_a(x, y) - W_b(x, y). \end{aligned}$$

The proof is complete. \square

2.2. Corollary. *If, in particular, $\lambda = u + iv$, $\lambda' = \bar{\lambda} = u - iv$, ($u, v \in \mathbb{R}$), then we can take $y(t, \lambda) = x(t, \bar{\lambda})$ and (2.6) yields*

$$(2.7) \quad 2v \int_a^b |x(t, \lambda)|^2 \nabla t = i\{W_a(x, \bar{x}) - W_b(x, \bar{x})\}. \quad \square$$

Let $\varphi(t, \lambda)$, $\theta(t, \lambda)$ be two solutions of Equation (2.5) satisfying the initial conditions

$$(2.8) \quad \varphi(a, \lambda) = \sin \alpha, \quad \varphi^{[\Delta]}(a, \lambda) = -\cos \alpha,$$

$$(2.9) \quad \theta(a, \lambda) = \cos \alpha, \quad \theta^{[\Delta]}(a, \lambda) = \sin \alpha,$$

where $0 \leq \alpha < \pi$. Then, since the Wronskian of any two solutions of (2.5) does not depend on t , we get

$$W_t(\varphi, \theta) = W_a(\varphi, \theta) = \sin^2 \alpha + \cos^2 \alpha = 1.$$

Then φ , θ are linearly independent solutions of (2.5), φ , $\varphi^{[\Delta]}$, θ , $\theta^{[\Delta]}$ are entire functions of λ which are continuous in (t, λ) . These solutions are real for real λ . Every solution y of (2.5) except for φ is, up to a constant multiple, of the form

$$(2.10) \quad y = \theta + l\varphi$$

for some number l which will depend on λ .

Take now a point $b \in (a, \infty)_{\mathbb{T}}$, and consider the boundary condition

$$(2.11) \quad y(b) \cos \beta + y^{[\Delta]}(b) \sin \beta = 0, \quad (0 \leq \beta < \pi),$$

and ask what must l be like in order that the solution y , (2.10), satisfy (2.11). If denote the corresponding value of l by $l_b(\lambda)$, then we find that

$$l_b(\lambda) = -\frac{\theta(b, \lambda) \cot \beta + \theta^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda) \cot \beta + \varphi^{[\Delta]}(b, \lambda)}.$$

Let us take any complex number z and introduce the function

$$(2.12) \quad l = l_b(\lambda, z) = -\frac{\theta(b, \lambda)z + \theta^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)z + \varphi^{[\Delta]}(b, \lambda)}.$$

If b and λ are fixed, and z varies, (2.12) may be written as

$$(2.13) \quad l = \frac{Az + B}{Cz + D},$$

where $A = -\theta(b, \lambda)$, $B = -\theta^{[\Delta]}(b, \lambda)$, $C = \varphi(b, \lambda)$, and $D = \varphi^{[\Delta]}(b, \lambda)$. Since

$$AD - BC = W_b(\varphi, \theta) = 1 \neq 0,$$

the linear-fractional transformation (2.13) is a one-to-one conformal mapping which transforms circles into circles; straight lines being considered as circles with infinite radii. Besides, applying (2.7) to the solution $\varphi(t, \lambda)$, and taking into account (2.8) - by virtue of which $W_a(\varphi, \bar{\varphi}) = 0$ - we have

$$2v \int_a^b |\varphi(t, \lambda)|^2 \nabla t = -i\varphi(b, \lambda)\bar{\varphi}^{[\Delta]}(b, \lambda) + i\varphi^{[\Delta]}(b, \lambda)\bar{\varphi}(b, \lambda),$$

which implies that $\varphi(b, \lambda) \neq 0$ and $\varphi^{[\Delta]}(b, \lambda) \neq 0$ if $\Im\lambda = v \neq 0$. Therefore, if $\Im\lambda = v \neq 0$, then $l_b(\lambda, z)$ varies on a circle $C_b(\lambda)$ with a finite radius in the l -plane, as z varies over the real axis of the z -plane.

The center and the radius of the circle $C_b(\lambda)$ will be defined as follows. The center of the circle is the symmetric point of the point at infinity with respect to the circle. Thus if we set

$$l_b(\lambda, z') = \infty \text{ and } l_b(\lambda, z'') = \text{the center of } C_b(\lambda),$$

then z'' must be the symmetric point of z' with respect to the real axis of the z -plane, namely $z'' = \bar{z}'$. On the other hand,

$$l_b \left(\lambda, -\frac{\varphi^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)} \right) = \infty.$$

Therefore, the center of the circle $C_b(\lambda)$ is given by

$$l_b \left(\lambda, -\frac{\bar{\varphi}^{[\Delta]}(b, \lambda)}{\bar{\varphi}(b, \lambda)} \right) = -\frac{W_b(\theta, \bar{\varphi})}{W_b(\varphi, \bar{\varphi})}.$$

The radius $r_b(\lambda)$ of the circle $C_b(\lambda)$ is equal to the distance between the center of $C_b(\lambda)$ and the point $l_b(\lambda, 0)$ on the circle $C_b(\lambda)$. Hence,

$$r_b(\lambda) = \left| \frac{\theta^{[\Delta]}(b, \lambda)}{\varphi^{[\Delta]}(b, \lambda)} - \frac{W_b(\theta, \bar{\varphi})}{W_b(\varphi, \bar{\varphi})} \right| = \left| \frac{W_b(\theta, \varphi)}{W_b(\varphi, \bar{\varphi})} \right|.$$

On the other hand, by virtue of (2.8) and (2.9), $W_b(\theta, \varphi) = W_a(\theta, \varphi) = -1$. Further, by virtue of (2.7) and (2.8), we have

$$(2.14) \quad 2v \int_a^b |\varphi(t, \lambda)|^2 \nabla t = iW_a(\varphi, \bar{\varphi}) - iW_b(\varphi, \bar{\varphi}) = -iW_b(\varphi, \bar{\varphi}),$$

where $v = \Im\lambda$. Therefore, we obtain

$$(2.15) \quad r_b(\lambda) = \frac{1}{2|v| \int_a^b |\varphi(t, \lambda)|^2 \nabla t}, \quad \Im\lambda = v \neq 0.$$

Since $\theta(b, \lambda)\varphi^{[\Delta]}(b, \lambda) - \varphi(b, \lambda)\theta^{[\Delta]}(b, \lambda) = W_b(\theta, \varphi) = -1 \neq 0$, the transformation (2.12) has a unique inverse which is given by

$$(2.16) \quad z = -\frac{\varphi^{[\Delta]}(b, \lambda)l + \theta^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)l + \theta(b, \lambda)}.$$

We shall now prove the following statement.

2.3. Lemma. *If $v = \Im\lambda > 0$, then the interior of the circle $C_b(\lambda)$ is mapped onto the lower half plane of the z -plane by the transformation (2.16), and the exterior of the circle $C_b(\lambda)$ is mapped onto the upper half plane of the z -plane.*

Proof. Since the real axis of the z -plane is the image of the circle $C_b(\lambda)$ by the transformation (2.16), the interior of $C_b(\lambda)$ is mapped onto either the upper half plane or the lower half plane of the z -plane, and further, the point at infinity of the l -plane is mapped

onto the point $-\varphi^{[\Delta]}(b, \lambda)/\varphi(b, \lambda)$ of the z -plane. On the other hand, by making use of (2.14),

$$\begin{aligned} \Im \left(-\frac{\varphi^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)} \right) &= \frac{i}{2} \left\{ \frac{\varphi^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)} - \frac{\overline{\varphi^{[\Delta]}(b, \lambda)}}{\overline{\varphi(b, \lambda)}} \right\} = -\frac{i}{2} \frac{W_b(\varphi, \overline{\varphi})}{|\varphi(b, \lambda)|^2} \\ &= \frac{v}{|\varphi(b, \lambda)|^2} \int_a^b |\varphi(t, \lambda)|^2 \nabla t > 0. \end{aligned}$$

This means that $-\varphi^{[\Delta]}(b, \lambda)/\varphi(b, \lambda)$ belongs to the upper half plane of the z -plane. Hence the point at infinity, which is not contained in the interior of $C_b(\lambda)$, is mapped into the upper half plane. This proves the lemma. \square

2.4. Lemma. *If $v = \Im \lambda > 0$ then l belongs to the interior of the circle $C_b(\lambda)$ if and only if*

$$\int_a^b |\theta(t, \lambda) + l\varphi(t, \lambda)|^2 \nabla t < -\frac{\Im l}{v},$$

and l lies on the circle $C_b(\lambda)$ if and only if

$$\int_a^b |\theta(t, \lambda) + l\varphi(t, \lambda)|^2 \nabla t = -\frac{\Im l}{v}.$$

Proof. In view of Lemma 2.3, if $\Im \lambda = v > 0$ then l belongs to the interior of the circle $C_b(\lambda)$ if and only if $\Im z < 0$, that is, $i(z - \overline{z}) > 0$. From (2.16) it follows that

$$\begin{aligned} i(z - \overline{z}) &= i \left\{ -\frac{\varphi^{[\Delta]}(b, \lambda)l + \theta^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)l + \theta(b, \lambda)} + \frac{\overline{\varphi^{[\Delta]}(b, \lambda)}\overline{l} + \overline{\theta^{[\Delta]}(b, \lambda)}}{\overline{\varphi(b, \lambda)}\overline{l} + \overline{\theta(b, \lambda)}} \right\} \\ &= \frac{iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi})}{|\varphi(b, \lambda)l + \theta(b, \lambda)|^2}. \end{aligned}$$

Therefore, $\Im z < 0$ if and only if

$$iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) > 0.$$

By Formula (2.7) with $x = \theta + l\varphi$, we have

$$2v \int_a^b |\theta + l\varphi|^2 \nabla t = i \{ W_a(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) - W_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) \}.$$

Further, by (2.8) and (2.9) we have $W_a(\theta, \overline{\varphi}) = -1$, $W_a(\varphi, \overline{\theta}) = 1$, and $W_a(\theta, \overline{\theta}) = W_a(\varphi, \overline{\varphi}) = 0$. Therefore,

$$\begin{aligned} W_a(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) &= W_a(\theta, \overline{\theta}) + \overline{l}W_a(\theta, \overline{\varphi}) + lW_a(\varphi, \overline{\theta}) + |l|^2 W_a(\varphi, \overline{\varphi}) \\ &= l - \overline{l} = 2i\Im l. \end{aligned}$$

Consequently,

$$2v \int_a^b |\theta + l\varphi|^2 \nabla t = -2\Im l - iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}),$$

and the statements of the lemma follows. \square

2.5. Remark. It is easy to see that Lemma 2.4 also holds when $v = \Im \lambda < 0$. In both the cases $v > 0$ and $v < 0$ the sign of $\Im l$ is opposite to the sign of v .

2.6. Lemma. *If $v = \Im \lambda \neq 0$, and $0 < b < b'$ then*

$$\tilde{C}_{b'}(\lambda) \subset \tilde{C}_b(\lambda),$$

where $\tilde{C}_b(\lambda)$ is the set composed of the circle $C_b(\lambda)$ and its interior.

Proof. If l belongs to the interior of the circle $C_{b'}(\lambda)$ or is on $C_{b'}(\lambda)$, then taking into account Lemma 2.4, we have

$$\int_a^b |\theta + l\varphi|^2 \nabla t \leq \int_a^{b'} |\theta + l\varphi|^2 \nabla t \leq -\frac{\Im l}{v}.$$

Hence the lemma follows by using Lemma 2.4 again. □

Lemma 2.6 implies that, if $v = \Im \lambda \neq 0$, then the set

$$\bigcap_{b>a} \tilde{C}_b(\lambda) = C_\infty(\lambda)$$

is either a point or a closed circle with a nonzero finite radius.

2.7. Definition. According as $C_\infty(\lambda)$ is a point or a circle, the equation (2.5) is said to be in the *limit point case* or the *limit circle case*.

According to this definition, the classification seems to depend on the $p(t)$, $q(t)$, and λ . However, it is independent of λ and depends only on $p(t)$, $q(t)$, as is shown in the next section.

Let $m = m(\lambda)$ be the limit point $C_\infty(\lambda)$, or any point on the limit circle $C_\infty(\lambda)$. Then, for any $b \in (a, \infty)_{\mathbb{T}}$ we have

$$\int_a^b |\theta(t, \lambda) + m(\lambda)\varphi(t, \lambda)|^2 \nabla t \leq -\frac{\Im m(\lambda)}{v}.$$

Hence,

$$\int_a^\infty |\theta(t, \lambda) + m(\lambda)\varphi(t, \lambda)|^2 \nabla t \leq -\frac{\Im m(\lambda)}{v}.$$

Denote by $L^2_{\nabla}(a, \infty)$ the space of all complex-valued ∇ -measurable (see [14]) functions $f : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{C}$ such that

$$\int_a^\infty |f(t)|^2 \nabla t < \infty.$$

Similarly, we can introduce the space $L^2_{\Delta}(a, \infty)$.

Thus, we have obtained the following theorem.

2.8. Theorem. *For all non-real values of λ there exists a solution*

$$\psi(t, \lambda) = \theta(t, \lambda) + m(\lambda)\varphi(t, \lambda)$$

of Equation (2.5) such that $\psi \in L^2_{\nabla}(a, \infty)$. □

In the limit circle case the radius $r_b(\lambda)$ tends to a finite nonzero limit as $b \rightarrow \infty$. Then (2.15) implies that in this case also $\varphi \in L^2_{\nabla}(a, \infty)$. Therefore, in the limit circle case all solutions of Equation (2.5) belong to $L^2_{\nabla}(a, \infty)$ for $\Im \lambda \neq 0$, because in this case both $\varphi(t, \lambda)$ and $\theta(t, \lambda) + m(\lambda)\varphi(t, \lambda)$ belong to $L^2_{\nabla}(a, \infty)$, and this identifies the limit circle case. We will see below in Theorem 3.1 that in the limit circle case all solutions of Equation (2.5) belong to $L^2_{\nabla}(a, \infty)$ also for all real values of λ .

In the limit point case, $r_b(\lambda)$ tends to zero as $b \rightarrow \infty$, and from (2.15) this implies that $\varphi(t, \lambda)$ is not of the class $L^2_{\nabla}(a, \infty)$. Therefore in this situation there is only one solution of class $L^2_{\nabla}(a, \infty)$ for $\Im \lambda \neq 0$. Note that in the limit point case the equation may not have any nontrivial solution of class $L^2_{\nabla}(a, \infty)$ for real values of λ . For example, (for $\lambda = 0$), the equation $-y^{\Delta \nabla} = 0$ has the general solution $y(t) = c_1 + c_2 t$, and evidently this solution belongs to $L^2_{\nabla}(a, \infty)$ only for $c_1 = c_2 = 0$.

3. Invariance of the limit point and limit circle properties

In the previous section the expressions “limit point case” and “limit circle case” were applied to particular values of λ , but in fact if the limit is a circle for any complex λ it is a circle for every complex λ . In the present section we prove this property.

3.1. Theorem. *If every solution of $Ly = \lambda_0 y$ is of class $L^2_{\nabla}(a, \infty)$ for some complex number λ_0 , then for an arbitrary complex number λ every solution of $Ly = \lambda y$ is of class $L^2_{\nabla}(a, \infty)$.*

Proof. It is given that two linearly independent solutions $y_1(t)$ and $y_2(t)$ of $Ly = \lambda_0 y$ are of class $L^2_{\nabla}(a, \infty)$. Let $\chi(t)$ be any solution of $Ly = \lambda y$, which can be written as

$$Ly = \lambda_0 y + (\lambda - \lambda_0)y.$$

By multiplying y_1 by a constant if necessary (to achieve $W_t(y_1, y_2) = 1$) the variation of constants formula (2.4) yields

$$(3.1) \quad \chi(t) = c_1 y_1(t) + c_2 y_2(t) + (\lambda - \lambda_0) \int_c^t [y_1(t)y_2(s) - y_1(s)y_2(t)]\chi(s)\nabla s,$$

where c_1, c_2 are constants and c is any fixed point in $[a, \infty)_{\mathbb{T}}$. Let us introduce the notation

$$\|\chi\|_{c,t} = \left\{ \int_c^t |\chi(s)|^2 \nabla s \right\}^{1/2} \quad \text{for } t \in [a, \infty)_{\mathbb{T}} \text{ with } t \geq c.$$

Next, let M be such that $\|y_1\|_{c,t} \leq M, \|y_2\|_{c,t} \leq M$ for all $t \in [a, \infty)_{\mathbb{T}}$ with $t \geq c$. Such a constant M exists because y_1 and y_2 are of class $L^2_{\nabla}(a, \infty)$. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned} & \left| \int_c^t [y_1(t)y_2(s) - y_1(s)y_2(t)]\chi(s)\nabla s \right| \\ & \leq |y_1(t)| \int_c^t |y_2(s)| |\chi(s)| \nabla s + |y_2(t)| \int_c^t |y_1(s)| |\chi(s)| \nabla s \\ & \leq M(|y_1(t)| + |y_2(t)|) \|\chi\|_{c,t}. \end{aligned}$$

Using this in (3.1) yields

$$|\chi(t)| \leq |c_1| |y_1(t)| + |c_2| |y_2(t)| + |\lambda - \lambda_0| M(|y_1(t)| + |y_2(t)|) \|\chi\|_{c,t}.$$

Hence applying the Minkowski inequality (note that the usual Cauchy-Schwarz and Minkowski inequalities hold on time scales, see [7, Chapter 6]), we get

$$\|\chi\|_{c,t} \leq (|c_1| + |c_2|)M + 2|\lambda - \lambda_0| M^2 \|\chi\|_{c,t}.$$

If c is chosen large enough so that $|\lambda - \lambda_0| M^2 < 1/4$, then

$$\|\chi\|_{c,t} \leq 2(|c_1| + |c_2|)M.$$

Since the right side of this inequality is independent of t , it follows that $\chi \in L^2_{\nabla}(a, \infty)$, and the theorem is proved. \square

4. A criterion for the limit point case

In this section we present a simple criterion for the limit point case. For the usual Sturm-Liouville equation this was established earlier by Putnam [22].

4.1. Theorem. *If p is arbitrary and $q \in L^2_{\nabla}(a, \infty)$, then Equation (2.5) is in the limit point case.*

Proof. It is sufficient to show that the equation

$$(4.1) \quad -[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = 0, \quad t \in (a, \infty)_{\mathbb{T}},$$

does not have two linearly independent solutions belonging to $L^2_{\nabla}(a, \infty)$.

If y is such a solution, then, because of the condition $q \in L^2_{\nabla}(a, \infty)$, the function $(y^{[\Delta]})^\nabla = qy$ belongs to $L^1_{\nabla}(a, \infty)$ by the Cauchy-Schwarz inequality. Therefore the limit

$$\lim_{t \rightarrow \infty} y^{[\Delta]}(t) = y^{[\Delta]}(t_0) + \int_{t_0}^{\infty} (y^{[\Delta]})^\nabla(s) \nabla s$$

exists and is finite. Hence the function $y^{[\Delta]}(t)$ is bounded as $t \rightarrow \infty$.

Now let y_1, y_2 be two linearly independent solutions of Equation (4.1). Then

$$y_1(t)y_2^{[\Delta]}(t) - y_1^{[\Delta]}(t)y_2(t) = c \neq 0.$$

If $y_1 \in L^2_{\nabla}(a, \infty)$ and $y_2 \in L^2_{\nabla}(a, \infty)$, then $y_1^{[\Delta]}$ and $y_2^{[\Delta]}$ are bounded, and so the function $y_1y_2^{[\Delta]} - y_1^{[\Delta]}y_2 = c \neq 0$ also belongs to $L^2_{\nabla}(a, \infty)$, which is impossible. The theorem is proved. \square

5. Limit point and limit circle for second order dynamic equations with delta derivatives

In this section, we consider an equation of the form

$$(5.1) \quad -[p(t)y^\Delta(t)]^\Delta + q(t)y^\sigma(t) = \lambda y^\sigma(t), \quad t \in [a, \infty)_{\mathbb{T}},$$

where $p(t)$ is a real-valued Δ -differentiable function on $[a, \infty)_{\mathbb{T}}$ with piecewise continuous Δ -derivative $p^\Delta(t)$, $p(t) \neq 0$ for all t , $q(t)$ is a real-valued piecewise continuous function on $[a, \infty)_{\mathbb{T}}$, λ is a complex parameter (spectral parameter), σ denotes the forward jump operator on \mathbb{T} and $y^\sigma(t) = y(\sigma(t))$.

Let L_1 denote the linear operator defined by

$$L_1x = -(px^\Delta)^\Delta + qx^\sigma.$$

5.1. Lemma. *If $L_1x(t, \lambda) = \lambda x^\sigma(t, \lambda)$ and $L_1y(t, \lambda') = \lambda' y^\sigma(t, \lambda')$, then for any $b \in (a, \infty)_{\mathbb{T}}$,*

$$(5.2) \quad (\lambda' - \lambda) \int_a^b xy \nabla t = W_a(x, y) - W_b(x, y),$$

where $W_t(x, y)$ is the Wronskian of x and y defined by (2.2).

Proof. Using the relationship

$$(5.3) \quad \int_a^b f(t) \nabla t = \int_a^b f(\sigma(t)) \Delta t,$$

between the nabla and delta integrals (see [15, Theorem 11] or [16, Theorem 2.2]), where f is a continuous (in the time scale topology) function, and the product rule for Δ -differentiation,

$$[f(t)g(t)]^\Delta = f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t),$$

we have

$$\begin{aligned} (\lambda' - \lambda) \int_a^b xy \nabla t &= (\lambda' - \lambda) \int_a^b x^\sigma y^\sigma \Delta t = \int_a^b (x^\sigma L_1 y - y^\sigma L_1 x) \Delta t \\ &= - \int_a^b [x^\sigma (py^\Delta)^\Delta - y^\sigma (px^\Delta)^\Delta] \Delta t = - \int_a^b (xpy^\Delta - ypx^\Delta)^\Delta \Delta t \\ &= - \int_a^b [W_t(x, y)]^\Delta \Delta t = W_a(x, y) - W_b(x, y). \end{aligned}$$

The proof is complete. □

Now all the contents of Section 2, beginning with Corollary 2.2 may be repeated without any alteration, and in particular the following statements hold.

Let $\varphi(t, \lambda), \theta(t, \lambda)$ be two solutions of Equation (5.1) satisfying the initial conditions

$$\begin{aligned} \varphi(a, \lambda) &= \sin \alpha, & \varphi^{[\Delta]}(a, \lambda) &= -\cos \alpha, \\ \theta(a, \lambda) &= \cos \alpha, & \theta^{[\Delta]}(a, \lambda) &= \sin \alpha, \end{aligned}$$

where $0 \leq \alpha < \pi$. Let $b \in (a, \infty)_{\mathbb{T}}$, take any complex number z and introduce the linear-fractional function

$$l = l_b(\lambda, z) = - \frac{\theta(b, \lambda)z + \theta^{[\Delta]}(b, \lambda)}{\varphi(b, \lambda)z + \varphi^{[\Delta]}(b, \lambda)}$$

of z . If $\Im \lambda \neq 0$, then $l_b(\lambda, z)$ varies on a circle $C_b(\lambda)$ with a finite radius in the l -plane as z varies over the real axis of the z -plane. If $\Im \lambda \neq 0$ and $0 < b < b'$, then $\tilde{C}_{b'}(\lambda) \subset \tilde{C}_b(\lambda)$, where $\tilde{C}_b(\lambda)$ is the set composed of the circle $C_b(\lambda)$ and its interior. Therefore, if $\Im \lambda \neq 0$, the set

$$\bigcap_{b>a} \tilde{C}_b(\lambda) = C_\infty(\lambda)$$

is either a point or a closed circle with a nonzero finite radius. According as $C_\infty(\lambda)$ is a point or a circle, the equation (5.1) is said to be in the limit point case or the limit circle case.

For all non-real values of λ there exists a solution

$$\psi(t, \lambda) = \theta(t, \lambda) + m(\lambda)\varphi(t, \lambda)$$

of Equation (5.1) such that $\psi \in L^2_{\nabla}(a, \infty)$. In the limit circle case all solutions of Equation (5.1) belong to $L^2_{\nabla}(a, \infty)$ for $\Im \lambda \neq 0$, and this identifies the limit circle case. We will see below in Theorem 5.2 that in the limit circle case all solutions of Equation (5.1) belong to $L^2_{\nabla}(a, \infty)$ also for all real values of λ . In the limit point case, there is only one solution of class $L^2_{\nabla}(a, \infty)$ for $\Im \lambda \neq 0$.

5.2. Theorem. *If every solution of $L_1 y = \lambda_0 y^\sigma$ is of class $L^2_{\nabla}(a, \infty)$ for some complex number λ_0 , then for an arbitrary complex number λ every solution of $L_1 y = \lambda y^\sigma$ is of class $L^2_{\nabla}(a, \infty)$.*

Proof. It is given that two linearly independent solutions $y_1(t)$ and $y_2(t)$ of $L_1 y = \lambda_0 y^\sigma$ are of class $L^2_{\nabla}(a, \infty)$. Let $\chi(t)$ be any solution of $L_1 y = \lambda y^\sigma$, which can be written as

$$L_1 y = \lambda_0 y^\sigma + (\lambda - \lambda_0) y^\sigma.$$

By multiplying y_1 by a constant if necessary (to achieve $W_t(y_1, y_2) = 1$), the variation of constants formula (see [7]) and the relationship (5.3) yield

$$\begin{aligned} \chi(t) &= c_1 y_1(t) + c_2 y_2(t) + (\lambda - \lambda_0) \int_c^t [y_1(t)y_2(\sigma(s)) - y_1(\sigma(s))y_2(t)]\chi(\sigma(s))\Delta s \\ &= c_1 y_1(t) + c_2 y_2(t) + (\lambda - \lambda_0) \int_c^t [y_1(t)y_2(s) - y_1(s)y_2(t)]\chi(s)\nabla s, \end{aligned}$$

where c_1, c_2 are constants and c is any fixed point in $[a, \infty)_{\mathbb{T}}$. The proof may be completed as in the proof of Theorem 3.1. \square

5.3. Theorem. *If p is arbitrary and $q \in L^2_{\Delta}(a, \infty)$, then Equation (5.1) is in the limit point case.*

Proof. It is sufficient to show that the equation

$$(5.4) \quad -[p(t)y^{\Delta}(t)]^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad t \in [a, \infty)_{\mathbb{T}},$$

does not have two linearly independent solutions belonging to $L^2_{\nabla}(a, \infty)$.

If y is such a solution, then, because of the condition $q \in L^2_{\Delta}(a, \infty)$, the function $(y^{[\Delta]})^{\Delta} = qy^{\sigma}$ belongs to $L^1_{\Delta}(a, \infty)$ by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_a^{\infty} |(y^{[\Delta]})^{\Delta}| \Delta t &= \int_a^{\infty} |qy^{\sigma}| \Delta t \leq \sqrt{\int_a^{\infty} |q|^2 \Delta t} \sqrt{\int_a^{\infty} |y^{\sigma}|^2 \Delta t} \\ &= \sqrt{\int_a^{\infty} |q|^2 \Delta t} \sqrt{\int_a^{\infty} |y|^2 \nabla t} < \infty. \end{aligned}$$

Therefore the limit

$$\lim_{t \rightarrow \infty} y^{[\Delta]}(t) = y^{[\Delta]}(t_0) + \int_{t_0}^{\infty} (y^{[\Delta]})^{\Delta}(s)\Delta s$$

exists and is finite. Hence the function $y^{[\Delta]}(t)$ is bounded as $t \rightarrow \infty$.

Now let y_1, y_2 be two linearly independent solutions of Equation (5.4). Then

$$y_1(t)y_2^{[\Delta]}(t) - y_1^{[\Delta]}(t)y_2(t) = c \neq 0.$$

If $y_1 \in L^2_{\nabla}(a, \infty)$ and $y_2 \in L^2_{\nabla}(a, \infty)$, then $y_1^{[\Delta]}$ and $y_2^{[\Delta]}$ are bounded, and so the function $y_1 y_2^{[\Delta]} - y_1^{[\Delta]} y_2 = c \neq 0$ also belongs to $L^2_{\nabla}(a, \infty)$ which is impossible. The theorem is proved. \square

6. Concluding remarks

- (1) Since the conditions in Theorem 4.1 and Theorem 5.3 are given in terms of the delta and nabla integrals, let us present the explicit form of these integrals for some standard examples of time scales. Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $f : \mathbb{T} \rightarrow \mathbb{C}$ a function.

- (i) If $\mathbb{T} = [1, \infty)_{\mathbb{R}}$, then $\sigma(t) = t$ and $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$, the ordinary derivative of f at t , and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t) dt,$$

the ordinary integral.

(ii) If $\mathbb{T} = \mathbb{N} = \{1, 2, 3, \dots\} = [1, \infty)_{\mathbb{N}}$, then $\sigma(t) = t + 1$, and

$$f^{\Delta}(t) = f(t+1) - f(t), \quad f^{\nabla}(t) = f(t) - f(t-1), \quad t > 1,$$

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k), \quad \int_a^b f(t)\nabla t = \sum_{k=a+1}^b f(k).$$

(iii) If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, then $\sigma(t) = qt$, and

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad f^{\nabla}(t) = \frac{f(t) - f(q^{-1}t)}{(1-q^{-1})t}, \quad t > 1,$$

$$\int_a^b f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k), \quad \int_a^b f(t)\nabla t = (1-q^{-1}) \sum_{k=m+1}^n q^k f(q^k),$$

provided $a = q^m$, $b = q^n$ with $m, n \in \mathbb{N}_0$ and $m < n$.

(iv) Let \mathbb{T} be a time scale of the form

$$\mathbb{T} = \{t_k : k \in \mathbb{N}_0\} \text{ with } 0 < t_0 < t_1 < t_2 < \dots \text{ and } \lim_{k \rightarrow \infty} t_k = \infty.$$

Then $\sigma(t_k) = t_{k+1}$ for all $k \in \mathbb{N}_0$, and we have

$$f^{\Delta}(t_k) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k}, \quad f^{\nabla}(t_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}, \quad k > 0,$$

$$\int_{t_m}^{t_n} f(t)\Delta t = \sum_{k=m}^{n-1} f(t_k)(t_{k+1} - t_k), \quad \int_{t_m}^{t_n} f(t)\nabla t = \sum_{k=m+1}^n f(t_k)(t_k - t_{k-1}),$$

where $m, n \in \mathbb{N}_0$ with $m < n$.

(2) In connection with the conditions of Theorem 4.1 and Theorem 5.3, note that using the above presented explicit form of the delta and nabla integrals it is easy to check that the function $q(t) = 1/t$ belongs to both spaces $L_{\Delta}^2(a, \infty)$ and $L_{\nabla}^2(a, \infty)$, for each of the time scales $[1, \infty)_{\mathbb{R}}$, \mathbb{N} and $q^{\mathbb{N}_0}$ ($q > 1$). On the other hand, the same function belongs to $L_{\nabla}^2(a, \infty)$, but does not belong to $L_{\Delta}^2(a, \infty)$ for the time scale

$$\mathbb{T} = \{t_k : k \in \mathbb{N}_0\} \text{ with } t_k = 2^{(2^k)},$$

so that Theorem 4.1 applies and Theorem 5.3 does not in this case. Indeed, we have

$$\int_2^{\infty} q^2(t)\nabla t = \sum_{k=1}^{\infty} q^2(t_k)(t_k - t_{k-1}) = \sum_{k=1}^{\infty} 2^{-(2^{k+1})} [2^{(2^k)} - 2^{(2^{k-1})}]$$

$$= \sum_{k=1}^{\infty} [2^{-(2^k)} - 2^{-3(2^{k-1})}] < \infty,$$

$$\int_2^{\infty} q^2(t)\Delta t = \sum_{k=0}^{\infty} q^2(t_k)(t_{k+1} - t_k) = \sum_{k=1}^{\infty} 2^{-(2^{k+1})} [2^{(2^{k+1})} - 2^{(2^k)}]$$

$$= \sum_{k=1}^{\infty} [1 - 2^{-(2^k)}] = \infty.$$

For improper integrals on time scales we refer the reader to [6] or [8, Section 5.6].

(3) Above we considered dynamic equations of the forms (1.1) and (1.2). The concepts of limit point and limit circle cases can be introduced and investigated in a similar way for dynamic equations of the forms

$$(6.1) \quad -[p(t)y^\nabla(t)]^\Delta + q(t)y(t) = \lambda y(t), \quad t \in (a, \infty)_{\mathbb{T}},$$

and

$$(6.2) \quad -[p(t)y^\nabla(t)]^\nabla + q(t)y^\rho(t) = \lambda y^\rho(t), \quad t \in (\sigma(a), \infty)_{\mathbb{T}},$$

where σ and ρ denote the forward and backward jump operators on \mathbb{T} and $y^\rho(t) = y(\rho(t))$. The Wronskian associated with equations (6.1) and (6.2) has the form

$$\widetilde{W}_t(y_1, y_2) = p(t)[y_1(t)y_2^\nabla(t) - y_1^\nabla(t)y_2(t)], \quad t \in [\sigma(a), \infty)_{\mathbb{T}},$$

and an analogue of Lemmas 2.1 and 5.1 is

$$(\lambda' - \lambda) \int_{\sigma(a)}^b xy \Delta t = W_{\sigma(a)}(x, y) - W_b(x, y).$$

For equations (6.1) and (6.2), in (2.8) and (2.9) we should take $\sigma(a)$ instead of a and ∇ instead of Δ (the ∇ -derivative is not defined at a if a is right-scattered). In the limit circle case all solutions of Equation (6.1), as well as of Equation (6.2), belong to $L^2_\Delta(a, \infty)$ for all complex values of λ , and this identifies the limit circle case.

In the limit point case, there is only one solution of class $L^2_\Delta(a, \infty)$ for $\Im \lambda \neq 0$. If $q \in L^2_\Delta(a, \infty)$, then Equation (6.1) is in the limit point case; whereas, if $q \in L^2_\nabla(a, \infty)$, then Equation (6.2) is in the limit point case. We see that if the second derivative in the dynamic equation is Δ (respectively, ∇), then the condition for the limit point case is $q \in L^2_\Delta(a, \infty)$ (respectively $q \in L^2_\nabla(a, \infty)$).

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