Hacettepe Journal of Mathematics and Statistics
Volume 39 (3) (2010), 393 – 401

ON THE LOGARITHMIC INTEGRAL

Brian Fisher*† and Biljana Jolevska-Tuneska‡

Received 29 : 09 : 2009 : Accepted 02 : 03 : 2010

Abstract

The logarithmic integral $\mathrm{li}(x)$ and its associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of these functions and other functions are then found.

Keywords: Logarithmic integral, Distribution, Convolution, Neutrix, Neutrix convolution.

2000 AMS Classification: 33 B 10, 46 F 10.

1. Introduction

The *logarithmic integral* $\mathrm{li}(x)$, see Abramowitz and Stegun [1], is defined by

$$\begin{aligned}\mathrm{li}(x) &= \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x|<1,\\ \mathrm{PV}\displaystyle\int_0^x \frac{dt}{\ln t}, & \text{for } x>1,\\ \mathrm{PV}\displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } x<-1\end{cases}\\ &= \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x|<1,\\ \lim_{\epsilon\to 0^+}\left[\displaystyle\int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^x \frac{dt}{\ln t}\right], & \text{for } x>1,\\ \lim_{\epsilon\to 0^+}\left[\displaystyle\int_0^{-1+\epsilon} \frac{dt}{\ln|t|} + \int_{-1-\epsilon}^x \frac{dt}{\ln|t|}\right], & \text{for } x<-1\end{cases}\end{aligned}$$

where PV denotes the Cauchy principal value of the integral. We will therefore write

$$\mathrm{li}(x) = \mathrm{PV}\int_0^x \frac{dt}{\ln|t|}$$

*Department of Mathematics, University of Leicester, Leicester, LE1 7RH, England. E-mail: fbr@le.ac.uk

†Corresponding Author

‡Faculty of Electrical Engineering and Informational Technologies, Karpos II bb, Skopje, Republic of Macedonia. E-mail: biljanaj@feit.ukim.edu.mk

for all values of x.

The associated functions $\mathrm{li}_+(x)$ and $\mathrm{li}_-(x)$ are now defined by

$$\mathrm{li}_+(x) = H(x)\,\mathrm{li}(x),\ \ \mathrm{li}_-(x) = H(-x)\,\mathrm{li}(x),$$

where $H(x)$ denotes Heaviside's function.

The distribution $\ln^{-1}|x|$ is defined by

$$\ln^{-1}|x| = \mathrm{li}'(x)$$

and its associated distributions $\ln^{-1} x_+$ and $\ln^{-1} x_-$ are defined by

$$\ln^{-1} x_+ = H(x)\ln^{-1}|x| = \mathrm{li}'_+(x),\ \ \ln^{-1} x_- = H(-x)\ln^{-1}|x| = \mathrm{li}'_-(x).$$

The classical definition of the convolution of two functions f and g is as follows:

1.1. Definition. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists, and

$$f * g = g * f, \tag{1.1}$$

and if $(f*g)'$ and $f*g'$ (or $f'*g$) exists, then

$$(f*g)' = f*g' \quad (\text{or } f'*g). \tag{1.2}$$

Definition 1.1 can be extended to define the convolution $f*g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [6].

1.2. Definition. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f*g$ is defined by the equation

$$\langle (f*g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support, or
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f*g$ exists by this definition then equations (1.1) and (1.2) are satisfied.

The above definition of the convolution is rather restrictive and so a neutrix convolution was defined in [3]. In order to define the neutrix convolution, we first of all let τ be a function in $\mathcal{D}$, see [7], satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

The following definition of the non-commutative neutrix convolution was given in [3].

1.3. Definition. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *non-commutative neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}_{n\in\mathbb{N}}$, provided the limit h exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [2], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n : \lambda > 0,\ r = 1, 2, \ldots,$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It is easily seen that any results proved with the original Definitions 1.1 and 1.2 of the convolution hold with Definition 1.3 of the neutrix convolution. The following results proved in [3] hold, first showing that the neutrix convolution is a generalization of the convolution.

1.4. Theorem. *Let f and g be distributions in $\mathcal{D}'$, satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g. \qquad \square$$

1.5. Theorem. *Let f and g be distributions in $\mathcal{D}'$, and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

If $\operatorname*{N-lim}_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ *exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathcal{D}$, then $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h. \qquad \square$$

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s \operatorname{li}(n^r)$ and $n^s \ln^{-r} n$, $(n > 1)$ for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$.

2. Main Results

Before proving our main results, we need the following lemmas.

2.1. Lemma.

$$\operatorname{li}(x^r) = \mathrm{PV}\int_0^x \frac{t^{r-1}dt}{\ln|t|}. \tag{2.1}$$

Proof. Making the substitution $t = u^r$, we have

$$\operatorname{li}(x^r) = \mathrm{PV}\int_0^{x^r} \frac{dt}{\ln|t|} = \mathrm{PV}\int_0^x \frac{u^{r-1}du}{\ln|u|},$$

proving Equation (2.1). $\square$

2.2. Lemma.

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} \tau_n(t)\operatorname{li}(t)(x-t)^r\,dt = 0 \tag{2.2}$$

for $r = 1, 2, \ldots$.

Proof. We note that $\ln(t) > 1$ when $t > e$ and so

$$\operatorname{li}(t) = \operatorname{li}(e) + \int_e^t \frac{du}{\ln u} < \operatorname{li}(e) + t - e.$$

Thus, when $|x| < n - 1$, we have

$$\begin{aligned}\int_n^{n+n^{-n}} \tau_n(t) \operatorname{li}(t)(x-t)^r \, dt &< \int_n^{n+n^{-n}} \operatorname{li}(t)(x-t)^r \, dt \\ &< (2n + n^{-r} - 1)^r \int_n^{n+n^{-n}} \operatorname{li}(t) \, dt \\ &< 4^r n^r n^{-n} [\operatorname{li}(e) - e + 2n + n^{-n}],\end{aligned}$$

and Equation (2.2) follows. □

2.3. Lemma.

$$\operatorname*{N-lim}_{n\to\infty} \operatorname{li}[(x+n)^r] = 0, \tag{2.3}$$

$$\operatorname*{N-lim}_{n\to\infty} n^r \operatorname{li}[(x+n)] = 0, \tag{2.4}$$

for $r = 1, 2, \ldots$.

Proof. With $x > 1$, and putting $f(x) = \operatorname{li}(x^r)$, we have

$$f'(x) = \frac{rx^{r-1}}{\ln x}.$$

It follows that $f^{(k)}(x)$ is of the form

$$f^{(k)}(x) = \sum_{j=1}^{k} \frac{\alpha_{kj}}{x^{k-r} \ln^j x} \tag{2.5}$$

and

$$f^{(r+1)}(n+c) = O(n^{-1}), \ (c \geq 0). \tag{2.6}$$

By Taylor's Theorem, we have

$$\begin{aligned}f(x+n) &= \sum_{k=0}^{r} \frac{x^k}{k!} f^{(k)}(n) + \frac{x^{r+1}}{(r+1)!} f^{(r+1)}(n+\xi x) \\ &= \sum_{k=0}^{r} \sum_{j=1}^{k} \frac{\alpha_{kj}}{n^{k-r} \ln^j n} \frac{x^k}{k!} + O(n^{-1})\end{aligned}$$

and Equation (2.3) follows from Equations (2.5) and (2.6).

Equation (2.4) follows similarly. □

We now prove a number of results involving the convolution. First of all we have

2.4. Theorem. *The convolutions* $\operatorname{li}_+(x) * x_+^r$ *and* $\ln^{-1} x_+ * x_+^r$ *exist, and*

$$\operatorname{li}_+(x) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i \operatorname{li}_+(x^{r-i+2}), \tag{2.7}$$

$$\ln^{-1} x_+ * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i \operatorname{li}_+(x^{r-i+1}) \tag{2.8}$$

for $r = 0, 1, 2, \ldots$.

Proof. It is obvious that $\mathrm{li}_+(x) * x_+^r = 0$ if $x < 0$.

When $x > 0$, we have

$$\begin{aligned}
\mathrm{li}_+(x) * x_+^r &= \mathrm{PV}\int_0^x (x-t)^r \int_0^t \frac{du}{\ln u}\,dt \\
&= \mathrm{PV}\int_0^x \frac{1}{\ln u}\int_u^x (x-t)^r\,dt\,du \\
&= \mathrm{PV}\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1}x^i\binom{r+1}{i}\int_0^x \frac{u^{r-i+1}}{\ln u}\,du \\
&= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i\,\mathrm{li}_+(x^{r-i+2}),
\end{aligned}$$

on using Equation (2.1), and Equation (2.7) is proved.

Now, using Equation (1.2) and (2.7), we get

$$\begin{aligned}
\ln^{-1} x_+ * x_+^r &= r\,\mathrm{li}_+(x) * x_+^{r-1} \\
&= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}x^i\,\mathrm{li}_+(x^{r-i+1}),
\end{aligned}$$

proving Equation (2.8). □

2.5. Corollary. *The convolutions* $\mathrm{li}_-(x) * x_-^r$ *and* $\ln^{-1} x_- * x_-^r$ *exist, and*

$$\mathrm{li}_-(x) * x_-^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+2}x^i\,\mathrm{li}_-(x^{r-i+2}), \tag{2.9}$$

$$\ln^{-1} x_- * x_-^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i+1}x^i\,\mathrm{li}_-(x^{r-i+1}) \tag{2.10}$$

for $r = 0, 1, 2, \ldots$.

Proof. Equations (2.9) and (2.10) are obtained applying a similar procedure as used in obtaining equations (2.7) and (2.8). □

2.6. Theorem. *The neutrix convolutions* $\mathrm{li}_+(x) \circledast x^r$ *and* $\ln^{-1} x_+ \circledast x^r$ *exist, and*

$$\mathrm{li}_+(x) \circledast x^r = 0, \tag{2.11}$$

$$\ln^{-1} x_+ \circledast x^r = 0 \tag{2.12}$$

for $r = 0, 1, 2, \ldots$.

Proof. We put $[\mathrm{li}_+(x)]_n = \mathrm{li}_+(x)\tau_n(x)$. Then the convolution $[\mathrm{li}_+(x)]_n * x^r$ exists, and

$$[\mathrm{li}_+(x)]_n * x^r = \int_0^n \mathrm{li}(t)(x-t)^r\,dt + \int_n^{n+n^{-n}} \tau_n(t)\,\mathrm{li}(t)(x-t)^r\,dt, \tag{2.13}$$

where

$$\begin{aligned}\int_0^n \mathrm{li}(t)(x-t)^r\,dt &= \mathrm{PV}\int_0^n (x-t)^r \int_0^t \frac{du}{\ln u}\,dt \\ &= \mathrm{PV}\int_0^n \frac{1}{\ln u}\int_u^n (x-t)^r\,dt\,du \\ &= \mathrm{PV}\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1}x^i\binom{r+1}{i}\int_0^n \frac{u^{r-i+1}-n^{r-i+1}}{\ln u}\,du \\ &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i\big[\,\mathrm{li}(n^{r-i+2})-n^{r-i+1}\,\mathrm{li}(n)\big].\end{aligned}$$

Thus from Lemma 2.3 we have

$$\text{(2.14)}\qquad \mathrm{N}\!-\!\lim_{n\to\infty}\int_0^n \mathrm{li}(t)(x-t)^r\,dt = 0.$$

Equation (2.11) now follows using Lemma 2.2, Equations (2.13) and (2.14).

Differentiating Equation (2.11) and using Theorem 1.5 we get

$$\text{(2.15)}\qquad \ln^{-1}x_+ \circledast x^r = \mathrm{N}\!-\!\lim_{n\to\infty}[\mathrm{li}_+(x)\tau_n'(x)] * x^r,$$

where on integration by parts we have

$$\begin{aligned}[\mathrm{li}_+(x)\tau_n'(x)] * x^r &= \int_n^{n+n^{-n}} \tau_n'(t)\,\mathrm{li}(t)(x-t)^r\,dt \\ &= -\,\mathrm{li}(n)(x-n)^r - \int_n^{n+n^{-n}} \ln^{-1}(t)(x-t)^r\tau_n(t)\,dt \\ &\quad + r\int_n^{n+n^{-n}} \mathrm{li}(t)(x-t)^{r-1}\tau_n(t)\,dt.\end{aligned}\qquad\text{(2.16)}$$

It is clear that

$$\text{(2.17)}\qquad \lim_{n\to\infty}\int_n^{n+n^{-n}} \ln^{-1}(t)(x-t)^r\tau_n(t)\,dt = 0,$$

so Equation (2.12) follows from Lemma 2.2 and Equations (2.15), (2.16) and (2.17). □

2.7. Corollary. *The neutrix convolutions* $\mathrm{li}_-(x) \circledast x^r$ *and* $\ln^{-1} x_- \circledast x^r$ *exist, and*

$$\text{(2.18)}\qquad \mathrm{li}_-(x) \circledast x^r = 0,$$

$$\text{(2.19)}\qquad \ln^{-1} x_- \circledast x^r = 0$$

for $r = 0, 1, 2, \ldots$.

Proof. Equations (2.18) and (2.19) are obtained applying a similar procedure as in the case of Equations (2.11) and (2.12). □

2.8. Corollary. *The neutrix convolutions* $\mathrm{li}(x) \circledast x^r$ *and* $\ln^{-1}|x| \circledast x^r$ *exist, and*

$$\text{(2.20)}\qquad \mathrm{li}(x) \circledast x^r = 0,$$

$$\text{(2.21)}\qquad \ln^{-1}|x| \circledast x^r = 0$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (2.20) follows on adding Equations (2.18) and (2.11), and Equation (2.21) follows on adding Equations (2.12) and (2.19). □

2.9. Corollary. *The neutrix convolutions* $\mathrm{li}_+(x) \circledast x_-^r$, $\mathrm{li}_-(x) \circledast x_+^r$, $\ln^{-1} x_+ \circledast x_-^r$ *and* $\ln^{-1} x_- \circledast x_+^r$ *exist, and*

$$\mathrm{li}_+(x) \circledast x_-^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i x^i \, \mathrm{li}_+(x^{r-i+2}), \tag{2.22}$$

$$\mathrm{li}_-(x) \circledast x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{i+1} x^i \, \mathrm{li}_-(x^{r-i+2}), \tag{2.23}$$

$$\ln^{-1} x_+ \circledast x_-^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^i \, \mathrm{li}_+(x^{r-i+1}), \tag{2.24}$$

$$\ln^{-1} x_- \circledast x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+1} x^i \, \mathrm{li}_-(x^{r-i+1}), \tag{2.25}$$

for $r = 0, 1, 2, \ldots$.

Proof. Using that $x^r = x_+^r + (-1)^r x_-^r$, and the fact that the neutrix convolution product is distributive with respect to addition, we have

$$\mathrm{li}_+(x) \circledast x^r = \mathrm{li}_+(x) * x_+^r + (-1)^r \, \mathrm{li}_+(x) \circledast x_-^r.$$

Equation (2.22) follows from Equations (2.7) and (2.11). Equation (2.23) is obtained by applying a similar procedure as in the case of Equation (2.22).

Equation (2.24) follows from Equations (2.8) and (2.12), and Equation (2.25) is obtained by applying a similar procedure as in the case of Equation (2.24). □

2.10. Theorem. *The neutrix convolutions* $x^r \circledast \mathrm{li}_+(x)$ *and* $x^r \circledast \ln^{-1} x_+$ *exist, and*

$$x^r \circledast \mathrm{li}_+(x) = 0, \tag{2.26}$$

$$x^r \circledast \ln^{-1} x_+ = 0 \tag{2.27}$$

for $r = 0, 1, 2, \ldots$.

Proof. We put $(x^r)_n = x^r \tau_n(x)$ for $r = 0, 1, 2, \ldots$. Then the convolution $(x^r)_n * \mathrm{li}_+(x)$ exists, and

$$(x^r)_n * \mathrm{li}_+(x) = \int_0^{x+n} \mathrm{li}(t)(x-t)^r \, dt + \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t) \, \mathrm{li}(t)(x-t)^r \, dt, \tag{2.28}$$

where

$$\begin{aligned}
\int_0^{x+n} \mathrm{li}(t)(x-t)^r \, dt &= PV \int_0^{x+n} (x-t)^r \int_0^t \frac{du}{\ln u} \, dt \\
&= \mathrm{PV} \int_0^{x+n} \frac{1}{\ln u} \int_u^{x+n} (x-t)^r \, dt \, du \\
&= \mathrm{PV} \frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i+1} x^i \binom{r+1}{i} \int_0^{x+n} \frac{u^{r-i+1}}{\ln u} \, du \\
&\qquad - \mathrm{PV} \frac{(-n)^{r+1}}{r+1} \int_0^{x+n} \frac{du}{\ln u} \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i \, \mathrm{li}\left[(x+n)^{r-i+2}\right] \\
&\qquad - \frac{(-n)^{r+1}}{r+1} \, \mathrm{li}(x+n).
\end{aligned}$$

Thus, on using Lemma 2.3, we have

$$\text{(2.29)} \quad \operatorname*{N-lim}_{n\to\infty} \int_0^{x+n} \operatorname{li}(t)(x-t)^r\,dt = 0.$$

Further, using Lemma 2.2 it is easily seen that

$$\text{(2.30)} \quad \lim_{n\to\infty} \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t)\operatorname{li}(t)(x-t)^r\,dt = 0,$$

and Equation (2.26) follows from Equations (2.28), (2.29) and (2.30).

Differentiating Equation (2.26) gives Equation (2.27). □

2.11. Corollary. *The neutrix convolutions $x^r \circledast \operatorname{li}_-(x)$ and $x^r \circledast \ln^{-1} x_-$ exist, and*

$$\text{(2.31)} \quad x^r \circledast \operatorname{li}_-(x) = 0,$$

$$\text{(2.32)} \quad x^r \circledast \ln^{-1} x_- = 0$$

for $r = 0, 1, 2, \ldots$.

Proof. Equations (2.31) and (2.32) are obtained by applying a similar procedure as for Equations (2.26) and (2.27). □

2.12. Corollary. *The neutrix convolutions $x^r \circledast \operatorname{li}(x)$ and $x^r \circledast \ln^{-1}|x|$ exist, and*

$$\text{(2.33)} \quad x^r \circledast \operatorname{li}(x) = 0,$$

$$\text{(2.34)} \quad x^r \circledast \ln^{-1}|x| = 0$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (2.33) follows on adding Equations (2.31) and (2.26), and Equation (2.34) follows on adding Equations (2.27) and (2.32). □

2.13. Corollary. *The neutrix convolutions $x_-^r \circledast \operatorname{li}_+(x)$, $x_+^r \circledast \operatorname{li}_-(x)$, $x_-^r \circledast \ln^{-1} x_+$ and $x_+^r \circledast \ln^{-1} x_-$ exist, and*

$$\text{(2.35)} \quad x_-^r \circledast \operatorname{li}_+(x) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^i \operatorname{li}_+(x^{r-i+2}),$$

$$\text{(2.36)} \quad x_+^r \circledast \operatorname{li}_-(x) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{i+1} x^i \operatorname{li}_-(x^{r-i+2}),$$

$$\text{(2.37)} \quad x_-^r \circledast \ln^{-1} x_+ = \frac{1}{r+1}\sum_{i=0}^{r}\binom{r}{i}(-1)^i x^i \operatorname{li}_+(x^{r-i+1}),$$

$$\text{(2.38)} \quad x_+^r \circledast \ln^{-1} x_- = \frac{1}{r+1}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1} x^i \operatorname{li}_-(x^{r-i+1})$$

for $r = 0, 1, 2, \ldots$.

Proof. Equation (2.35) follows from Equations (2.7) and (2.26) on noting that

$$x^r \circledast \operatorname{li}_+(x) = x_+^r * \operatorname{li}_+(x) + (-1)^r x_-^r \circledast \operatorname{li}_+(x).$$

Equation (2.36) is obtained by arguing as in the case of Equation (2.35). Equation (2.37) follows from Equations (2.8) and (2.27).

Equation (2.38) is obtained by arguing as in case of Equation (2.37). □

For further results involving the convolution the reader is referred to [4] and [5].

Acknowledgement. The authors would like to thank the referee for his help in improving this paper.

References

[1] Abramowitz, M. and Stegun, I. A. (Eds). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 9th printing* (New York: Dover, p. 879, 1972).

[2] van der Corput, J. G. *Introduction to the neutrix calculus*, J. Analyse Math. **7**, 291–398, 1959-60.

[3] Fisher, B. *Neutrices and the convolution of distributions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **17**, 119–135, 1987.

[4] Fisher, B., Jolevska-Tuneska, B. and Takači, A. *On convolutions and neutrix convolutions involving the incomplete gamma function*, Integral Transforms and Special Functions **15** (5), 404–415, 2004.

[5] Jolevska-Tuneska, B. and Takači, A. *Results on the commutative neutrix convolution product of distributions*, Hacettepe J. Math. Stat. **37** (2), 135–141, 2008.

[6] Gel'fand, I. M. and Shilov, G. E. *Generalized functions, Vol. I* (Academic Press Chap. 1, 1964).

[7] Jones, D. S. *The convolution of generalized functions*, Quart. J. Math. Oxford **24** (2), 145–163, 1973.