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AN APPROACH TO NUMERICAL SEMIGROUPS

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Abstract

In this paper, we give some results on principal ideals of a numerical semigroup $S = \langle s_1, s_2, \ldots, s_p \rangle$ for $p \geq 2$, $p \in \mathbb{N}$. We also describe some relations between Apery subsets and ideals of S.

Keywords: Numerical semigroup, Apery set, Ideal, Gap.

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1. Introduction

A *numerical semigroup* S is a subset of $\mathbb{N}$ (the set of nonnegative integers) closed under addition, satisfying $0 \in S$ and for which $\mathbb{N} \setminus S$ has finitely many elements. For a numerical semigroup S, $A = \{s_1, s_2, \ldots, s_p\} \subset S$ is a *generating set* of S provided that $S = \langle s_1, s_2, \ldots, s_p \rangle = \{s_1k_1 + s_2k_2 + \cdots + s_pk_p : k_i \in \mathbb{N},\ 1 \leq i \leq p\}$. The set $A = \{s_1, s_2, \ldots, s_p\}$ is called a *minimal generating set* of if no proper subset is a generating set of S. It was observed in [1] that the set $\mathbb{N} \setminus S$ is finite if and only if $\gcd\{s_1, s_2, \ldots, s_p\} = 1$ (gcd stands for the greatest common divisor).

Another important invariant of S is the largest integer not belonging to S, known as the *Frobenius number* of S and denoted by $g(S)$, that is $g(S) = \max\{x \in \mathbb{Z} : x \notin S\}$ (see [6, 1]). We define

$$n(S) = \sharp(\{0, 1, \ldots, g(S)\} \cap S)$$

where $\sharp(A)$ denotes the cardinality of A. It is also well-known that

$$S = \{0, s_1, s_2, \ldots, s_{n-1}, s_n = g(S) + 1, \rightarrow \ldots\}$$

where $\rightarrow$ means that every integer greater than $g(S) + 1$ belongs to S, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1, 2, \ldots, n$.

For $m \in S \setminus \{0\}$, the Apery set of m in S is the set $\mathrm{Ap}(S, m) = \{s \in S : s - m \notin S\}$. It can easily be proved that $\mathrm{Ap}(S, m)$ is formed by the smallest elements of S belonging to the different congruence classes mod m. According to this, we have $\sharp(\mathrm{Ap}(S, m)) = m$

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and $g(S) = \max(\mathrm{Ap}\,(S,m)) - m$. Various aspects and properties of Apery sets are given in [2, 3].

The elements of $\mathbb{N} \setminus S$, denoted by $H(S)$, are called the *gaps* of S (see [6]). A subset I of S is an *ideal* if $I + S \subseteq I$ (that is, for all $x \in I$ and $s \in S$, the element $x + s$ is in I). An ideal I of S is *generated* by $A \subset S$ if $I = A + S$. We also say that the ideal I is *finitely generated* if there exists a finite $A \subseteq S$ such that $I = A + S$.

We say that I is *principal* if it can be generated by a single element. That is, there exists $x_0 \in S$ such that $I = \{x_0\} + S = \{x_0 + s : s \in S\}$. We usually write $I = [x_0]$ instead of $I = \{x_0\} + S$ (see [5]). The elements of $H(I) = S \setminus I$ are called the *gaps* of I. If I and J are ideals of S, we define their *ideal sum* by $I + J = \{i + j : i \in I,\ j \in J\}$ (see [1]).

The contents of this study are organized as follows. In section 2, we give some results concerning the sum, union and intersection of principal ideals of S. In particular, the main goal of this section is to prove Theorem 2.5. Furthermore, the aim of Section 3 is to give some relations between the Apery subsets and the principal ideals of S.

Throughout this paper, we will assume the numerical semigroup S satisfies

$$S = \langle s_1, s_2, \ldots, s_p\rangle = \{s_1k_1 + s_2k_2 + \cdots + s_pk_p : k_i \in \mathbb{N},\ 1 \le i \le p\},$$

and that its principal ideals are I_i, for $i = 1, 2, \ldots, p$ ($p \ge 2$, $p \in \mathbb{N}$), respectively.

2. Some results for principal ideals of numerical semigroups

In this section, we give some results concerning the sum, union and intersection of principal ideals of a numerical semigroup S. In particular, we obtain elements belonging to the intersection of the principal ideals of S which are not in the sum of the principal ideals of S.

2.1. Lemma. $\sum\limits_{i=1}^{p} I_i \subset I_i$, *where* $\sum\limits_{i=1}^{p} I_i = \Big[\sum\limits_{i=1}^{p} s_i\Big]$ *and* $s_i \in S$.

Proof. If $x \in \sum\limits_{i=1}^{p} I_i = \Big[\sum\limits_{i=1}^{p} s_i\Big]$, then there exists $s \in S$ such that $x = \sum\limits_{i=1}^{p} s_i + s$. Thus, we find $x \in S$. Therefore, we get $\sum\limits_{i=1}^{p} I_i \subset S \implies \sum\limits_{\substack{i=1,\\ k\neq i}}^{p} I_i + I_k \subset S + I_k \subset I_k,\ 1 \le k \le p.$ □

We obtain the following result from Lemma 2.1.

2.2. Corollary. $\sum\limits_{i=1}^{p} I_i \subset \bigcap\limits_{i=1}^{p} I_i$.

2.3. Lemma. *Let S and I_i be a numerical semigroup and principal ideals of S, respectively. Then $s_p \notin I_i$ for $i = 1, 2, \ldots, p-1$.*

Proof. If $s_p \in I_i$, then there exists $s \in S$ such that $s_i + s = s_p$ for $i = 1, 2, \ldots, p-1$. Thus it follows that $S = \langle s_1, s_2, \ldots, s_{p-1}\rangle$, which is a contradiction since $A = \{s_1, s_2, \ldots, s_p\}$ is a minimal generating set of S. □

2.4. Lemma. *Let S and I_i be a numerical semigroup and principal ideals of S, respectively. Then* $\bigcup\limits_{i=1}^{p-1} I_i \subseteq S \setminus \{0, s_p\}$.

Proof. If $x \in \bigcup\limits_{i=1}^{p-1} I_i$, then it follows that $x \neq s_p$ and $x \neq 0$ from definition of principal ideal of S, and Lemma 2.3. □

2.5. Theorem. *Let S be a numerical semigroup, I_i and $g(S)$ be its principal ideals and Frobenius number, respectively. Then*

$$g(S) + \sum_{i=1}^{p} s_i \in \bigcap_{i=1}^{p} I_i \setminus \sum_{i=1}^{p} I_i.$$

Proof. Firstly, we show that $g(S) + \sum_{i=1}^{p} s_i \in \bigcap_{i=1}^{p} I_i$:

$$g(S) + \sum_{i=1}^{p} s_i = s_1 + (g(S) + s_2 + s_3 + \cdots + s_p) \in I_1,$$
$$\text{since } (g(S) + s_2 + s_3 + \cdots + s_p) \in S,$$
$$g(S) + \sum_{i=1}^{p} s_i = s_2 + (g(S) + s_1 + s_3 + \cdots + s_p) \in I_2,$$
$$\text{since } (g(S) + s_1 + s_3 + \cdots + s_p) \in S,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$g(S) + \sum_{i=1}^{p} s_i = s_p + (g(S) + s_1 + s_2 + \cdots + s_{p-1}) \in I_p,$$
$$\text{since } (g(S) + s_1 + s_2 + \cdots + s_{p-1}) \in S.$$

Now, we must show that $g(S)+ \sum_{i=1}^{p} s_i \notin \sum_{i=1}^{p} I_i$. Suppose on the contrary that $g(S) + \sum_{i=1}^{p} s_i \in \sum_{i=1}^{p} I_i$. Then, there exists $s \in S$ such that $g(S)+ \sum_{i=1}^{p} s_i = s_1+s_2+s_3+\cdots+s_p+s$. Thus, we get $g(S) = s \in S$, which is a contradiction. □

2.6. Example. Let us consider the numerical semigroup S given by $S = \langle 4, 7, 9\rangle = \{0, 4, 7, 8, 9, 11, \to \ldots\}$. The Frobenius number of S is $g(S) = 10$. Then the principal ideals of S are described by:

$$I = [4] = 4 + S = \{4, 8, 11, 12, 13, 15, \to \ldots\},$$
$$J = [7] = 7 + S = \{7, 11, 14, 15, 16, 18, \to \ldots\}, \text{ and}$$
$$K = [9] = 9 + S = \{9, 13, 16, 17, 18, 20, \to \ldots\}.$$

In this case, we find that

$$I + J + K = [20] = \{20, 24, 27, 28, 29, 31, \to \ldots\} \subset I, J, K,$$
$$I \cup J = \{4, 7, 8, 11, \to \ldots\} \subseteq S \setminus \{0, 9\},$$
$$I \cap J \cap K = \{16, 18, 20, 21, \to \ldots\} \supset I + J + K,$$

and $g(S) + s_1 + s_2 + s_3 = 10 + 4 + 7 + 9 = 30 \in I \cap J \cap K$ but 30 not in $[4 + 7 + 9]$.

3. The relation between principal ideals and Apery sets

In this section, we obtain some relations between the principal ideals $I_i = \left[s_i\right]$ and the Apery sets $\mathrm{Ap}\,(S, s_i) = \{s \in S : s - s_i \notin S\}$ for $1 \le i \le p$.

3.1. Lemma. $\mathrm{Ap}\,(S, s_i) \subseteq I_i^c$ *for each i, $1 \le i \le p$.*

Proof. We must show that $I_i \cap \mathrm{Ap}\,(S, s_i) = \emptyset$. Suppose that $I_i \cap \mathrm{Ap}\,(S, s_i) \neq \emptyset$ for some $i \in \{1, 2, \ldots, p\}$. If $x \in \left[s_i\right] \cap \mathrm{Ap}\,(S, s_i)$, then $x = s_i + s$ for some $s \in S$, and $x - s_i \notin S$ for this i. But this is contradiction since $x \in S$. □

The following is a consequence of Lemma 3.1.

3.2. Corollary. *For each* $i \in \{1, 2, \ldots, p\}$ *the family* $\{[s_i],\ \mathrm{Ap}\,(S, s_i)\}$ *is a partition of* S.

Proof. Take $i \in \{1, 2, \ldots, p\}$. According to lemma 3.1, it is sufficient to show that $S = [s_i] \cup \mathrm{Ap}\,(S, s_i)$. It is clear that $[s_i] \cup \mathrm{Ap}\,(S, s_i) \subseteq S$, so take $x \in S$ with $x \notin [s_i]$. Then we have $x - s_i \notin S$, so $x \in \mathrm{Ap}\,(S, s_i)$ which gives the required result. □

3.3. Lemma. $\sum\limits_{i=1}^{p} s_i \notin \mathrm{Ap}\,(S, s_i)$ *for each* $i \in \{1, 2, \ldots, p\}$.

Proof. The result follows from the fact that $(\sum\limits_{i=1}^{p} s_i) - s_j = s_1 + s_2 + s_3 + \cdots + s_{j-1} + s_{j+1} \in S$ for each $i \in \{1, 2, \ldots, p\}$ and $2 \leq j \leq p - 1$. □

3.4. Lemma. $\mathrm{Ap}\,(S, s_i) \subset \mathrm{Ap}\,(S, \sum\limits_{i=1}^{p} s_i)$ *for each* $i \in \{1, 2, \ldots, p\}$.

Proof. For each $i \in \{1, 2, \ldots, p\}$, if $x \notin \mathrm{Ap}\,(S, \sum\limits_{i=1}^{p} s_i)$, then $x - s_1 - s_2 - \ldots - s_p \in S$. It follows that $x - s_i \in S$, and hence $x \notin \mathrm{Ap}\,(S, s_i)$. □

3.5. Lemma. $\mathrm{Ap}\,(S, s_i) = H(I_i)$ *for each* $i \in \{1, 2, \ldots, p\}$.

Proof. The result follows from the following observation: for each $i \in \{1, 2, \ldots, p\}$,

$$\begin{aligned} x \in \mathrm{Ap}\,(S, s_i) &\iff x - s_i \notin S \iff \forall\, s \in S,\ s \neq x - s_i \\ &\iff x \neq s_i + s \iff x \notin I_i \iff x \in H(I_i). \end{aligned}$$

□

The following result is a consequence of Lemma 3.5.

3.6. Corollary. $S \setminus \Big(\sum\limits_{i=1}^{p} I_i\Big) = \mathrm{Ap}\,\Big(S,\ \sum\limits_{i=1}^{p} s_i\Big)$. □

3.7. Lemma. $\bigcup\limits_{i=1}^{p} H\big(I_i\big) \subseteq H\Big(\sum\limits_{i=1}^{p} I_i\Big)$.

Proof. From Lemma 2.1 we have $\sum\limits_{i=1}^{p} I_i \subseteq I_i$, and so $H(I_i) \subseteq H\Big(\sum\limits_{i=1}^{p} I_i\Big)$ for each $i \in \{1, 2, \ldots, p\}$. Thus, we obtain $\bigcup\limits_{i=1}^{p} H\big(I_i\big) \subseteq H\Big(\sum\limits_{i=1}^{p} I_i\Big)$ □

3.8. Example. Let us consider a numerical semigroup S given by $S = \langle 5, 7, 9, 11, 13 \rangle = \{0, 5, 7, 9, \rightarrow \ldots\}$. The Frobenius number of S is $g(S) = 8$. The principal ideals I_i of S (for $i = 1, 2, 3, 4, 5$) are respectively;

$$\begin{aligned} &I_1 = [5] = \{5, 10, 12, 14, \rightarrow \ldots\}, \\ &I_2 = [7] = \{7, 12, 14, 16, \rightarrow \ldots\}, \\ &I_3 = [9] = \{9, 14, 16, 18, \rightarrow \ldots\}, \\ &I_4 = [11] = \{11, 16, 18, 20, \rightarrow \ldots\}, \text{ and,} \\ &I_5 = [13] = \{13, 18, 20, 22, \rightarrow \ldots\}. \end{aligned}$$

Now, the subsets $\mathrm{Ap}\,(S,\ s_i)$ of S (for $i = 1, 2, 3, 4, 5$) are respectively;

$$\begin{aligned}
\mathrm{Ap}\,(S, 5) &= \{s \in S : s - 5 \notin S\} = \{0, 7, 9, 11, 13\} \\
&= H(I_1), \\
\mathrm{Ap}\,(S, 7) &= \{s \in S : s - 7 \notin S\} = \{0, 5, 9, 10, 11, 13, 15\} \\
&= H(I_2), \\
\mathrm{Ap}\,(S, 9) &= \{s \in S : s - 9 \notin S\} = \{0, 5, 7, 10, 11, 12, 13, 15, 17\} \\
&= H(I_3), \\
\mathrm{Ap}\,(S, 11) &= \{s \in S : s - 11 \notin S\} = \{0, 5, 7, 9, 10, 12, 13, 14, 15, 17, 19\} \\
&= H(I_4),\ \text{and}, \\
\mathrm{Ap}\,(S, 13) &= \{s \in S : s - 13 \notin S\} = \{0, 5, 7, 9, 10, 11, 12, 14, 15, 16, 17, 19, 21\} \\
&= H(I_5).
\end{aligned}$$

From Corollary 3.2 , we can write

$$S = \big[s_i\big] \cup \mathrm{Ap}\,(S,\ s_i),\ \ \big[s_i\big] \cap \mathrm{Ap}\,(S,\ s_i) = \emptyset,\ \sum_{i=1}^{5} s_i = 45 \notin \mathrm{Ap}\,(S, s_i),$$

and

$$\mathrm{Ap}\,(S, s_i) \subset \mathrm{Ap}\,(S\ , 45),\ \text{ for } i = 1, 2, 3, 4, 5.$$

On the other hand, we have $S \setminus \sum_{i=1}^{5} I_i = \mathrm{Ap}\,(S,\ 45)$ and $\bigcup_{i=1}^{5} H(I_i) \subset H([45])$.

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