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NUMERICAL SOLUTION OF A QUASILINEAR PARABOLIC PROBLEM WITH PERIODIC BOUNDARY CONDITION

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Abstract

In this paper we study the one dimensional mixed problem, with Neumann and Dirichlet type periodic boundary conditions, for the quasilinear parabolic equation $\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x, u)$. We construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as on test examples. We demonstrate a numerical procedure for this problem on concrete examples, and also we obtain numerical solution by using the implicit finite difference algorithm.

Keywords: Quasilinear parabolic equation, Periodic boundary condition, Generalized solutions, Iteration algorithm.

2000 AMS Classification: 39 K 05.

1. Introduction

The heat conduction equation of an isotropic material is given in the following form

$$c\frac{\partial u}{\partial t} = k\Delta u + \varepsilon c\frac{\partial}{\partial t}\Delta u + q(t, x), \tag{1}$$

where $u = u(t, x)$ denotes the temperature of the material at the point x and the moment t. Moreover, the heat source is denoted by $q = q(t, x)$. In addition, the symbols k and ε denoted heat conduction and a small parameter, respectively [12, 11, 3].

Since the heat source q depends on $u = u(t, x)$, equation (1) turns into the following quasilinear pseudo-parabolic equation:

$$c\frac{\partial u}{\partial t} = k\Delta u + \varepsilon c\frac{\partial}{\partial t}\Delta u + q(t, x, u). \tag{2}$$

where the source function q is any continuous function.

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For $\varepsilon = 0$ and $\varepsilon \neq 0$, these equations with different boundary conditions were investigated by various researchers by using Fourier or different methods [1, 2, 5, 6, 7, 8]. In [8], the author proved the existence and uniqueness of the generalized solution of the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x,u),\ (t,x) \in D := \{0 < x < \pi,\ 0 < t < T\} \tag{3}$$

$$u(t,0) = u(t,\pi) = 0,\ t \in [0,T] \tag{4}$$

$$u(0,x) = \varphi(x),\ x \in [0,\pi]\,. \tag{5}$$

In technical applications, the following boundary conditions

$$u(t,0) = u(t,\pi),\ t \in [0,T]\,, \quad u_x(t,0) = u_x(t,\pi),\ t \in [0,T]$$

are encountered very often. Moreover, from a technical point of view, these boundary conditions are more difficult.

In [4], the authors Ciftci and Halilov investigated quasilinear parabolic equations with periodic boundary conditions by using Fourier series with variable coefficients. They proved the existence, uniqueness and convergence of the weak generalized solution of the following mixed problem for a quasilinear parabolic equation with the given source term $f = f(t,x,u)$:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x,u),\ (t,x) \in D := \{0 < x < \pi,\ 0 < t < T\} \tag{6}$$

$$u(t,0) = u(t,\pi),\ t \in [0,T] \tag{7}$$

$$u_x(t,0) = u_x(t,\pi),\ t \in [0,T] \tag{8}$$

$$u(0,x) = \varphi(x),\ x \in [0,\pi]\,, \tag{9}$$

where $\varphi(x)$ and $f(t,x,u)$ are given functions on $[0,\pi]$ and $\bar{D} \times (-\infty,\infty)$, respectively.

In this study, we construct an iteration algorithm for the solution of the problem (6)-(9). We analyze computationally the convergence of the iteration algorithm, as well as on test examples. We demonstrate the numerical procedure for the problem (6)-(9) on concrete examples and finally, obtain a numerical solution by using the implicit finite difference algorithm.

2. Numerical Algorithm for the nonlinear problem (6)–(9)

There are some competing numerical methods for solving problem of this kind, which known by the abbreviations IMEX (implicit-explicit), SS (split step), IF (integrating factor), SL (sliders), and ETD (exponential time-differencing) [10].

We construct an iteration algorithm for the quasilinear parabolic problem (6)-(9), using an implicit finite difference scheme for the following linearized problem:

$$\frac{\partial u^{(n)}}{\partial t} - a^2 \frac{\partial^2 u^{(n)}}{\partial x^2} = f(t,x,u^{(n-1)}),\ (t,x) \in D := \{0 < x < \pi,\ 0 < t < T\} \tag{10}$$

$$u^{(n)}(t,0) = u^{(n)}(t,\pi),\ t \in [0,T] \tag{11}$$

$$u_x^{(n)}(t,0) = u_x^{(n)}(t,\pi),\ t \in [0,T] \tag{12}$$

$$u^{(n)}(0,x) = \varphi(x),\ x \in [0,\pi]\,. \tag{13}$$

Let $u^{(n)}(t,x) = v(t,x)$ and $f(t,x,u^{(n-1)}) = \widetilde{f}(t,x)$. Then the linearized quasilinear parabolic problem (10)–(13) can be rewritten in the form of an initial-periodic boundary value problem for a linear parabolic problem as follows:

$$\frac{\partial v}{\partial t} - a^2 \frac{\partial^2 v}{\partial x^2} = \widetilde{f}(t,x),\ (t,x) \in D := \{0 < x < \pi,\ 0 < t < T\}, \tag{14}$$

$$v(t,0) = v(t,\pi),\ t \in [0,T], \tag{15}$$

$$v_x(t,0) = v_x(t,\pi),\ t \in [0,T], \tag{16}$$

$$v(0,x) = \varphi(x),\ x \in [0,\pi]. \tag{17}$$

To solve the linearized problem (6), we define the uniform space and time grids

$$\overline{w}_h^t = \Big\{x_i \in [0,\pi],\ t_j \in [0,T] : x_i = ih_x,\ i = \overline{0, N_x}, \\ h_x = \frac{\pi}{N_x},\ t_j = jh_t,\ j = \overline{0, N_t},\ h_t = \frac{T}{N_t}\Big\},$$

and use the standard finite difference approximations

$$u_{x,ij} := \frac{u_{i+1,j} - u_{i,j}}{h_x},\ u_{t,ij} := \frac{u_{i,j+1} - u_{i,j}}{h_t},\ u_{ij} := u(t_j, x_i), \\ i = \overline{1, N_x - 1},\ j = \overline{0, N_t - 1}$$

of the partial derivatives u_x, u_t, where the constants $h_x > 0$ and $h_t > 0$ are the grid parameters.

For the numerical solution of the linear parabolic problem (14)–(17), we use the following implicit monotone difference scheme:

$$\begin{aligned} &\frac{v_{i,j+1} - v_{i,j}}{h_t} - \frac{a^2}{h_x^2}(v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) = \widetilde{f}_{i,j+1}, \\ &\qquad\qquad i = \overline{1, N_x - 1},\ j = \overline{0, N_t - 1}, \\ &v_{i,0} = \varphi_i,\ i = \overline{0, N_x};\quad v_{0,j} = v_{N_x,j},\ v_{x,0j} = v_{x,N_x j},\ j = \overline{0, N_t}, \end{aligned} \tag{18}$$

which has accuracy $\mathrm{O}(h_x^2 + h_t)$ on the uniform grid w_h^t [13].

3. Numerical Test Problems

3.1. Numerical Solution of the Linear Problem. First, we consider the linear case. That is, the source function $f(t,x,u)$ in equation (6) depends only on variables x and t:

$$f(t,x,u) = f(t,x),\ x \in [0,\pi],\ t \in [0,T].$$

Then the function

$$u(t,x) = \exp(-t)\sin^2 x,\ (t,x) \in D,$$

is the analytical solution of the linear parabolic problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t,x),\ (t,x) \in D, \tag{19}$$

$$u(t,0) = u(t,\pi) = \mu_1(t),\ t \in [0,T], \tag{20}$$

$$u_x(t,0) = u_x(t,\pi) = \mu_2(t),\ t \in [0,T], \tag{21}$$

$$u(0,x) = \varphi(x),\ x \in [0,\pi], \tag{22}$$

with the given input data

$$\mu_1(t) = 0,$$

$$\mu_2(t) = 0,$$

$$\varphi(x) = \sin^2(x)$$

and the source term

$$f(t,x) = -(\sin^2 x + 2a^2 \cos(2x)) \exp(-t).$$

Table 1 shows the absolute sup-norm errors obtained for different grid parameters N_x, N_t.

Table l. Numerical results for the linear problem (19)–(22) on different grids

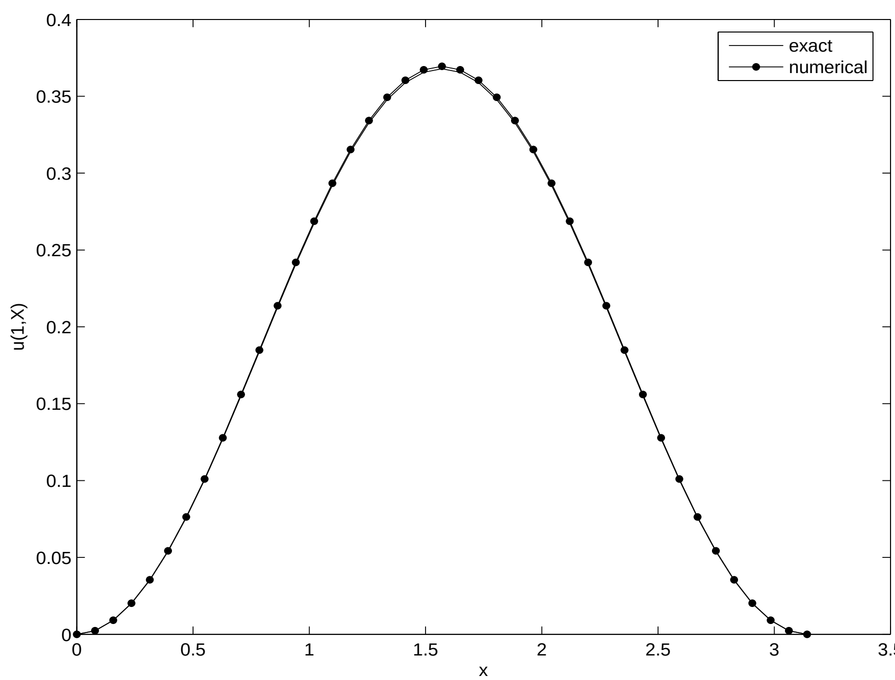
$h_x\,(N_x)$	$h_t\,(N_t)$	$R = \frac{h_t}{h_x^2}$	$\varepsilon_v := \Vert u - u_h \Vert_\infty$	n_t
0.1571 (21)	0.05 (21)	2.0264	0.0109	–
0.1571 (21)	0.025 (41)	1.0132	0.0070	0.6619
0.1571 (21)	0.0125 (81)	0.5066	0.0051	0.4651
0.0785 (51)	0.02 (51)	3.2423	0.0039	–
0.0785 (51)	0.01 (101)	1.6211	0.0024	0.7105
0.0785 (51)	0.005 (201)	0.8106	0.0016	0.5892

The last column gives the approximation error

$$n_t(N_x) = \log\left(\frac{\varepsilon_t^{(1)}}{\varepsilon_t^{(2)}}\right) \Big/ \log\left(\frac{N_t^{(2)}}{N_t^{(1)}}\right),$$

for the uniform space and time grids w_h^t used. The last line of the table shows that the smallest absolute error $\varepsilon_v = 0.0016$ is obtained on the grid of size $N_x \times N_t = 51 \times 201$, with $R = 0.8106$. Figure 1 shows the exact and numerical solutions of the problem (19)–(22) for T=1.

Figure 1. Exact and Numerical Solutions of the Linear Problem (19)–(22) for the final time $T = 1$



3.2. Numerical Solution of the Nonlinear Problem. In this example, we apply the linearization scheme (18) to the nonlinear problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t,x,u),\ (t,x) \in D = (0,\pi) \times (0,T) \tag{23}$$

$$u(t,0) = u(t,\pi) = \mu_1(t),\ t \in [0,T] \tag{24}$$

$$u_x(t,0) = u_x(t,\pi) = \mu_2(t),\ t \in [0,T] \tag{25}$$

$$u(0,x) = \varphi(x),\ x \in [0,\pi]. \tag{26}$$

The analytical solution

$$u(t,x) = \exp(t - \sin 2x),\ (t,x) \in D,$$

of the problem (23)–(26) corresponds to the source term

$$f(t,x,u) = (1 - 4\sin 2x - 4\cos^2 2x)u$$

and to appropriately chosen functions $\varphi(x) = \exp(-\sin 2x)$, $\mu_1(t) = \exp(t)$, and $\mu_2(t) = -2\exp(t)$.

The nonlinear problem (23)-(26) was solved by applying the iteration scheme (18). The condition

$$\varepsilon_{it} := \left\| u_n^{(n+1)} - u_n^{(n)} \right\|_\infty,$$

with $\varepsilon_{it} = 10^{-8}$, was used as a stopping criteria for the iteration process. Numerical results obtained on different grids are shown in Table 2.

Table 2. Numerical results for the Nonlinear Problem (23)–(26) on different grids

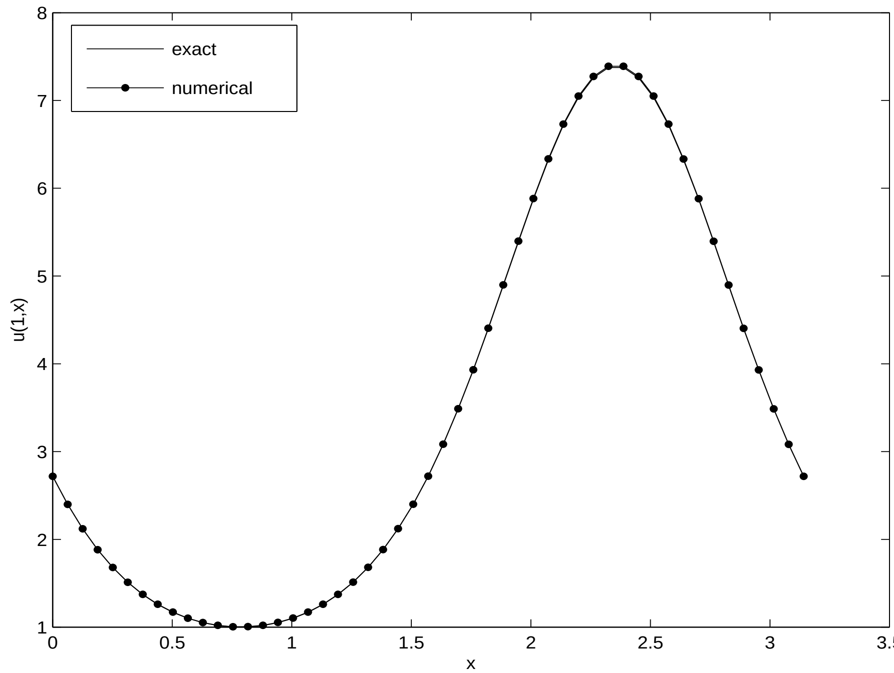
$h_x\,(N_x)$	$h_t\,(N_t)$	$R = \frac{h_t}{h_x^2}$	$\varepsilon_u := \left\|u - u_h^{(n)}\right\|_\infty$	iteration number	n_t
0.1571 (21)	0.0500 (21)	2.0264	0.1167	2	–
0.1571 (21)	0.0250 (41)	1.0132	0.0908	2	0.3751
0.1571 (21)	0.0125 (81)	0.5066	0.0778	2	0.2269
0.0785 (51)	0.0200 (51)	3.2423	0.0312	2	–
0.0785 (51)	0.0100 (101)	1.6211	0.0207	2	0.6004
0.0785 (51)	0.0050 (201)	0.8106	0.0154	2	0.4298

The absolute sup-norm errors obtained for different grid parameters $N_x \times N_t$ are given in the fourth column of the table. As seen from these results, the errors in this case are almost same as in the linear case given in Table 1 for the problem (19)–(23). Further, by increasing the gird size the approximation error n_t tends to 1, which agrees with the order of approximation of the difference scheme (19)–(23), with respect to the time mesh parameter h_t.

Figure 2 shows the exact and numerical solutions of the nonlinear problem (23)–(26) for $T = 1$.

It is clear from these results that accuracy of the above discrete model is sufficiently high.

Figure 2. Exact and Numerical Solutions of the Nonlinear Problem (23)–(26) for the final time $T = 1$



4. Conclusion

In this paper, we studied a quasilinear parabolic problem with periodic boundary condition. We constructed an iteration algorithm to obtain numerical solutions of the problem by solving the linearized problem (14)–(17) based on the monotone finite difference scheme (18). The computational results presented are consistent with the theoretical results.

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