

SOME PROPERTIES OF THE PSI AND POLYGAMMA FUNCTIONS[§]

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Abstract

In this paper, some monotonicity and concavity results of several functions involving the psi and polygamma functions are proved, and then some known inequalities are extended and generalized.

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1. Introduction

It is common knowledge that the classical Euler gamma function

$$(1.1) \quad \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt,$$

the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ and $x > 0$ are important special functions, play a central role in the theory of special functions, and have extensive applications in many branches, for example, probability, statistics, physics, engineering, and other mathematical sciences. In the recent past, numerous papers appeared providing inequalities for the gamma and various related functions. A detailed list of references is given in [12]. More literature can be found in the celebrated books [4, 15, 21]. In addition, some new results can be found in recently published papers such as, for example, [2, 3, 6, 7, 8, 9, 10, 13, 14, 20], and closely-related references therein.

In [8, Theorem 2.1], it was discovered that if $a \leq -\ln 2$ and $b \geq 0$, then

$$(1.2) \quad a - \ln(e^{1/x} - 1) < \psi(x) < b - \ln(e^{1/x} - 1)$$

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holds for $x > 0$. From this, the double inequality

$$(1.3) \quad \ln\left(\frac{\pi^2}{6}\right) - \ln[e^{1/(n+1)} - 1] < H_n < \gamma - \ln[e^{1/(n+1)} - 1]$$

for $n \in \mathbb{N}$ was deduced in [8, Corollary 2.2], where

$$(1.4) \quad H_n = \sum_{k=1}^n \frac{1}{k}$$

stands for the well-known harmonic numbers and γ represents Euler-Mascheroni's constant.

In [3, pp. 386–388], the inequality (1.3) was sharpened as

$$(1.5) \quad 1 + \ln(\sqrt{e} - 1) - \ln(e^{1/(n+1)} - 1) \leq H_n < \gamma - \ln(e^{1/(n+1)} - 1)$$

for $n \in \mathbb{N}$, by proving that the function

$$(1.6) \quad \phi(x) = \psi(x) + \ln(e^{1/x} - 1)$$

is strictly increasing on $(0, \infty)$.

In [7, Theorem 2.8], also by substantially proving the increasing monotonicity of $\phi(x)$, the inequality (1.2) was sharpened as follows: The inequality (1.2) is valid on $(0, \infty)$ if and only if $a \leq -\gamma$ and $b \geq 0$.

The first aim of this paper is to provide an alternative proof for the increasingly monotonic property of $\phi(x)$, and to present the strictly concave property of $\phi(x)$ as follows.

1.1. Theorem. *The function $\phi(x)$ defined by (1.6) is not only strictly increasing but also strictly concave on $(0, \infty)$, with*

$$(1.7) \quad \lim_{x \rightarrow 0^+} \phi(x) = -\gamma \quad \text{and} \quad \lim_{x \rightarrow \infty} \phi(x) = 0.$$

As direct consequences of the proof of Theorem 1.1, the following two inequalities for the tri-gamma function $\psi'(x)$ and the tetra-gamma function $\psi''(x)$ are deduced.

1.2. Corollary. *For $x > 0$, we have*

$$(1.8) \quad \psi'(x) > \frac{e^{1/x}}{(e^{1/x} - 1)x^2} \quad \text{and} \quad \psi''(x) < \frac{e^{1/x}[1 - 2x(e^{1/x} - 1)]}{(e^{1/x} - 1)^2 x^4}.$$

The second aim of this paper is to extend Theorem 1.1 to the following necessary and sufficient conditions.

1.3. Theorem. *For $\theta > 0$, let*

$$(1.9) \quad \phi_\theta(x) = \psi(x) + \ln(e^{\theta/x} - 1)$$

on $(0, \infty)$. Then

- (1) *The function $\phi_\theta(x)$ is strictly increasing if and only if $0 < \theta \leq 1$ and strictly decreasing if $\theta \geq 2$;*
- (2) *The function $\phi_\theta(x)$ is strictly concave if $0 < \theta \leq 1$ and strictly convex if $\theta \geq 2$;*
- (3) *$\lim_{x \rightarrow \infty} \phi_\theta(x) = \ln \theta$ and*

$$(1.10) \quad \lim_{x \rightarrow 0^+} \phi_\theta(x) = \begin{cases} -\gamma, & \theta = 1, \\ \infty, & \theta > 1, \\ -\infty, & 0 < \theta < 1. \end{cases}$$

As straightforward consequences of the proof of Theorem 1.3, the following inequalities for the tri-gamma function $\psi'(x)$ and the tetra-gamma function $\psi''(x)$ are presented, which extend the two inequalities in Corollary 1.2.

1.4. Corollary. *For $x > 0$, inequalities*

$$(1.11) \quad \psi'(x) > \frac{\theta e^{\theta/x}}{x^2(e^{\theta/x} - 1)} \quad \text{and} \quad \psi''(x) \leq \frac{\theta e^{\theta/x} [\theta - 2x(e^{\theta/x} - 1)]}{x^4(e^{\theta/x} - 1)^2}$$

hold if $0 < \theta \leq 1$ and the reverse if $\theta \geq 2$.

In [7, Theorem 2.6], inequality

$$(1.12) \quad -\gamma + x\psi'\left(\frac{x}{2}\right) < \psi(x+1) < -\gamma + x\psi'(\sqrt{x+1} - 1)$$

for $x > 0$ was shown. The following careful observation reveals that inequality (1.12) is not valid: On taking $x = 1$, the left-hand side inequality in (1.12) is reduced to $-\gamma + \frac{\pi^2}{2} < 1 - \gamma$ which clearly does not hold true. After checking the proof of [7, Theorem 2.6], it is found that inequality (1.12) can be corrected and extended as the following theorem.

1.5. Theorem. *If $x > 0$,*

$$(1.13) \quad -\gamma + x\psi'\left(1 + \frac{x}{2}\right) < \psi(x+1) < -\gamma + x\psi'(\sqrt{x+1}).$$

If $-1 < x < 0$, inequality (1.13) is reversed.

As a generalization of Theorem 1.5, the following monotonicity properties are obtained.

1.6. Theorem. *The functions*

$$(1.14) \quad f(x) = \psi(x+1) - x\psi'\left(1 + \frac{x}{2}\right)$$

and

$$(1.15) \quad g(x) = x\psi'(\sqrt{x+1}) - \psi(x+1)$$

are both strictly increasing on $(-1, \infty)$, with limits

$$\lim_{x \rightarrow (-1)^+} f(x) = -\infty,$$

$$\lim_{x \rightarrow (-1)^+} g(x) = 1 + \gamma - \frac{\pi^2}{6}$$

and

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty.$$

1.7. Remark. Making use of the difference equation (2.2) below, the functions $f(x)$ and $g(x)$ defined in (1.14) and (1.15) for $x \in (-1, \infty)$ can be rewritten as

$$(1.16) \quad f(x) = \begin{cases} \psi(x) - x\psi'\left(\frac{x}{2}\right) + \frac{5}{x}, & x \neq 0 \\ -\gamma, & x = 0 \end{cases}$$

and

$$(1.17) \quad g(x) = \begin{cases} x\psi'(\sqrt{x+1}) - \psi(x) - \frac{1}{x}, & x \neq 0 \\ \gamma, & x = 0. \end{cases}$$

As an immediate consequence of the proof of Theorem 1.6, the monotonicity property of the function

$$(u^2 - 1)\psi'(u) - \psi(u^2)$$

on $(-1, \infty)$ is derived as follows.

1.8. Theorem. For $x > -1$, the function

$$(1.18) \quad h(x) = \begin{cases} (x^2 - 1)\psi'(x) - \psi(x^2), & x \neq 0 \\ 1 + \gamma - \frac{\pi^2}{6}, & x = 0 \end{cases}$$

is strictly increasing on $(-1, \infty)$, with

$$(1.19) \quad \lim_{x \rightarrow -1^+} h(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} h(x) = \infty.$$

1.9. Remark. We conjecture that the function $h(x)$ is strictly concave on $(-1, 1)$ and strictly convex on $(1, \infty)$.

Finally, Theorem 1.6 can be generalized as follows.

1.10. Theorem. If i is a positive odd integer, then the function

$$(1.20) \quad f_i(x) = \psi^{(i)}(x+1) - x\psi^{(i+1)}\left(1 + \frac{x}{2}\right)$$

is strictly decreasing on $(-1, \infty)$; If i is a positive even integer, then the function $f_i(x)$ is strictly increasing on $(-1, \infty)$; For all $i \in \mathbb{N}$, the limits

$$(1.21) \quad \lim_{x \rightarrow -1^+} f_i(x) = (-1)^{i+1}\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} f_i(x) = 0$$

hold true.

1.11. Remark. Similar to the monotonic properties of the function $f_i(x)$, we propose the following conjecture: If i is a positive odd integer, then the function

$$(1.22) \quad g_i(x) = \psi^{(i)}(x+1) - x\psi^{(i+1)}(\sqrt{x+1})$$

is strictly decreasing on $(-1, \infty)$; If i is a positive even integer, then the function $g_i(x)$ is strictly increasing on $(-1, \infty)$; For all $i \in \mathbb{N}$, the limits

$$(1.23) \quad \lim_{x \rightarrow -1^+} g_i(x) = (-1)^{i+1}\infty \quad \text{and} \quad \lim_{x \rightarrow \infty} g_i(x) = 0$$

are valid, except $\lim_{x \rightarrow \infty} g_1(x) = 1$.

Direct calculation yields

$$\begin{aligned} [g_i(x)]' &= \psi^{(i+1)}(x+1) - \psi^{(i+1)}(\sqrt{x+1}) - \frac{x}{2\sqrt{x+1}}\psi^{(i+2)}(\sqrt{x+1}) \\ &= \frac{2u[\psi^{(i+1)}(u^2) - \psi^{(i+1)}(u)] - (u^2 - 1)\psi^{(i+2)}(u)}{2u} \\ &= \frac{[\psi^{(i)}(u^2) - (u^2 - 1)\psi^{(i+1)}(u)]'}{2u}, \end{aligned}$$

where $u = \sqrt{x+1} > 0$ for $x > -1$. Therefore, in order to verify above conjecture, it is sufficient to show the monotonic properties of the function

$$(1.24) \quad \psi^{(i)}(u^2) - (u^2 - 1)\psi^{(i+1)}(u)$$

on $(0, \infty)$.

1.12. Remark. It is also natural to pose the following open problem: For $i, k \in \mathbb{N}$ and positive numbers $\alpha, \beta, \delta, \lambda, \mu$ and τ , what about the monotonicities and convexities of the more general function

$$(1.25) \quad \varphi_{i,k}(x) = \psi^{(i-1)}(x + \alpha) - (x + \beta)^k \psi^{(i)}(\lambda(x + \delta)^\mu + \tau)$$

on an appropriate interval where it is defined?

2. Lemmas

The following lemmas are useful for the proofs of some of our theorems.

2.1. Lemma. [16, p. 526, Lemma 2.1] *If $f(x)$ is a function defined in an infinite interval I such that*

$$f(x) - f(x + \varepsilon) > 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x) = \delta$$

for some $\varepsilon > 0$, then $f(x) > \delta$ on I .

Proof. By induction, for any $x \in I$,

$$f(x) > f(x + \varepsilon) > f(x + 2\varepsilon) > \dots > f(x + k\varepsilon) \rightarrow \delta$$

as $k \rightarrow \infty$. The proof of Lemma 2.1 is complete. □

2.2. Lemma. [1] *For $x > 0$ and $k \in \mathbb{N}$,*

$$(2.1) \quad \psi(x) - \ln x + \frac{1}{x} = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt,$$

$$(2.2) \quad \psi^{(k-1)}(x+1) = \psi^{(k-1)}(x) + \frac{(-1)^{k-1}(k-1)!}{x^k}.$$

Recall from [15, Chapter XIII] and [22, Chapter IV] that a function f is said to be *completely monotonic* on an interval I if it has derivatives of all orders on I and

$$(2.3) \quad 0 \leq (-1)^n f^{(n)}(x) < \infty$$

for $x \in I$ and $n \geq 0$. The well known Bernstein's Theorem [22, p. 161] states that a function f is completely monotonic on $(0, \infty)$ if and only if

$$(2.4) \quad f(x) = \int_0^\infty e^{-xs} d\mu(s),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. This expresses that a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of the measure μ .

2.3. Lemma. [17, Theorem 1.3] *The function*

$$\psi(x) - \ln x + \frac{\alpha}{x}$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 1$ and

$$\ln x - \frac{\alpha}{x} - \psi(x)$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq \frac{1}{2}$. Consequently, inequalities

$$(2.5) \quad \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}$$

hold for $x \in (0, \infty)$ and $k \in \mathbb{N}$.

2.4. Remark. Recall from [5, 9, 13, 18] that a positive function f on an interval I is called *logarithmically completely monotonic* if it satisfies

$$(2.6) \quad (-1)^k [\ln f(x)]^{(k)} \geq 0$$

for $k \in \mathbb{N}$ on I . The results in Lemma 2.3 can also be concluded from the necessary and sufficient conditions established in [11] on $\alpha \in \mathbb{R}$ for the function

$$(2.7) \quad \frac{e^x \Gamma(x)}{x^{x-\alpha}}$$

and its reciprocal to be logarithmically completely monotonic on $(0, \infty)$.

2.5. Lemma. [1, pp. 259–260] For $z \neq -1, -2, -3, \dots$,

$$(2.8) \quad \psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}.$$

For $n \in \mathbb{N}$ and $z \neq 0, -1, -2, \dots$,

$$(2.9) \quad \psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}.$$

3. Proofs of theorems and corollaries

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. Direct calculation gives

$$(3.1) \quad \phi'(x) = \psi'(x) - \frac{e^{1/x}}{(e^{1/x} - 1)x^2},$$

$$(3.2) \quad \phi''(x) = \psi''(x) + \frac{2e^{1/x}}{(e^{1/x} - 1)x^3} + \frac{e^{1/x}}{(e^{1/x} - 1)x^4} - \frac{e^{2/x}}{(e^{1/x} - 1)^2 x^4}$$

and

$$\lim_{x \rightarrow \infty} \phi'(x) = \lim_{x \rightarrow \infty} \phi''(x) = 0.$$

From the difference equation (2.2), it is deduced that

$$(3.3) \quad \begin{aligned} \phi'(x) - \phi'(x+1) &= \frac{1}{x^2} + \frac{1}{[1 - 1/e^{1/(x+1)}](x+1)^2} - \frac{1}{(1 - 1/e^{1/x})x^2} \\ &= \frac{e^{1/(x+1)}}{[e^{1/(x+1)} - 1](x+1)^2} - \frac{1}{(e^{1/x} - 1)x^2}. \end{aligned}$$

It is easy to see that $\phi'(x) - \phi'(x+1) > 0$ for $x > 0$ is equivalent to

$$(3.4) \quad x^2(e^{1/x} - 1) > (x+1)^2[1 - e^{-1/(x+1)}].$$

This can be expanded and simplified as

$$\sum_{k=3}^{\infty} \frac{1}{k!} \left[\frac{1}{x^{k-2}} + \frac{(-1)^k}{(x+1)^{k-2}} \right] > 0$$

which is clearly valid. By Lemma 2.1, it is concluded that the function $\phi'(x)$ is positive and $\phi(x)$ is strictly increasing on $(0, \infty)$.

By utilization of (3.3) and differentiation, it is acquired directly that

$$\begin{aligned} \phi''(x) - \phi''(x+1) &= [\phi'(x) - \phi'(x+1)]' \\ &= \frac{e^{1/(x+1)}[3 + 2x - 2(x+1)e^{1/(x+1)}]}{[e^{1/(x+1)} - 1]^2(x+1)^4} - \frac{(1-2x)e^{1/x} + 2x}{(e^{1/x} - 1)^2x^4} \end{aligned}$$

for $x > 0$. It is obvious that the fact $\phi''(x) - \phi''(x+1) < 0$ is equivalent to

$$\left[\frac{e^{1/x} - 1}{e^{1/(x+1)} - 1} \right]^2 > \frac{1}{e^{1/(x+1)}} \left(\frac{x+1}{x} \right)^4 \frac{e^{1/x} - 2x(e^{1/x} - 1)}{1 - 2(1+x)[e^{1/(x+1)} - 1]}.$$

Considering (3.4), in order to prove the above inequality, it is sufficient to show

$$\frac{1}{e^{1/(x+1)}} > \frac{e^{1/x} - 2x(e^{1/x} - 1)}{1 - 2(1+x)[e^{1/(x+1)} - 1]},$$

which is equivalent to

$$3 + 2x - 2e^{1/(x+1)} - (2x+1)e^{1/(x+1)+1/x} \triangleq h(x) < 0.$$

Straightforward computation gives

$$\begin{aligned} h'(x) &= \frac{2x^2e^{1/(x+1)} + 2(x+1)^2x^2 + (4x^2 + 4x + 1 - 2x^4)e^{1/(x+1)+1/x}}{x^2(x+1)^2}, \\ h''(x) &= -\frac{2(2x+3)x^4 + e^{1/x}(8x^5 + 26x^4 + 36x^3 + 24x^2 + 8x + 1)}{x^4(x+1)^4e^{-1/(x+1)}} \\ &< 0, \end{aligned}$$

and $\lim_{x \rightarrow \infty} h'(x) = 0$. Hence, the function $h'(x)$ for $x > 0$ is decreasing and positive, and then the function $h(x)$ for $x > 0$ is increasing. From $\lim_{x \rightarrow \infty} h(x) = -4$, it is deduced that $h(x) < -4 < 0$ for $x > 0$. Consequently, utilizing Lemma 2.1, it is concluded that $\phi''(x) < 0$ on $(0, \infty)$. The concavity of $\phi(x)$ is proved.

From (2.1), it follows that

$$\begin{aligned} \phi(x) &= \ln x - \frac{1}{x} + \ln(e^{1/x} - 1) + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt \\ &= \ln \frac{(e^{1/x} - 1)}{1/x} - \frac{1}{x} + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt \\ &\rightarrow 0 \end{aligned}$$

as $x \rightarrow \infty$. Employing (2.2) for $i = 1$ reveals

$$\begin{aligned} \phi(x) &= \psi(x) + \frac{1}{x} + \ln(e^{1/x} - 1) - \frac{1}{x} \\ &= \psi(x+1) + \ln(e^{1/x} - 1) - \ln e^{1/x} \\ &= \psi(x+1) + \ln(1 - e^{-1/x}) \\ &\rightarrow \psi(1) = -\gamma \end{aligned}$$

as $x \rightarrow 0^+$. The proof of Theorem 1.1 is complete. □

Proof of Corollary 1.2. These inequalities follow from the increasing monotonicity and concavity of $\phi(x)$ and formulas (3.1) and (3.2). □

Proof of Theorem 1.3. Using (2.1), the function $\phi_\theta(x)$ becomes

$$\begin{aligned}\phi_\theta(x) &= \ln x - \frac{1}{x} + \ln(e^{\theta/x} - 1) + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt \\ &= \ln \frac{(e^{\theta/x} - 1)}{1/x} - \frac{1}{x} + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt \\ &\rightarrow \ln \theta\end{aligned}$$

as $x \rightarrow \infty$. Employing (2.2) for $i = 1$ reveals

$$\begin{aligned}\phi_\theta(x) &= \psi(x) + \frac{1}{x} + \ln(e^{\theta/x} - 1) - \frac{1}{x} \\ &= \psi(x+1) + \ln(e^{\theta/x} - 1) - \ln e^{1/x} \\ &= \psi(x+1) + \ln(e^{(\theta-1)/x} - e^{-1/x}) \\ &\rightarrow \begin{cases} \psi(1) = -\gamma, & \theta = 1 \\ \infty, & \theta > 1 \\ -\infty, & 0 < \theta < 1 \end{cases}\end{aligned}$$

as $x \rightarrow 0^+$. The two limits in Theorem 1.3 are proved.

Easy calculation yields

$$(3.5) \quad \begin{aligned}\phi'_\theta(x) &= \psi'(x) - \frac{\theta e^{\theta/x}}{x^2(e^{\theta/x} - 1)} \\ &\triangleq \psi'(x) - \varphi(\theta, x),\end{aligned}$$

$$(3.6) \quad \begin{aligned}\phi''_\theta(x) &= \psi''(x) + \frac{\theta e^{\theta/x} [2x(e^{\theta/x} - 1) - \theta]}{x^4(e^{\theta/x} - 1)^2} \\ &= \psi''(x) - \frac{d\varphi(\theta, x)}{dx}.\end{aligned}$$

It is easy to verify that

$$\frac{d\varphi(\theta, x)}{d\theta} = \frac{e^{\theta/x}(e^{\theta/x} - 1 - \theta/x)}{x^2(e^{\theta/x} - 1)^2} > 0$$

and

$$\begin{aligned}\frac{d^2\varphi(\theta, x)}{d\theta dx} &= -\frac{e^{\theta/x} [(\theta/x)^2(e^{\theta/x} + 1) - 4(\theta/x)(e^{\theta/x} - 1) + 2(e^{\theta/x} - 1)^2]}{x^3(e^{\theta/x} - 1)^3} \\ &= -\frac{2\theta^2 e^{\theta/x} \{ (e^{\theta/x} - 1)/2 + [(e^{\theta/x} - 1)/(\theta/x) - 1]^2 \}}{x^5(e^{\theta/x} - 1)^3} \\ &< 0\end{aligned}$$

for $\theta > 0$ and $x > 0$. These imply that the functions $\varphi(\theta, x)$ and $\frac{d\varphi(\theta, x)}{dx}$ are increasing and decreasing as a function of $\theta > 0$, respectively. Consequently, by using the two inequalities in Corollary 1.2, it is concluded for $0 < \theta \leq 1$ and $x \in (0, \infty)$ that

$$\varphi(\theta, x) \leq \varphi(1, x) = \frac{e^{1/x}}{x^2(e^{1/x} - 1)} < \psi'(x)$$

and

$$\frac{d\varphi(\theta, x)}{dx} \geq \frac{d\varphi(1, x)}{dx} = \frac{e^{1/x} [1 - 2x(e^{1/x} - 1)]}{(e^{1/x} - 1)^2 x^4} > \psi''(x).$$

Hence the function $\phi'_\theta(x)$ is positive and $\phi''_\theta(x) < 0$ on $(0, \infty)$ for $0 < \theta \leq 1$, and the function $\phi_\theta(x)$ is increasing and concave on $(0, \infty)$ for $0 < \theta \leq 1$.

In order that $\phi'_\theta(x) < 0$, by the right-hand side inequality in (2.5), it is sufficient to prove

$$\frac{1}{x} + \frac{1}{x^2} - \frac{\theta e^{\theta/x}}{x^2(e^{\theta/x} - 1)} \leq 0,$$

which is equivalent to

$$1 + \frac{u}{\theta} - \frac{ue^u}{e^u - 1} \leq 0$$

for $u = \frac{\theta}{x} > 0$. Therefore, it suffices to let

$$\theta \geq \frac{u(e^u - 1)}{1 + (u - 1)e^u} \triangleq \delta(u)$$

for $u > 0$. Since $\delta(u)$ is decreasing and $1 < \delta(u) < 2$ on $(0, \infty)$, when $\theta \geq 2$, the function $\phi_\theta(x)$ is decreasing on $(0, \infty)$.

In order that $\phi''_\theta(x) > 0$, by the right-hand side inequality in (2.5), it is sufficient to show

$$\frac{\theta e^{\theta/x} [2x(e^{\theta/x} - 1) - \theta]}{x^4(e^{\theta/x} - 1)^2} \geq \frac{1}{x^2} + \frac{2}{x^3},$$

which is equivalent to

$$\frac{2ue^u(e^u - 1 - u/2)}{(e^u - 1)^2} \geq 1 + \frac{2u}{\theta}$$

for $u = \frac{\theta}{x} > 0$. Therefore, it suffices to let

$$\frac{2}{\theta} \leq \frac{1}{u} \left[\frac{2ue^u(e^u - 1 - u/2)}{(e^u - 1)^2} - 1 \right] \triangleq \rho(u)$$

for $u > 0$. Since $\rho(u)$ is increasing and $1 < \rho(u) < 2$ on $(0, \infty)$, it is sufficient to let $\theta \geq 2$.

If $\phi_\theta(x)$ is increasing on $(0, \infty)$, then $\phi'_\theta(x) > 0$ means that

$$[x^2\psi'(x) - \theta]e^{\theta/x} > x^2\psi'(x).$$

It is well known that $\psi'(x) > 0$ on $(0, \infty)$, thus it is necessary that $\theta < x^2\psi'(x)$. Lemma 2.3 for $k = 1$ gives

$$\psi'(x) < \frac{1}{x} + \frac{1}{x^2}$$

on $(0, \infty)$, and then $\theta < x + 1$ on $(0, \infty)$. Hence, the required necessary condition $\theta \leq 1$ is proved. \square

Proof of Corollary 1.4. These inequalities follow directly from the monotonicity and convexity of $\phi_\theta(x)$ and formulas (3.5) and (3.6). \square

Proof of Theorem 1.5. By (2.8), it follows that

$$(3.7) \quad \psi(x + 1) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x} \right)$$

for $x > -1$. By the mean value theorem for differentiation, it is obvious that there exists a number $\mu = \mu(k) = \mu(k, x)$ for $k \in \mathbb{N}$ such that $-1 < \mu(k) < x$ and

$$(3.8) \quad \frac{1}{k} - \frac{1}{k+x} = \frac{x}{[k + \mu(k)]^2}.$$

Employing (3.8) in (3.7) leads to

$$(3.9) \quad \psi(x+1) = -\gamma + x \sum_{k=1}^{\infty} \frac{1}{[k + \mu(k)]^2}$$

for $x > -1$. From (3.8), it is deduced that

$$\mu(k) = \sqrt{k(k+x)} - k.$$

It is not difficult to show that the mapping $k \rightarrow \mu(k)$ is strictly increasing on $[1, \infty)$ with

$$\mu(1) = \sqrt{1+x} - 1 > -1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \mu(k) = \frac{x}{2}.$$

Hence, from (3.9) and (2.9), it is concluded that the inequality

$$\begin{aligned} x\psi' \left(1 + \frac{x}{2} \right) &= x\psi' \left(1 + \lim_{k \rightarrow \infty} \mu(k) \right) \\ &= x \sum_{k=1}^{\infty} \frac{1}{[k + \lim_{k \rightarrow \infty} \mu(k)]^2} \\ &< \gamma + \psi(x+1) \\ &< x \sum_{k=1}^{\infty} \frac{1}{[k + \mu(1)]^2} \\ &= x\psi'(1 + \mu(1)) \\ &= x\psi'(\sqrt{x+1}) \end{aligned}$$

holds for $x > 0$ and reverses for $-1 < x < 0$. The proof of Theorem 1.5 is finished. \square

Proof of Theorem 1.6. Direct computation and utilization of the mean value theorem for differentiation gives

$$\begin{aligned} f'(x) &= \psi'(x+1) - \psi' \left(1 + \frac{x}{2} \right) - \frac{x}{2} \psi'' \left(1 + \frac{x}{2} \right) \\ &= \frac{x}{2} \psi''(1 + \xi(x)) - \frac{x}{2} \psi'' \left(1 + \frac{x}{2} \right) \\ &= \frac{x}{2} \left[\psi''(1 + \xi(x)) - \psi'' \left(1 + \frac{x}{2} \right) \right], \end{aligned}$$

where $\xi(x)$ is between $\frac{x}{2}$ and x for $x > -1$. Since $\psi''(x)$ is strictly increasing on $(0, \infty)$, it follows clearly that $f'(x) > 0$ for $x \neq 0$. Hence, the function $f(x)$ is strictly increasing on $(-1, \infty)$.

A standard argument leads to

$$\begin{aligned} g'(x) &= \psi'(\sqrt{x+1}) - \psi'(x+1) + \frac{x}{2\sqrt{x+1}} \psi''(\sqrt{x+1}) \\ &= \frac{2u[\psi'(u) - \psi'(u^2)] + (u^2 - 1)\psi''(u)}{2u} \\ &= \frac{[(u^2 - 1)\psi'(u) - \psi(u^2)]'}{2u} \end{aligned}$$

for $x > -1$ and $u = \sqrt{x+1} > 0$. Utilization of formulas (2.8) and (2.9) and direct differentiation gives

$$\begin{aligned}
 (u^2 - 1)\psi'(u) - \psi(u^2) &= (u^2 - 1) \sum_{i=0}^{\infty} \frac{1}{(u+i)^2} + \gamma - \sum_{i=0}^{\infty} \left(\frac{1}{1+i} - \frac{1}{u^2+i} \right) \\
 &= \gamma + (u^2 - 1) \sum_{i=0}^{\infty} \left[\frac{1}{(u+i)^2} - \frac{1}{(i+1)(u^2+i)} \right] \\
 (3.10) \qquad &= \gamma + \sum_{i=1}^{\infty} \frac{i(u^2 - 1)(u - 1)^2}{(i+1)(u+i)^2(u^2+i)}
 \end{aligned}$$

and

$$(3.11) \quad [(u^2 - 1)\psi'(u) - \psi(u^2)]' = \sum_{i=1}^{\infty} \frac{2i(u-1)^2(u^3 + 2u^2 + 2iu + i)}{(i+u)^3(u^2+i)^2} > 0$$

for $u > 0$. Hence, the function $g(x)$ is strictly increasing on $(-1, \infty)$.

It is apparent that

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \psi(x+1) + \psi'\left(\frac{1}{2}\right) = -\infty.$$

By (2.5) for $k = 1$ and $\lim_{x \rightarrow \infty} \psi(x) = \infty$, it is clear that $\lim_{x \rightarrow \infty} f(x) = \infty$.

By using (3.10), it is easy to see that

$$\begin{aligned}
 \lim_{x \rightarrow -1^+} g(x) &= \lim_{u \rightarrow 0^+} [(u^2 - 1)\psi'(u) - \psi(u^2)] \\
 &= \gamma + \lim_{u \rightarrow 0^+} \sum_{i=1}^{\infty} \frac{i(u^2 - 1)(u - 1)^2}{(i+1)(u+i)^2(u^2+i)} \\
 &= \gamma - \sum_{i=1}^{\infty} \frac{1}{(i+1)i^2} \\
 &= 1 + \gamma - \frac{\pi^2}{6}
 \end{aligned}$$

and

$$\begin{aligned}
 \lim_{x \rightarrow \infty} g(x) &= \lim_{u \rightarrow \infty} [(u^2 - 1)\psi'(u) - \psi(u^2)] \\
 &= \gamma + \lim_{u \rightarrow \infty} \sum_{i=1}^{\infty} \frac{i(u^2 - 1)(u - 1)^2}{(i+1)(u+i)^2(u^2+i)} \\
 &= \gamma + \sum_{i=1}^{\infty} \frac{i}{i+1} \\
 &= \infty.
 \end{aligned}$$

The proof of Theorem 1.6 is complete. □

Proof of Theorem 1.8. It is not difficult to see that the factor $u^3 + 2u^2 + 2iu + i$ for $i \in \mathbb{N}$ in formula (3.11) is positive if and only if $u > -1$. Therefore, the function $h(x)$ is strictly increasing on $(-1, \infty)$.

The limits can be derived from (2.8) and (2.9). □

Proof of Theorem 1.10. It is obvious that

$$\begin{aligned} [f_i(x)]' &= \psi^{(i+1)}(x+1) - \psi^{(i+1)}\left(1 + \frac{x}{2}\right) - \frac{x}{2}\psi^{(i+2)}\left(1 + \frac{x}{2}\right) \\ &= \frac{x}{2}\left[\psi^{(i+2)}(1 + \eta(x)) - \psi^{(i+2)}\left(1 + \frac{x}{2}\right)\right], \end{aligned}$$

where $\eta(x)$ is between $\frac{x}{2}$ and x . This means $(-1)^i[f_i(x)]' \geq 0$ for $i \in \mathbb{N}$, and then the monotonicities of $f_i(x)$ on $(-1, \infty)$ for $i \in \mathbb{N}$ are proved.

The two limits can be deduced easily from inequality (2.5). \square

3.1. Remark. This paper is a slightly revised version of the preprint [19].

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