Hacettepe Journal of Mathematics and Statistics
Volume 39 (2) (2010), 233 – 240

BASES OF $[0,1]$-MATROIDS[§]

Chun-E. Huang[*†‡] and Fu-Gui Shi[*]

Received 19 : 10 : 2009 : Accepted 14 : 12 : 2009

Abstract

In this paper, a characterization of $[0,1]$-matroids is given. It is proved that a $[0,1]$-matroid is equivalent to a hereditary fuzzy pre-matroid, and that a perfect $[0,1]$-matroid is equivalent to a Goetschel-Voxman fuzzy matroid. It is proved that there is a one-to-one correspondence between the family of closed perfect $[0,1]$-matroids on E and the set of their fuzzy bases.

Keywords: $[0,1]$-matroid, Basis, Perfect [0, 1]-matroid, Closed $[0,1]$-matroid.

2000 AMS Classification: 03 E 70, 05 B 35, 52 B 40.

1. Introduction

Matroids were introduced by Whitney [9] in 1935. They generalize both graphs and matrices and play an important role in mathematics, especially in applied mathematics. Matroids are precisely the structures for which the very simple and efficient greedy algorithm works.

In 1988, R. Goetschel and W. Voxman [1] introduced the concept of fuzzy matroids. Subsequently, many scholars researched Goetschel-Voxman fuzzy matroids (see [2, 3, 4, 5], etc.).

Recently, Shi [7] introduced a new approach to the fuzzification of matroids, namely M-fuzzifying matroids. This approach preserves many basic properties of crisp matroids, and an M-fuzzifying matroid and its M-fuzzifying rank function are in one-to-one correspondence. Further, Shi [8] presented the concept of an (L,M)-fuzzy matroid, which is a wider generalization of M-fuzzifying matroids. An $(L,2)$-fuzzy matroid is called an L-matroid, and can be applied to fuzzy algebras and fuzzy graphs.

[§]This project is supported by the National Natural Science Foundation of China (10971242).
[*]Department of Mathematics, School of Science, Beijing Institute of Technology, Beijing 100081, P. R. China.
E-mail: (C.-E. Huang) hchune@yahoo.com (F.-G Shi) fuguishi@bit.edu.cn f.g.shi@263.net
[†]School of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan 411201, P. R. China.
[‡]Corresponding Author.

In this paper, characterizations of $[0,1]$-matroids are given. In particular, it is proved that a $[0,1]$-matroid is equivalent to a hereditary fuzzy pre-matroid, and that a perfect $[0,1]$-matroid is equivalent to a Goetschel-Voxman fuzzy matroid; see Theorem 3.2 and Corollary 3.5.

The family of fuzzy bases of a $[0,1]$-matroid is also considered. A $[0,1]$-matroid $(E,\mathfrak{I})$ does not generally have a basis; however, when $(E,\mathfrak{I})$ is closed and perfect, its base exists; see [2, Theorem 1.10] or Lemma 4.2. Indeed, it is proved that there exists a one-to-one correspondence between the family of closed perfect $[0,1]$-matroids on E and the set of their fuzzy bases; see Theorem 4.9.

2. Preliminaries

In the sequel, we shall consider $[0,1]$-matroids. Let E be a finite set, denote the power set of E by 2^E, and for any $A \subseteq E$, denote by $|A|$ and χ_A the cardinality and the characteristic function of A, respectively.

A fuzzy set A on E is a mapping $A : E \to [0,1]$. The family of all fuzzy sets on E is denoted by $[0,1]^E$. For any $A, B \in [0,1]^E$, the relation $A \leq B$ is defined by $A(x) \leq B(x)$ for all $x \in E$. Let $A \in 2^E$ and $a \in [0,1]$. The fuzzy set $a \wedge A$ on E is defined by

$$(a \wedge A)(x) = \begin{cases} a, & x \in A, \\ 0, & x \notin A. \end{cases}$$

For $A, B \in [0,1]^E$, $a \in (0,1]$ and $\mathcal{A} \subseteq [0,1]^E$, we use the following operations and notations:

$$\operatorname{supp} A = \{x \in E : A(x) > 0\}; \qquad m(A) = \inf\{A(x) : x \in \operatorname{supp} A\};$$
$$(A \vee B)(x) = \max\{A(x), B(x)\}; \qquad (A \wedge B)(x) = \min\{A(x), B(x)\};$$
$$A_{[a]} = \{x \in E : A(x) \geq a\}; \qquad \mathcal{A}[a] = \{A_{[a]} : A \in \mathcal{A}\};$$
$$\operatorname{Max}(\mathcal{A}) = \{A \in [0,1]^E : A \leq B \in \mathcal{A} \text{ implies } A = B\};$$
$$\operatorname{Low}(\mathcal{A}) = \{A \in [0,1]^E : \text{ there exists } B \in \mathcal{A} \text{ such that } A \leq B\}.$$

2.1. Definition. [9] Let E be a finite set and $\mathfrak{I} \subseteq 2^E$. If $\mathfrak{I}$ satisfies the following statements:

(I1) $\emptyset \in \mathfrak{I}$;
(I2) If $A, B \in 2^E$, $A \subseteq B$ and $B \in \mathfrak{I}$, then $A \in \mathfrak{I}$;
(I3) If $A, B \in \mathfrak{I}$ and $|B| > |A|$, then there exists $e \in B - A$ such that $A \cup \{e\} \in \mathfrak{I}$,

then the pair $(E,\mathfrak{I})$ is called a (*crisp*) *matroid*.

2.2. Definition. [1] Suppose that E is a finite set and $\mathfrak{I} \subseteq [0,1]^E$ a non-empty family of fuzzy sets satisfying:

(i) If $A, B \in [0,1]^E$, $A \leq B$, and $B \in \mathfrak{I}$, then $A \in \mathfrak{I}$;
(ii) If $A, B \in \mathfrak{I}$ and $|\operatorname{supp} A| < |\operatorname{supp} B|$, then there exists $C \in \mathfrak{I}$ such that
  (a) $A < C \leq A \vee B$;
  (b) $m(C) \leq \min(m(A), m(B))$.

Then the pair $(E,\mathfrak{I})$ is called a *Goetschel-Voxman fuzzy matroid*.

2.3. Definition. [11] Let A be a fuzzy set on E. The mapping $|A| : \mathbb{N} \to [0,1]$ defined by for any $n \in \mathbb{N}$ by

$$|A|(n) = \bigvee\{a \in (0,1] : |A_{[a]}| \geq n\}$$

is called the *cardinality of* A, where "$\bigvee$" denotes the supremum.

2.4. Lemma. [8] *For any fuzzy set* $A \in [0,1]^E$, *it holds that* $|A|_{[a]} = |A_{[a]}|$ *for any* $a \in (0,1]$. □

2.5. Definition. [8] Let E be a finite set and $\mathcal{I} \subseteq [0,1]^E$. If $\mathcal{I}$ satisfies the following statements:

(LI1) (Non-empty) $\chi_\emptyset \in \mathcal{I}$;

(LI2) (Hereditary property) If $A, B \in [0,1]^E$, $A \leq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$;

(LI3) (Exchange property) If $A, B \in \mathcal{I}$ and $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$, then there exists $e_b \in F(A,B)$ such that $(b \wedge A_{[b]}) \vee e_b \in \mathcal{I}$, where

$$F(A,B) = \{e_b \in E : A(e) < b \leq B(e)\},$$

then the pair $(E, \mathcal{I})$ is called a $[0,1]$-*matroid*.

2.6. Theorem. [8] *Let* E *be a finite set and* $\mathcal{I} \subseteq [0,1]^E$. *If* $(E, \mathcal{I})$ *is a* $[0,1]$-*matroid, then* $(E, \mathcal{I}[a])$ *is a matroid for each* $a \in (0,1]$. □

3. Some properties and a characterization of $[0,1]$-matroids

If $(E, \mathcal{I})$ is a $[0,1]$-matroid, then, by Theorem 2.6, $(E, \mathcal{I}[a])$ is a matroid for any $a \in (0,1]$. Since E is a finite set, there is at most a finite number of matroids on E. Thus there is a finite sequence $0 < a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$ such that

(i) If $b, c \in (a_{i-1}, a_i)$ $(1 \leq i \leq r)$, then $\mathcal{I}[b] = \mathcal{I}[c]$;

(ii) If $a_{i-1} < b < a_i < c < a_{i+1} (1 \leq i \leq r-1)$, then $\mathcal{I}[c] \subsetneq \mathcal{I}[b]$.

The sequence $0 < a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$ is called the $[0,1]$-*fundamental sequence of* $\mathcal{I}$.

3.1. Theorem. *Let* E *be a finite set and* $0 < a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$ *a finite sequence. For any* $b \in (0,1]$, *suppose that* $(E, \mathcal{I}[b])$ *is a matroid on* E *such that the following statements hold:*

(i) *For all* $a, c \in (0,1]$ *with* $a < c$, $\mathcal{I}[a] \supseteq \mathcal{I}[c]$;

(ii) $\mathcal{I}[a_i] \subsetneq \mathcal{I}[a_{i-1}]$, $(1 \leq i \leq r)$;

(iii) *For all* $a, c \in (a_{i-1}, a_i)$, $(1 \leq i \leq r)$, $\mathcal{I}[a] = \mathcal{I}[c]$.

We define

$$\mathcal{I}^* = \{A \in [0,1]^E : A_{[a]} \in \mathcal{I}[a] \text{ for each } a \in (0,1]\}.$$

Then $(E, \mathcal{I}^*)$ *is a* $[0,1]$-*matroid.*

Proof. We show that $\mathcal{I}^*$ is a $[0,1]$-matroid as follows.

(LI1) Since $\mathcal{I}[a]$ is a matroid for any $a \in (0,1]$, we have $\emptyset \in \mathcal{I}[a]$. This implies $\chi_\emptyset \in \mathcal{I}^*$.

(LI2) If $A, B \in [0,1]^E$, $A \leq B$ and $B \in \mathcal{I}^*$, then $B_{[a]} \in \mathcal{I}[a]$ for each $a \in (0,1]$. Hence, $A_{[a]} \in \mathcal{I}[a]$, so $A \in \mathcal{I}^*$.

(LI3) If $A, B \in \mathcal{I}^*$ and $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$, then $|B|_{[b]} > |A|_{[b]}$. By Lemma 2.4, $|B_{[b]}| > |A_{[b]}|$. Since $\mathcal{I}[b]$ is a matroid and $A_{[b]}, B_{[b]} \in \mathcal{I}[b]$, there exists an element $e \in B_{[b]} - A_{[b]}$ such that $A_{[b]} \cup \{e\} \in \mathcal{I}[b]$. Obviously, we have $b \wedge (A_{[b]} \cup \{e\}) = (b \wedge A_{[b]}) \vee e_b$. For each $a \in (0,1]$, we can obtain

$$((b \wedge A_{[b]}) \vee e_b)_{[a]} = \begin{cases} \emptyset, & a > b, \\ A_{[b]} \cup \{e\}, & a \leq b. \end{cases}$$

Since $A_{[b]} \cup \{e\} \in \mathcal{I}[b] \subseteq \mathcal{I}[a]$ for any $a \leq b$, we obtain $((b \wedge A_{[b]}) \vee e_b)_{[a]} \in \mathcal{I}[a]$ for any $a \in (0,1]$. By the definition of $\mathcal{I}^*$, we have $(b \wedge A_{[b]}) \vee e_b \in \mathcal{I}^*$. Therefore $\mathcal{I}^*$ is a $[0,1]$-matroid. □

Next we give a characterization of $[0,1]$-matroids. We recall that in [5], Novak introduced the concept of a hereditary fuzzy pre-matroid and obtained several nontrivial results. Let $\mathfrak{I} \subseteq [0,1]^E$ be a nonempty family of fuzzy sets. The pair $(E, \mathfrak{I})$ is called a *hereditary fuzzy pre-matroid* if it satisfies the following statements:

(i) If $A, B \in [0,1]^E$, $A \le B$ and $B \in \mathfrak{I}$, then $A \in \mathfrak{I}$;
(ii) $(E, \mathfrak{I}[a])$ is a matroid for each $a \in (0,1]$.

3.2. Theorem. *Let $\mathfrak{I} \subseteq [0,1]^E$ be a family of fuzzy sets. Then $(E,\mathfrak{I})$ is a $[0,1]$-matroid if and only if $(E,\mathfrak{I})$ is a hereditary fuzzy pre-matroid.*

Proof. Suppose that $(E,\mathfrak{I})$ is a $[0,1]$-matroid. Then by (LI1), (LI2), and Theorem 2.6, $(E,\mathfrak{I})$ is a hereditary fuzzy pre-matroid.

Conversely, suppose that $(E,\mathfrak{I})$ is a hereditary fuzzy pre-matroid. Obviously, $(E,\mathfrak{I})$ satisfies (LI1) and (LI2). Now we show (LI3). Suppose that $A, B \in \mathfrak{I}$ and $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$. Then we obtain $|B_{[b]}| > |A_{[b]}|$ by Lemma 2.4, and $A_{[b]}, B_{[b]} \in \mathfrak{I}[b]$. Since $(E,\mathfrak{I})$ is a hereditary fuzzy pre-matroid, $(E,\mathfrak{I}[b])$ is a matroid. Thus there exists $e \in B_{[b]} - A_{[b]}$ such that $A_{[b]} \cup \{e\} \in \mathfrak{I}[b]$. We can obtain a fuzzy set $G \in \mathfrak{I}$ such that

$$G_{[b]} = A_{[b]} \cup \{e\} \text{ and } b \wedge (A_{[b]} \cup \{e\}) = b \wedge G_{[b]} \le G \in \mathfrak{I}.$$

By (LI2), $(b \wedge A_{[b]}) \vee e_b \in \mathfrak{I}$. Hence, $(E,\mathfrak{I})$ is a $[0,1]$-matroid. □

In [5], Novak presented the concept of a *perfect hereditary fuzzy pre-matroid* as a hereditary fuzzy pre-matroid $(E,\mathfrak{I})$ satisfying the following statement:

(P) If $A, B \in \mathfrak{I}$, and for all $a \in (0,1]$, $A_{[a]} \subseteq B_{[a]}$ or $A_{[a]} \supseteq B_{[a]}$, then $A \vee B \in \mathfrak{I}$.

3.3. Lemma. *Let $\mathfrak{I} \subseteq [0,1]^E$ be a family of fuzzy sets satisfying the hereditary property (LI2). Then for each $A \in [0,1]^E$ and $a \in (0,1]$, $a \wedge A_{[a]} \in \mathfrak{I}$ if and only if $A_{[a]} \in \mathfrak{I}[a]$.*

Proof. Let $A \in [0,1]^E$ and $a \in (0,1]$. If $a \wedge A_{[a]} \in \mathfrak{I}$, then $(a \wedge A_{[a]})_{[a]} = A_{[a]} \in \mathfrak{I}[a]$. Conversely, if $A_{[a]} \in \mathfrak{I}[a]$, then there is $G \in \mathfrak{I}$ such that $A_{[a]} = G_{[a]}$. Since $a \wedge G_{[a]} \le G$, (LI2) implies that $a \wedge G_{[a]} \in \mathfrak{I}$, so $a \wedge A_{[a]} \in \mathfrak{I}$. □

By [5, Lemma 1] and Lemma 3.3, we can easily obtain the following result.

3.4. Theorem. *Let $(E,\mathfrak{I})$ be a hereditary fuzzy pre-matroid. Then $(E,\mathfrak{I})$ is perfect if and only if $A \in \mathfrak{I}$ whenever $A \in [0,1]^E$ and $A_{[a]} \in \mathfrak{I}[a]$ for all $a \in (0,1]$.* □

It follows from this result and [10, Theorem 3.3] that the notion of perfect hereditary fuzzy pre-matroid coincides with that of perfect $[0,1]$ matroid given in [10, Definition 3.1].

In [5], Novak proved that a perfect hereditary fuzzy pre-matroid is equivalent to a Goetschel-Voxman fuzzy matroid. Thus, we obtain the following.

3.5. Corollary. *Let E be a finite set and $\mathfrak{I} \subseteq [0,1]^E$. Then $(E,\mathfrak{I})$ is a perfect $[0,1]$-matroid if and only if $(E,\mathfrak{I})$ is a Goetschel-Voxman fuzzy matroid.* □

3.6. Corollary. *Let E be a finite set and $\mathfrak{I} \subseteq [0,1]^E$. Then $(E,\mathfrak{I})$ is a Goetschel-Voxman fuzzy matroid if and only if $\mathfrak{I}$ satisfies the statements* (LI1), (LI2), (LI3) *and* (P). □

The following result from [1] can also be obtained from Theorem 3.1 and Lemma 3.3.

3.7. Theorem. *Let $(E,\mathfrak{I})$ be a perfect $[0,1]$-matroid. Define a family of subsets of $[0,1]^E$ by*

$$\mathfrak{I}^* = \{A \in [0,1]^E : A_{[a]} \in \mathfrak{I}[a] \ \ \textit{for each} \ \ a \in (0,1]\}.$$

Then $\mathfrak{I}^* = \mathfrak{I}$. □

4. Bases of $[0,1]$-matroids

In this section, we discuss bases of $[0,1]$-matroids. We define the family of fuzzy bases of a $[0,1]$-matroid $(E,\mathcal{I})$ by $\mathcal{B}_{\mathcal{I}} = \mathrm{Max}\,(\mathcal{I})$. Each element of $\mathcal{B}_{\mathcal{I}}$ is called a *fuzzy basis* of $\mathcal{I}$.

In Corollary 3.5 it is shown that the Goetschel-Voxman fuzzy matroids are equivalent to perfect $[0,1]$-matroids. Hence, results concerning the family of fuzzy bases of a Goetschel-Voxman fuzzy matroid apply equally to the family of fuzzy bases of a perfect $[0,1]$-matroid. In the following, we do not distinguish a Goetschel-Voxman fuzzy matroid from a perfect $[0,1]$-matroid.

As is seen in [2, Example 1.6], a perfect $[0,1]$-matroid $(E,\mathcal{I})$ does not necessarily have a fuzzy basis. In [2, Theorem 1.10], Goetschel and Voxman prove that when a matroid $(E,\mathcal{I})$ is closed and perfect, it has a family of fuzzy bases. It is natural to ask if these families of fuzzy bases can be used to characterize closed perfect $[0,1]$-matroids. This problem is solved in this section by showing that there exists a one-to-one correspondence between the closed perfect $[0,1]$-matroids on E and their families of fuzzy bases.

The notion of a closed Goetschel-Voxman fuzzy matroid can be generalized to that of a $[0,1]$-matroid as follows.

4.1. Definition. [10, Definition 3.7] Let $(E,\mathcal{I})$ be a $[0,1]$-matroid with $[0,1]$-fundamental sequence $0 < a_0 < a_1 < \cdots < a_{r-1} < a_r = 1$. If $\mathcal{I}[a] = \mathcal{I}[a_i]$ for all $a \in (a_{i-1}, a_i], (1 \le i \le r)$, then $(E,\mathcal{I})$ is called a *closed* $[0,1]$*-matroid*.

The next lemma comes from [2].

4.2. Lemma. [2] *Let* $(E,\mathcal{I})$ *be a perfect* $[0,1]$*-matroid. Then* $(E,\mathcal{I})$ *is closed if and only if for each* $A \in \mathcal{I}$*, there exists a fuzzy basis* $\beta \in \mathcal{I}$ *such that* $A \le \beta$*.* □

4.3. Theorem. *Let* $(E,\mathcal{I})$ *be a* $[0,1]$*-matroid. Then the following statements are equivalent:*

(i) $(E,\mathcal{I})$ *is closed and perfect;*
(ii) *If* $A \in [0,1]^E$ *and for all* $a \in (0,1]$*,* $b \wedge A_{[a]} \in \mathcal{I}$ *for all* $b \in (0,a)$*, then* $A \in \mathcal{I}$*.*

Proof. (i) $\Longrightarrow$ (ii) For $A \in [0,1]^E$ and any $a \in (0,1]$, suppose $b \wedge A_{[a]} \in \mathcal{I}$ for all $b \in (0,a)$. Take $a \in (0,1]$. Since $(E,\mathcal{I})$ is closed, applying condition $(*)$ of [10, Theorem 3.8] to $A_{[a]} \in 2^E$ gives $a \wedge A_{[a]} \in \mathcal{I}$. Hence, $A \in \mathcal{I}$ since $(E,\mathcal{I})$ is perfect. This establishes (ii).

(ii) $\Longrightarrow$ (i) Take $A \in [0,1]^E$ and suppose that $a \wedge A_{[a]} \in \mathcal{I}$ for all $a \in (0,1]$. Then for all $b \in (0,a)$ we have $b \wedge A_{[a]} \in \mathcal{I}$ by (LI2), so $A \in \mathcal{I}$ by (ii). Therefore, $(E,\mathcal{I})$ is perfect.

To prove $(E,\mathcal{I})$ is closed, take $A \in 2^E$ and arbitrary $a_0 \in (0,1]$, and suppose that $b \wedge A \in \mathcal{I}$ for all $b \in (0,a_0)$. By [10, Theorem 3.8], it suffices to show that $a_0 \wedge A \in \mathcal{I}$. However $a_0 \wedge A \in [0,1]^E$, and for all $a \in (0,1]$, $b \in (0,a)$, we have $b \wedge (a_0 \wedge A)_{[a]} \in \mathcal{I}$. Indeed if $a \le a_0$ we have $b \wedge (a_0 \wedge A)_{[a]} = b \wedge A \in \mathcal{I}$ by hypothesis since $b \in (0,a_0)$, while for $a_0 < a$ we have $b \wedge (a_0 \wedge A)_{[a]} = \chi_{\emptyset} \in \mathcal{I}$ by (LI1). Hence (ii) gives $a_0 \wedge A \in \mathcal{I}$, as required. □

4.4. Theorem. *Let* $(E,\mathcal{I})$ *be a closed perfect* $[0,1]$*-matroid and* $\mathcal{B}_{\mathcal{I}}$ *the family of fuzzy bases. Then*

(B1) *For each* $a \in (0,1]$*,* $\mathrm{Max}\,(\mathcal{B}_{\mathcal{I}}[a]) = \mathcal{B}_{\mathcal{I}[a]}$*, where* $\mathcal{B}_{\mathcal{I}[a]}$ *denotes the family of bases of* $(E,\mathcal{I}[a])$*;*
(B2) *Let* $A \in [0,1]^E$*. Suppose that there exists* $\beta_a \in \mathcal{B}_{\mathcal{I}}$ *such that* $a \wedge A_{[a]} \le \beta_a$ *for each* $a \in (0,1]$*. Then there exists a fuzzy set* $\beta \in \mathcal{B}_{\mathcal{I}}$ *such that* $A \le \beta$*.*

Proof. (B1) By (LI1) and Lemma 4.2, we know that $\mathcal{B}_\mathcal{I}$ is non-empty.

For each $a \in (0,1]$, we first show that $\mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a]) \subseteq \mathcal{B}_{\mathcal{I}[a]}$. Let $B \in \mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a])$. Obviously, $B \in \mathcal{B}_\mathcal{I}[a]$. Then there exists $\beta \in \mathcal{B}_\mathcal{I}$ such that $B = \beta_{[a]}$. This implies $B \in \mathcal{I}[a]$. Suppose that $B \notin \mathcal{B}_{\mathcal{I}[a]}$. Then there is $B' \in \mathcal{B}_{\mathcal{I}[a]}$ such that $B \subsetneq B'$. Hence, $a \wedge B < a \wedge B' \in \mathcal{I}$. By Lemma 4.2, there exists a fuzzy basis β' of $(E, \mathcal{I})$ such that $a \wedge B' \leq \beta'$. Thus, $B \subsetneq B' = \beta'_{[a]} \in \mathcal{I}[a]$, which contradicts $B \in \mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a])$. Therefore $\mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a]) \subseteq \mathcal{B}_{\mathcal{I}[a]}$.

Conversely, let $B \in \mathcal{B}_{\mathcal{I}[a]}$. Then there exists $\beta' \in \mathcal{I}$ such that $B = \beta'_{[a]}$. Hence, $a \wedge \beta'_{[a]} \in \mathcal{I}$. By Lemma 4.2, there is $\beta \in \mathcal{B}_\mathcal{I}$ such that $a \wedge \beta'_{[a]} \leq \beta$. This implies

$$B = (a \wedge \beta'_{[a]})_{[a]} \subseteq \beta_{[a]} \in \mathcal{I}[a].$$

By $B \in \mathcal{B}_{\mathcal{I}[a]}$, $B = \beta_{[a]}$, which implies $B \in \mathcal{B}_\mathcal{I}[a]$. By $B \in \mathcal{B}_{\mathcal{I}[a]}$ and $\mathcal{B}_\mathcal{I}[a] \subseteq \mathcal{I}[a]$, $B \in \mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a])$. Hence, $\mathcal{B}_{\mathcal{I}[a]} \subseteq \mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a])$.

(B2) Let $A \in [0,1]^E$. Suppose that there exists $\beta_a \in \mathcal{B}_\mathcal{I}$ such that $a \wedge A_{[a]} \leq \beta_a$ for each $a \in (0,1]$. By the hereditary property (LI2) of $\mathcal{I}$ and $\beta_a \in \mathcal{I}$, $a \wedge A_{[a]} \in \mathcal{I}$. Since $(E, \mathcal{I})$ is perfect, we have $A \in \mathcal{I}$. By Lemma 4.2, there exists a basis $\beta \in \mathcal{B}_\mathcal{I}$ such that $A \leq \beta$. □

4.5. Remark. In the equality in Theorem 4.4 (B1), $\mathrm{Max}\,(\mathcal{B}_\mathcal{I}[a])$ cannot be replaced by $\mathcal{B}_\mathcal{I}[a]$. This can be seen from the following example.

4.6. Example. Let $E = \{a, b, c\}$ and

$$\mathcal{I}[0.5] = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\},\ \mathcal{I}[1] = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}.$$

Then $(E, \mathcal{I}[0.5])$ and $(E, \mathcal{I}[1])$ are matroids and $\mathcal{I}[1] \subsetneq \mathcal{I}[0.5]$. Let

$$\mathcal{I}[r] = \begin{cases} \mathcal{I}[0.5], & r \in (0, 0.5], \\ \mathcal{I}[1], & r \in (0.5, 1], \end{cases}.$$

and

$$\mathcal{I} = \{A \in [0,1]^E : A_{[r]} \in \mathcal{I}[r] \text{ for each } r \in (0,1]\}.$$

By Theorem 3.1, we know that $(E, \mathcal{I})$ is a $[0,1]$-matroid. Obviously, $(E, \mathcal{I})$ is perfect and closed. By the definition of fuzzy base of a $[0,1]$-matroid, we can deduce that the family of fuzzy bases of $(E, \mathcal{I})$ is composed of the following two fuzzy sets on E.

$$A(x) = \begin{cases} 1, & x = a, \\ 0.5, & x = b, \\ 0, & x = c, \end{cases},\quad B(x) = \begin{cases} 1, & x = a, \\ 0, & x = b, \\ 1, & x = c. \end{cases}$$

However, $\mathcal{B}_\mathcal{I}[1] = \{\{a\}, \{a,c\}\} \supsetneq \{\{a,c\}\} = \mathcal{B}_{\mathcal{I}[1]}$, so $\mathcal{B}_\mathcal{I}[1] \neq \mathcal{B}_{\mathcal{I}[1]}$.

4.7. Theorem. *Let E be a finite set and $\mathcal{B}$ be a non-empty family of fuzzy sets on E. If $\mathcal{B}$ satisfies the following statements:*

- (B1) *For each $a \in (0,1]$, $\mathrm{Max}\,(\mathcal{B}[a])$ is the family of bases for some matroid on E;*
- (B2) *Let $A \in [0,1]^E$. For each $a \in (0,1]$, if $a \wedge A_{[a]} \leq \beta_a$, where $\beta_a \in \mathcal{B}$, then there exists a fuzzy set $\beta \in \mathcal{B}$ such that $A \leq \beta$.*

Then $\mathcal{I}_\mathcal{B} = \mathrm{Low}\,\mathcal{B}$ is a closed perfect $[0,1]$-matroid and $\mathcal{B}_{\mathcal{I}_\mathcal{B}} = \mathcal{B}$.

Proof. (LI1) Since $\mathcal{B}$ is non-empty, obviously, we have $\chi_\emptyset \in \mathcal{I}_\mathcal{B}$.

(LI2) Let $A, B \in [0,1]^E$ and $A \leq B$. If $B \in \mathcal{I}_\mathcal{B}$, then there exists $\beta \in \mathcal{B}$ such that $B \leq \beta$. Hence, $A \leq \beta$, so $A \in \mathcal{I}_\mathcal{B}$.

(LI3) Suppose that $A, B \in \mathcal{I}_{\mathcal{B}}$ and $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$. Now we show that there exists an element $e_b \in F(A,B)$ such that $(b \wedge A_{[b]}) \vee e_b \in \mathcal{I}_{\mathcal{B}}$, where $F(A,B) = \{e_b \in E : A(e) < b \leq B(e)\}$.

If $b = |B|(n) > |A|(n)$ for some $n \in \mathbb{N}$, then $|B|_{[b]} > |A|_{[b]}$. By Lemma 2.4, we have $|B_{[b]}| > |A_{[b]}|$. By $A, B \in \mathcal{I}_{\mathcal{B}}$, there exist $\beta_A, \beta_B \in \mathcal{B}$ such that $A \leq \beta_A$ and $B \leq \beta_B$. It follows that $A_{[b]} \subseteq (\beta_A)_{[b]}$ and $B_{[b]} \subseteq (\beta_B)_{[b]}$. Let $\mathcal{I}[a] = \mathrm{Low}\,(\mathrm{Max}\,(\mathcal{B}[a]))$. By (B1), $\mathcal{I}[a]$ is a matroid for all $a \in (0,1]$. By $A_{[b]}, B_{[b]} \in \mathcal{I}[b]$ and $|B_{[b]}| > |A_{[b]}|$, there exists an element $e \in B_{[b]} - A_{[b]}$ such that $A_{[b]} \cup \{e\} \in \mathcal{I}[b]$. Hence, we obtain $B' \in \mathrm{Max}\,(\mathcal{B}[b])$ satisfying $A_{[b]} \cup \{e\} \subseteq B'$. By $B' \in \mathrm{Max}\,(\mathcal{B}[b])$, there is $\beta \in \mathcal{B}$ such that $B' = \beta_{[b]}$. Thus

$$(b \wedge A_{[b]}) \vee e_b = b \wedge (A_{[b]} \cup \{e\}) \leq b \wedge B' = b \wedge \beta_{[b]} \leq \beta \in \mathcal{I}_{\mathcal{B}}.$$

By the hereditary property of $\mathcal{I}_{\mathcal{B}}$, $(b \wedge A_{[b]}) \vee e_b \in \mathcal{I}_{\mathcal{B}}$. Therefore, $(E, \mathcal{I}_{\mathcal{B}})$ is a $[0,1]$-matroid.

Let $A \in [0,1]^E$. Suppose that $a \wedge A_{[a]} \in \mathcal{I}_{\mathcal{B}}$ for each $a \in (0,1]$. Then we can obtain a fuzzy set β_a satisfying $a \wedge A_{[a]} \leq \beta_a$. By (B2), there exists a fuzzy set $\beta \in \mathcal{B}$ such that $A \leq \beta$. Hence, $A \in \mathcal{I}_{\mathcal{B}}$, so $(E, \mathcal{I}_{\mathcal{B}})$ is perfect. By $\mathcal{I}_{\mathcal{B}} = \mathrm{Low}\,(\mathcal{B})$ and Lemma 4.2, $(E, \mathcal{I}_{\mathcal{B}})$ is closed. Thus $(E, \mathcal{I}_{\mathcal{B}})$ is a closed perfect $[0,1]$-matroid and

$$\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathrm{Max}\,(\mathcal{I}_{\mathcal{B}}) = \mathrm{Max}\,(\mathrm{Low}\,\mathcal{B}) = \mathcal{B}. \qquad \square$$

4.8. Theorem. *Let $\mathcal{I}$ be a closed perfect $[0,1]$-matroid. We have $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$.*

Proof. The proof is trivial and it is omitted. $\square$

4.9. Theorem. *Let $\mathscr{I}$ be the set of all closed perfect $[0,1]$-matroids on E and $\mathscr{B} \subseteq 2^{([0,1]^E)}$ the set of all members satisfying the statements (B1) and (B2). Then there is a one-to-one correspondence between $\mathscr{I}$ and $\mathscr{B}$.*

Proof. By Theorem 4.4, we can define a mapping $f : \mathscr{I} \to \mathscr{B}$ by

$$f(\mathcal{I}) = \mathrm{Max}\,\mathcal{I}.$$

for any $\mathcal{I} \in \mathscr{I}$. On the one hand, suppose that $\mathcal{I}_1, \mathcal{I}_2 \in \mathscr{I}$ with $\mathcal{I}_1 \neq \mathcal{I}_2$, $\mathcal{B}_1 = \mathrm{Max}\,\mathcal{I}_1$ and $\mathcal{B}_2 = \mathrm{Max}\,\mathcal{I}_2$. By Theorem 4.4 and Theorem 4.8, we have

$$\mathcal{I}_1 = \mathcal{I}_{\mathcal{B}_{\mathcal{I}_1}} = \mathcal{I}_{\mathcal{B}_1} \neq \mathcal{I}_{\mathcal{B}_2} = \mathcal{I}_{\mathcal{B}_{\mathcal{I}_2}} = \mathcal{I}_2.$$

Hence, $\mathcal{B}_1 \neq \mathcal{B}_2$, so f is injective. On the other hand, for any $\mathcal{B} \in \mathscr{B}$, by Theorem 4.7, there is a closed perfect $[0,1]$-matroid $\mathcal{I}_{\mathcal{B}} \in \mathscr{I}$ such that $\mathcal{B} = \mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = f(\mathcal{I}_{\mathcal{B}})$. Thus f is surjective. Therefore f is a bijection. $\square$

4.10. Remark. In crisp matroid theory, all bases of a matroid have the same cardinality. However, this is not true for a closed and perfect $[0,1]$-matroid, as can be seen from the following example.

4.11. Example. Let $(E, \mathcal{I})$ be the closed and perfect $[0,1]$-matroid of Example 4.6 and A, B its two fuzzy bases. We can obtain the cardinalities of A and B as follows:

$$|A|(n) = \begin{cases} 1, & n = 0, 1, \\ 0.5, & n = 2, \\ 0, & n \geq 3, \end{cases}, \quad |B|(n) = \begin{cases} 1, & n = 0, 1, 2, \\ 0, & n \geq 3. \end{cases}$$

Obviously $|A| \neq |B|$.

4.12. Remark. One can discuss the cardinalities of fuzzy bases of closed perfect $[0,1]$-matroid in the same way as for those of Goetschel-Voxman fuzzy matroid [2]. The only difference is that the cardinality of a fuzzy set is a fuzzy subset of $\mathbb{N}$.

Acknowledgements. The authors would like to thank the reviewers for their valuable comments and suggestions.

References

[1] Goetschel, R. and Voxman, W. *Fuzzy matroids*, Fuzzy Sets and Systems **27**, 291–302, 1988.

[2] Goetschel, R. and Voxman, W. *Bases of fuzzy matroids*, Fuzzy Sets and Systems **31**, 253–261, 1989.

[3] Goetschel, R. and Voxman, W. *Fuzzy rank functions*, Fuzzy Sets and Systems **42**, 245–258, 1991.

[4] Novak, L. A. *On fuzzy independence set systems*, Fuzzy Sets and Systems **91**, 365–374, 1997.

[5] Novak, L. A. *On Goetschel and Voxman fuzzy matroids*, Fuzzy Sets and Systems **117**, 407–412, 2001.

[6] Oxley, J. G. *Matroid Theory* (Oxford university press, 1992).

[7] Shi, F.-G. *A new approach to the fuzzification of matroids*, Fuzzy Sets and Systems, **160**, 696–705, 2009.

[8] Shi, F.-G. (L, M)*-fuzzy matroids*, Fuzzy Sets and Systems **160**, 2387–2400, 2009.

[9] Whitney, H. *On the abstract properties of linear dependence*, Amer. J. Math. **57**, 509–533, 1935.

[10] Xin, X. and Shi, F.-G. *Rank functions for closed and perfect* $[0, 1]$*-matroids*, Hacettepe J. Math. Stat. **39** (1), 31–39, 2010.

[11] Zadeh, L. A. *A computational approach to fuzzy quantifiers in natural languages*, Comput. Math. Appl. **9**, 149–184, 1983.