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DIRECT LOCAL AND GLOBAL APPROXIMATION RESULTS FOR OPERATORS OF GAMMA TYPE

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Abstract

In this paper some direct local and global approximation results are obtained for the modified Gamma operators defined by A. Izgi and Büyükyazıcı (*Approximation and rate of approximation on unbounded intervals*, Kastamonu Edu. J. Okt. **11** (2), 451–460, 2003 (in Turkish)), and independently by H. Karsli (*Rate of convergence of a new Gamma Type Operator for functions with derivatives of bounded variation*, Math. Comput. Modelling **45** (5-6), 617–624, 2007). Furthermore, a Voronoskaya type theorem is given for these operators.

Keywords: Gamma operators, Positive linear operators, Local and global approximation, Voronovskaya type theorem.

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1. Introduction

Let $X_{\text{loc}}[0,\infty)$ be the class of all complex-valued and locally bounded functions on $[0,\infty)$. For $f \in X_{\text{loc}}[0,\infty)$, it is well-known that the classical Gamma operators G_n (see Lupas and Müller [12]) applied to f are defined as

$$G_n(f;x) = \int_0^\infty g_n(x,u) f\left(\frac{n}{u}\right) du, \tag{1.1}$$

where $g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n$. Several researchers have studied approximation properties of the operators (1.1) and its modifications (see e.g. [18, 2, 17, 16] etc.) in some function

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spaces. One of the important modifications of the Gamma operators is due to Mazhar [13], namely

$$F_n(f;\, x) := \int_0^\infty g_n(x,u)\,du \int_0^\infty g_{n-1}(u,t)f(t)\,dt$$
$$= \frac{(2n)!x^{n+1}}{n!(n-1)!}\int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)\,dt,\ n>1,\ x>0,$$

where $g_n(x,u)$ is the same function that was used by Lupas and Müller [12]. Recently, by using the techniques due to Mazhar, Izgi and Buyukyazici [9], Karsli [10] independently considered the following Gamma type linear and positive operators:

$$\begin{aligned} L_n(f;\, x) &:= \int_0^\infty g_{n+2}(x,u)\,du \int_0^\infty g_n(u,t)f(t)\,dt \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty \frac{t^n}{(x+t)^{2n+4}}f(t)\,dt,\ x>0, \end{aligned} \tag{1.2}$$

and obtained some approximation results. We also mention a very recent paper devoted to the approximation properties of these operators, namely [11]. A bi-dimensional extension of the operators (1.2) can be found in [8].

In addition, if f is right-side continuous at $x=0$, we define

$$L_n(f;\, 0) := f(0)\ (n=1,2,\ldots).$$

This paper is devoted to a study aimed at

(1) Obtaining local approximation results by using the first-second modulus of continuity, and obtaining pointwise convergence results,
(2) Giving global approximation results,
(3) Obtaining a Voronovskaya type theorem for the Gamma type operators defined by (1.2).

Before leaving the introduction section, it is useful to mention that there are some important studies devoted to local and global approximation results for different operators by Ditzian [4], Finta [7], Felten [6], Becker [1], Swiderski [15], and Duman and Özarslan [5, 14].

2. Lemmas

We now present certain results which will be used in the proofs of our main theorems. In [10], the author obtained the following results:

2.1. Lemma. *For any $p = 0,1,2,\ldots,n+2$, we have*

(i) $L_n(t^p;\, x) = \dfrac{(n+p)!(n+2-p)!}{n!(n+2)!}x^p,$

(ii) $L_n((t-x);\, x) = \dfrac{-1}{n+2}x,$

(iii) $L_n((t-x)^2;\, x) = \dfrac{2}{n+2}x^2,$

(iv) $L_n((t-x)^3;\, x) = \dfrac{6x^3}{n(n+2)},$

(v) $L_n((t-x)^4;\, x) = \dfrac{12(n+4)}{(n-1)n(n+2)}x^4,\ (n>1),$

(vi) $L_n((t-x)^6;x) = \dfrac{120(48+23n+n^2)}{(n-3)(n-2)(n-1)n(n+2)}x^4,\ (n>3).$ □

2.2. Lemma. *For all $x \in (0,\infty)$ and n sufficiently large, we have*

$$\lambda_n(x,t) = \int_0^t K_n(x,u)du \le \frac{1}{(x-t)^2}\frac{2}{n+2}x^2,\ 0 \le t < x,$$

and

$$1-\lambda_n(x,z) = \int_z^\infty K_n(x,u)\,du \le \frac{1}{(z-x)^2}\frac{2}{n+2}x^2,\ x < z < \infty,$$

where

$$K_n(x,u) = \begin{cases} \frac{(2n+3)!}{n!(n+2)!}\frac{x^{n+3}u^n}{(x+u)^{2n+4}} & 0 < u < \infty, \\ 0 & u = 0. \end{cases}$$ □

3. Local results

In this section we obtain some pointwise convergence results, and some local approximation results, by using the first and second modulus of continuity. We start with some basic definitions and properties, which will be used in the rest of this paper.

Let $B[0,+\infty)$, $C_B[0,+\infty)$ denote the space of all real valued bounded and continuous bounded functions on $[0,\infty)$, respectively, endowed with the usual supremum norm. Let us recall that the first modulus of continuity of f, denoted by $\omega(f,\delta)$, is defined to be

$$\omega(f,\delta) = \sup_{\substack{|y-x|\le\delta \\ x,y\in[0,+\infty)}} |f(y)-f(x)|,$$

and satisfies the following property:

$$(3.1)\qquad |f(t)-f(x)| \le \left[1+\frac{|t-x|}{\delta}\right]\omega(f,\delta).$$

By $C_B^2[0,\infty)$ we denote the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B(0,\infty)$. Then, the classical Peetre's K-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined respectively by

$$K(f,\delta) := \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\| + \delta\left\|g''\right\|\right\}$$

and

$$\omega_2(f,\delta) := \sup_{0<h\le\delta,\ x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|,$$

where $\delta > 0$. Then, by [3, Theorem 2.4], there exists a constant $C > 0$ such that

$$K(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right).$$

3.1. Theorem. *Take $f \in B[0,\infty)$ satisfying the growth condition $|f(t)| \le M$ for some absolute constant M. If f is continuous at $x_0 \in (0,\infty)$, and right-side continuous at $x_0 = 0$, then we have*

$$\lim_{(n,x)\to(\infty,x_0)} L_n(f;x) = f(x_0).$$

Proof. Assume that

$$x_0 \neq 0, \text{ and } 0 < x_0 - x < \delta.$$

We take $\delta\epsilon(0, x_0)$ such that $|t - x_0| < \delta$ implies $|f(t) - f(x_0)| < \varepsilon/2$. Hence we have from Lemma 2.1 (i) that

$$\begin{aligned}
|L_n(f;\, x) - f(x_0)| &= \left| \left[\left(\int_0^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{\infty} \right) K_n(x,t)\,[f(t) - f(x_0)]\; dt \right] \right| \\
&\leq \left| \int_0^{x_0-\delta} K_n(x,t)\,[f(t) - f(x_0)]\; dt \right| + \left| \int_{x_0-\delta}^{x_0+\delta} K_n(x,t)\,[f(t) - f(x_0)]\; dt \right| \\
&\qquad + \left| \int_{x_0+\delta}^{\infty} K_n(x,t)\,[f(t) - f(x_0)]\, dt \right| \\
&=: |I_1(n,x)| + |I_2(n,x)| + |I_3(n,x)|,
\end{aligned}$$

say. It is easy to see that

$$|I_2(n,x)| \leq \varepsilon/2.$$

Now we consider the term $|I_1(n,x)|$. Since $|f(t)| \leq M$ one can write

$$\left| \int_0^{x_0-\delta} K_n(x,t)\,[f(t) - f(x_0)]\, dt \right| \leq 2M \int_0^{x_0-\delta} K_n(x,t)\, dt.$$

From Lemma 2.2 we know that

$$\int_0^{x_0-\delta} K_n(x,t) dt \leq \frac{1}{(x - x_0 + \delta)^2} \frac{2}{n+2} x^2,$$

since $x_0 - \delta < x$. So we have

$$|I_1(n,x)| \leq 2M \frac{1}{(x - x_0 + \delta)^2} \frac{2}{n+2} x^2,$$

which tends to zero as $(n,x) \to (\infty, x_0)$,

As in the proof of $|I_1(n,x)|$, one has for $|I_3(n,x)|$,

$$|I_3(n,x)| \leq \left| \int_{x_0+\delta}^{\infty} K_n(x,t)\,[f(t) - f(x_0)]\; dt \right| \leq 2M \frac{1}{(x_0 + \delta - x)^2} \frac{2}{n+2} x^2,$$

which also tends to zero as $(n,x) \to (\infty, x_0)$. Since $\varepsilon > 0$ is arbitrary, the proof of the theorem is complete in this case.

If $x_0 = 0$, then the proof of the theorem is similar, so we omit it. □

3.2. Corollary. *If $f \in C_B[0,\infty)$, then*

$$\lim_{n\to\infty} L_n(f;\, x) = f(x),$$

and the convergence is uniform on every compact subsets of $[0,\infty)$. □

3.3. Theorem. *Let the sequence of operators $\{L_n(f;\, x)\}_{n\in\mathbb{N}}$ be defined by* (1.2)*. For all $f \in C_B[0,\infty)$ we have*

$$|L_n(f;\, x) - f(x)| \leq 2\omega\left(f, \left(\sqrt{\frac{2}{n+2}}\right) x\right).$$

Proof. Taking into account the fact that $L_n(1;\, x) = 1$, using the linearity of the operators (1.2) and in view of (3.1), we get for all $f \in C_B[0,\infty)$ that

$$|L_n(f;\, x) - f(x)| \le \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}}\, |f(t) - f(x)|\; dt$$
$$\le \left[1 + \frac{(2n+3)!x^{n+3}}{\delta_n n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}}\, |t-x|\; dt\right]\omega(f,\delta_n).$$

Applying the Cauchy-Schwarz inequality to the right-hand side of the last inequality, we can write

$$|L_n(f;\, x) - f(x)| \le \left[1 + \frac{1}{\delta_n}\left\{\frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}}\, (t-x)^2\; dt\right\}^{\frac{1}{2}}\right]\omega(f,\delta_n).$$

Finally, considering Lemma 2.1 (iii), we conclude that

$$|L_n(f;\, x) - f(x)| \le \left[1 + \frac{1}{\delta_n}\left(\frac{2}{n+2}\, x^2\right)^{\frac{1}{2}}\right]\omega(f,\delta_n).$$

Hence, choosing

$$\delta_n = \left(\sqrt{\frac{2}{n+2}}\right)x$$

we get the desired result. □

Using the first and the second modulus of continuity we have the following local approximation result.

3.4. Theorem. *For any $f \in C_B[0,\infty)$ and for every $x \in [0,\infty)$, $n \in \mathbb{N}$, we have*

$$|L_n(f;\, x) - f(x)| \le C\omega_2\left(f, \sqrt{\frac{2}{n+2}\, x^2 + \left(\frac{x}{n+2}\right)^2}\right) + \omega\left(f, \frac{x}{n+2}\right)$$

for some constant $C > 0$.

Proof. Using the operator $L_n f$ given by (1.2), define a new operator $T_n : C_B[0,\infty) \to C_B[0,\infty)$ as follows:

$$T_n(f;\, x) = L_n(f;\, x) - f\left(x - \frac{x}{n+2}\right) + f(x). \tag{3.2}$$

Then, by Lemma 2.1 (ii) we get

$$T_n(t-x;\, x) = 0. \tag{3.3}$$

Let $g \in C_B^2[0,\infty)$, the space of all functions having a continuous second derivative on $(0,\infty)$, and let $x \in (0,\infty)$. By the Taylor formula we may write:

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\, du,\ \ t \in (0,\infty).$$

Taking into account (3.3) and using (3.2), we get

$$
\begin{aligned}
|T_n(g;x)-g(x)| &= \left|T_n\left(g(t)-g(x);x\right)\right| \\
&= \left|g'(x)T_n((t-x);x)+T_n\Big(\int_x^t (t-u)g''(u)\,du;x\Big)\right| \\
&= \left|T_n\Big(\int_x^t (t-u)g''(u)\,du;x\Big)\right| \\
&= \left|L_n\Big(\int_x^t (t-u)g''(u)\,du;x\Big) - \int_x^{x-\frac{x}{n+2}} \Big(x-\frac{x}{n+2}-u\Big)g''(u)\,du\right|.
\end{aligned}
$$

Since

$$
\left|L_n\Big(\int_x^t (t-u)g''(u)\,du;x\Big)\right| \le \frac{\|g''\|}{2}L_n\big((t-x)^2;x\big)
$$

and

$$
\left|\int_x^{x-\frac{x}{n+2}} (x-\frac{x}{n+2}-u)g''(u)\,du\right| \le \frac{\|g''\|}{2}\left(\frac{x}{n+2}\right)^2
$$

we get

$$
|T_n(g;x)-g(x)| \le \frac{\|g''\|}{2}L_n\left((t-x)^2;x\right)+\frac{\|g''\|}{2}\left(\frac{x}{n+2}\right)^2.
$$

Hence, Lemma 2.1 (iii) implies that

$$
|T_n(g;x)-g(x)| \le \frac{\|g''\|}{2}\Big(\frac{2}{n+2}x^2+\Big(\frac{x}{n+2}\Big)^2\Big). \tag{3.4}
$$

Then, for any $f\in C_B[0,\infty)$, it follows from (3.4) that

$$
\begin{aligned}
|L_n(f;x)-f(x)| &\le |T_n(f-g;x)-(f-g)(x)| \\
&\qquad + |T_n(g;x)-g(x)| + \Big|f\Big(x-\frac{x}{n+2}\Big)-f(x)\Big| \\
&\le 4\,\|f-g\| + 4\Big(\frac{2}{n+2}x^2+\Big(\frac{x}{n+2}\Big)^2\Big)\,\|g''\| \\
&\qquad + \Big|f\Big(x-\frac{x}{n+2}\Big)-f(x)\Big|.
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
|L_n(f;x)-f(x)| &\le 4\Big\{\|f-g\| + \Big(\frac{2}{n+2}x^2+\Big(\frac{x}{n+2}\Big)^2\Big)\|g''\|\Big\} \\
&\qquad + \omega\Big(f,\frac{x}{n+2}\Big) \\
&\le 4K\Big(f,\frac{2}{n+2}x^2+\Big(\frac{x}{n+2}\Big)^2\Big) + \omega\Big(f,\frac{x}{n+2}\Big) \\
&\le C\omega_2\Big(f,\sqrt{\frac{2}{n+2}x^2+\Big(\frac{x}{n+2}\Big)^2}\Big) + \omega\Big(f,\frac{x}{n+2}\Big),
\end{aligned}
$$

which gives the result. □

4. Global results

In this section, we obtain various global results by using certain Lipschitz classes. We start with some basic definitions. Let $p \in \mathbb{N}_0 := \{0, 1, \ldots\}$ and define the weight function μ_p as follows:

$$\mu_0(x) := 1 \text{ and } \mu_p(x) := \frac{1}{1 + x^p} \text{ for } x \geq 0 \text{ and } p \in \mathbb{N}_0 \setminus \{0\}\,. \tag{4.1}$$

Then, we consider the following (weighted) subspace $C_p[0, \infty)$ of $C[0, \infty)$ generated by μ_p:

$$C_p[0, \infty) := \{f \in C[0, \infty) : \mu_p f \text{ is uniformly continuous and bounded on } [0, \infty)\}$$

endowed with the norm

$$\|f\|_p := \sup_{x \in [0,\infty)} \mu_p(x)\, |f(x)| \text{ for } f \in C_p[0, \infty).$$

We also consider the following Lipschitz classes:

$$\begin{aligned}
\Delta_h^2 f(x) &:= f(x + 2h) - 2f(x + h) + f(x),\\
\omega_p^2(f, \delta) &:= \sup_{h \in (0,\delta]} \left\|\Delta_h^2 f\right\|_p,\\
\omega_p^1(f, \delta) &:= \sup \{\mu_p(x)\, |f(t) - f(x)| : |t - x| \leq \delta \text{ and } t, x \geq 0\}\\
\operatorname{Lip}_p^2 \alpha &:= \left\{f \in C_p[0, \infty) : \omega_p^2(f;\, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+\right\},
\end{aligned}$$

where $h > 0$ and $0 < \alpha \leq 2$. From the above it follows that

$$\begin{aligned}
&\lim_{\delta \to 0^+} \omega_p^1(f, \delta) = 0,\\
&\lim_{\delta \to 0^+} \omega_p^2(f, \delta) = 0
\end{aligned} \tag{4.2}$$

for every $C_p[0, \infty)$.

Now we proceed with some auxiliary lemmas which will help to prove our main results.

4.1. Lemma. *For the operators L_n and for fixed p, $(p = 0, 1, 2, \ldots, n + 2)$ there exists a constant $M_p \geq 0$ such that*

$$\mu_p(x) L_n\left(\frac{1}{\mu_p};\, x\right) \leq M_p. \tag{4.3}$$

Furthermore, for all $f \in C_p[0, \infty)$ we have

$$\|L_n(f)\|_p \leq M_p\, \|f\|_p\,, \tag{4.4}$$

which guarantees that L_n maps $C_p[0, \infty)$ into $C_p[0, \infty)$.

Proof. For $p = 0$, (4.3) follows immediately. Now assume that $n + 2 \geq p \geq 1$. By Lemma 2.1 (i), we can find a constant M_p such that

$$\begin{aligned}
\mu_p(x) L_n\Big(\frac{1}{\mu_p};\, x\Big) &= \mu_p(x) \{L_n(e_0;\, x) + L_n(e_p;\, x)\}\\
&= \mu_p(x) \left\{1 + \frac{(n + p)!(n + 2 - p)!}{n!(n + 2)!} x^p\right\}\\
&\leq M_p \mu_p(x) \{x^p + 1\} = M_p,
\end{aligned}$$

where

$$M_p = \max\left\{\sup_n \frac{(n + p)!(n + 2 - p)!}{n!(n + 2)!}, 1\right\}.$$

On the other hand, for all $f \in C_p[0,\infty)$ and every $x \in (0,\infty)$, it follows that

$$\begin{aligned}\mu_p(x)\left|L_n(f;\,x)\right| &\le \mu_p(x)\frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty \frac{t^n}{(x+t)^{2n+4}}\left|f(t)\right|\,dt\\ &= \mu_p(x)\frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty \frac{t^n}{(x+t)^{2n+4}}\left|f(t)\right|\frac{\mu_p(t)}{\mu_p(t)}\,dt\\ &\le \|f\|_p\,\mu_p(x)L_n\left(\frac{1}{\mu_p};\,x\right)\\ &\le M_p\,\|f\|_p\,.\end{aligned}$$

Now taking the supremum over $x \in (0,\infty)$, and taking into account $L_n(f;\,0) = f(0)$, we get (4.4). □

4.2. Lemma. *For the operators L_n, and for fixed p, $(p = 0,1,2,\ldots,n+2)$, there exists a constant $C_p \ge 0$ such that*

$$\mu_p(x)L_n\left(\frac{(t-x)^2}{\mu_p(t)};\,x\right) \le C_p\frac{1}{n+2}x^2.$$

Proof. For $p = 0$ the result follows from Lemma 2.1 (i). Now let $p = 1$. Then, using Lemma 2.1 (iii)-(iv), we can write that

$$\begin{aligned}\mu_1(x)L_n\left(\frac{(t-x)^2}{\mu_1(t)};\,x\right) &= \mu_1(x)\left\{(1+x)L_n\left((t-x)^2;\,x\right) + L_n((t-x)^3;\,x)\right\}\\ &\le \mu_1(x)\left\{(1+x)\frac{2}{n+2}x^2 + \frac{6x^3}{n(n+2)}\right\}\\ &\le C_p\left[\frac{1}{n+2}x^2\right].\end{aligned}$$

Finally, assume that $p \ge 2$. Then, we get from Lemma 2.1 that

$$\begin{aligned}L_n\left(\frac{(t-x)^2}{\mu_p};\,x\right) &= L_n(t^{p+2};\,x) - 2xL_n(t^{p+1};\,x) + x^2L_n(t^p;\,x) + L_n\left((t-x)^2;\,x\right)\\ &= \frac{(n+p+2)!(n-p)!}{n!(n+2)!}x^{p+2} - 2\frac{(n+p+1)!(n+1-p)!}{n!(n+2)!}x^{p+2}\\ &\qquad + \frac{(n+p)!(n+2-p)!}{n!(n+2)!}x^{p+2} + \frac{2}{n+2}x^2\\ &= \frac{1}{n+2}\,x^2\{2 + [(n+p+2)(n+p+1) - 2(n+p+1)(n+1-p)\\ &\qquad +(n+2-p)(n+1-p)]\,\frac{(n+p)!(n-p)!}{n!(n+1)!}x^p\Big\}\\ &= \frac{1}{n+2}x^2\left\{2 + (2n+4p^2+2)\frac{(n+p)!(n-p)!}{2n!(n+1)!}x^p\right\}.\end{aligned}$$

Since the term $(2n+4p^2+2)\dfrac{(n+p)!(n-p)!}{2n!(n+1)!}$ is bounded, there exists a constant $C_p \ge 0$ such that

$$L_n\left(\frac{(t-x)^2}{\mu_p};\,x\right) \le C_p\frac{1}{n+2}x^2\left[1+x^p\right],$$

whence the result. □

Now, for $p \in \mathbb{N}$, consider the space

$$C_p^2[0,\infty) := \left\{ f \in C_p[0,\infty) : f'' \in C_p(0,\infty) \right\}.$$

Then we have the following result.

4.3. Lemma. *For the operators L_n, if $T_n(f;x) := L_n(f;x) - f\left(x - \frac{x}{n+2}\right) + f(x)$, then there exists a positive constant C_p such that, for all $x \in (0,\infty)$ and $n \in \mathbb{N}$, we have*

$$\mu_p(x)\left|T_n\left(g;x\right) - g(x)\right| \le C_p \left\|g''\right\|_p \frac{1}{n+2} x^2.$$

Proof. By the Taylor formula one can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \ \ t \in (0,\infty).$$

Then,

$$\begin{aligned}
|T_n(g;x) - g(x)| &= \left|T_n\left(g(t) - g(x);x\right)\right| \\
&= \left| T_n\bigg(\int_x^t (t-u)g''(u)\,du;x \bigg) \right| \\
&= \bigg| L_n\bigg(\int_x^t (t-u)g''(u)\,du;x \bigg) \\
&\qquad - \int_x^{x-\frac{x}{n+2}} (x - \frac{x}{n+2} - u)g''(u)\,du \bigg|.
\end{aligned}$$

Since

$$\left| \int_x^t (t-u)g''(u)du \right| \le \frac{\|g''\|_p (t-x)^2}{2} \left(\frac{1}{\mu_p(x)} + \frac{1}{\mu_p(t)} \right)$$

and

$$\left| \int_x^{x-\frac{x}{n+2}} (x - \frac{x}{n+2} - u)g''(u)\,du \right| \le \frac{\|g''\|_p}{2\mu_p(x)} \left(\frac{x}{n+2} \right)^2,$$

it follows from Lemma 4.2 that

$$\begin{aligned}
\mu_p(x)\left|T_n(g;x) - g(x)\right| &\le \frac{\|g''\|_p}{2} \left\{ L_n((t-x)^2;x) + \mu_p(x) L_n\left(\frac{(t-x)^2}{\mu_p(t)};x \right) \right\} \\
&\qquad + \frac{\|g''\|_p}{2} \left(\frac{x}{n+2} \right)^2 \\
&\le C_p \left\|g''\right\|_p \frac{1}{n+2} x^2.
\end{aligned}$$

The lemma is proved. □

The next theorem is the main result of this section.

4.4. Theorem. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in (0,\infty)$, there exists an absolute constant $N_p > 0$ such that*

$$\mu_p(x)\left|L_n\left(f;\, x\right) - f(x)\right| \le N_p \omega_p^2\Big(f, x\sqrt{\frac{1}{n+2}}\Big) + \omega_p^1\Big(f;\, \frac{x}{n+2}\Big),$$

where μ_p is the same as in (4.1).

In particular, if $f \in \mathrm{Lip}_p^2\,\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_p(x)\left|L_n\left(f;\, x\right) - f(x)\right| \le N_p\left(\frac{1}{n+2}x^2\right)^{\frac{\alpha}{2}} + \omega_p^1\Big(f;\, \frac{x}{n+2}\Big)$$

holds.

Proof. Let $p \in \mathbb{N}_0$, $f \in C_p[0,\infty)$ and $x \in [0,\infty)$ be fixed. We denote the Steklov means of f by f_h, $h > 0$. Here we recall that

$$f_h(y) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(y+s+t) - f(y+2(s+t))\}\,ds\,dt,$$

for $h > 0$ and $y \ge 0$. It is obvious that

$$f(y) - f_h(y) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(y)\,ds\,dt,$$

which guarantees

$$\|f - f_h\|_p \le \omega_p^2(f;\, h). \tag{4.5}$$

Furthermore, we have

$$f_h''(y) = \frac{1}{h^2}\left(8\Delta_{h/2}^2 f(y) - \Delta_h^2 f(y)\right),$$

which implies

$$\|f_h''\|_p \le \frac{9}{h^2}\omega_p^2(f;\, h). \tag{4.6}$$

Combining (4.5) with (4.6) we conclude that the Steklov means f_h corresponding to $f \in C_p[0,\infty)$ belongs to $C_p^2[0,\infty)$.

Passing to the proof of our main result we see that for any $n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned}|L_n(f;\, x) - f(x)| &\le T_n\left(|f(t) - f_h(t)|\,;\, x\right) + |f(x) - f_h(x)| \\ &\quad + |T_n\left(f_h;\, x\right) - f_h(x)| + \Big|f\Big(x - \frac{x}{n+2}\Big) - f(x)\Big|.\end{aligned}$$

Since $f_h \in C_p^2[0,\infty)$ by the above, it follows from Lemma 4.1 and Lemma 4.3 that

$$\begin{aligned}\mu_p(x)\left|L_n(f;\, x) - f(x)\right| &\le (M_p + 1)\,\|f - f_h\|_p + C_p\,\|f_h''\|_p\,\frac{1}{n+2}x^2 \\ &\quad + \mu_p(x)\Big|f\Big(x - \frac{x}{n+2}\Big) - f(x)\Big|.\end{aligned}$$

By (4.5) and (4.6), the last inequality yields that

$$\mu_p(x)\left|L_n(f;\, x) - f(x)\right| \le N_p\omega_p^2(f;\, h)\Big\{1 + \frac{1}{h^2}\Big(\frac{1}{n+2}x^2\Big)\Big\} + \omega_p^1\Big(f;\, \frac{x}{n+2}\Big).$$

Thus, choosing $h = \sqrt{\frac{1}{n+2}\,x^2}$, the first part of the proof is completed. The remainder of the proof can be easily obtained from the definition of the space $\mathrm{Lip}_p^2\,\alpha$. □

4.5. Theorem. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in (0,\infty)$, there exists an absolute constant $T_p > 0$ such that*

$$\mu_p(x)\left|L_n\left(f;\,x\right) - f(x)\right| \le \left\|f'\right\|_p \left(\sqrt{\frac{2x^2}{n+2}}\left(1+\sqrt{T_p}\right)\right),$$

where μ_p is the same as in (4.1).

Proof. We have

$$f(t) - f(x) = \int_x^t f'(u)\,du,$$

and hence

$$\left|f(t) - f(x)\right| \le \left\|f'\right\|_p \left|t-x\right| \left(\frac{1}{\mu_p(t)} + \frac{1}{\mu_p(x)}\right).$$

So one has

$$\mu_p(x)\left|L_n\left(f;\,x\right) - f(x)\right| \le \left\|f'\right\|_p \left(L_n\left(\left|t-x\right|;\,x\right) + \mu_p(x) L_n\left(\frac{\left|t-x\right|}{\mu_p(t)};\,x\right)\right).$$

It is easy to see that

$$L_n\left(\left|t-x\right|;\,x\right) \le \sqrt{L_n\left(1;\,x\right) L_n\left(\left(t-x\right)^2;\,x\right)} \le \sqrt{\frac{2x^2}{n+2}}$$

and

$$\mu_p(x) L_n\left(\frac{\left|t-x\right|}{\mu_p(t)};\,x\right) \le \sqrt{\mu_p^2(x) L_n\left(\left(t-x\right)^2;\,x\right) L_n\Big(\frac{1}{\mu_p^2(t)};\,x\Big)}.$$

Note that $\mu_p^2(t) \le \mu_{2p}(t)$ and $\mu_p^{-2}(t) \le \mu_{2p}^{-1}(t) + 2\mu_p^{-1}(t)$.

Therefore

$$\begin{aligned}\mu_p^2(x) L_n\Big(\frac{1}{\mu_p^2(t)};\,x\Big) &\le \mu_{2p}(x)\Big[L_n\Big(\frac{1}{\mu_{2p}(t)};\,x\Big) + 2L_n\Big(\frac{1}{\mu_p(t)};\,x\Big)\Big] \\ &\le \mu_{2p}(x) L_n\Big(\frac{1}{\mu_{2p}(t)};\,x\Big) + 2\mu_p(x) L_n\Big(\frac{1}{\mu_p(t)};\,x\Big).\end{aligned}$$

By Lemma 4.1 we get the desired result, namely

$$\mu_p^2(x) L_n\Big(\frac{1}{\mu_p^2(t)};\,x\Big) \le T_p,$$

where T_p is a positive constant depending on p. Thus one has

$$\mu_p(x)\left|L_n\left(f;\,x\right) - f(x)\right| \le \left\|f'\right\|_p \left(\sqrt{\frac{2x^2}{n+2}}\left(1+\sqrt{T_p}\right)\right).$$

□

5. A Voronovskaya type theorem

Now, we shall give a Voronovskaya type theorem for the operators L_n. From Theorem 4.4, Theorem 4.5 and using (4.2) we obtain

5.1. Corollary. *Let $g \in C_p[0,\infty)$. Then*

$$\lim_{n\to\infty} L_n(g;\,x) = g(x),$$

the convergence being uniform on every compact subset of $[0,\infty)$. □

5.2. Theorem. *Let $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f, f' \in C_p[0,\infty)$. We assume that $f''(x)$ exists at a fixed point $x \in (0,\infty)$. Then*

$$\lim_{n\to\infty} n\left[L_n(f;x) - f(x)\right] = -xf'(x) + 2x^2 f''(x).$$

holds.

Proof. By Taylor's formula we have

$$f(t) = f(x) + (t-x) f'(x) + (t-x)^2 \left[\frac{f''(x)}{2} + h(t-x)\right], \tag{5.1}$$

where $h(\cdot) \in C_p[0,\infty)$ with $h(y)$ converging to zero with y. If we use (5.1) in the representation (1.2), we can write the following equality:

$$\left[L_n(f;x) - f(x)\right] = f'(x) L_n((t-x);x) + \frac{f''(x)}{2} L_n\left((t-x)^2;x\right) + L_n\left((t-x)^2 h(t-x);x\right).$$

It is known from Lemma 2.1 that

$$L_n((t-x);x) = \frac{-x}{n+2},\; L_n\left((t-x)^2;x\right) = \frac{2x^2}{n+2}$$

and

$$L_n\left((t-x)^2 h(t-x);x\right) \le \sqrt{L_n\left((t-x)^4;x\right) L_n\left(h^2(t-x);x\right)}.$$

One has from Lemma 2.1 (v) that

$$L_n\left((t-x)^4;x\right) \le A_4 \frac{x^4}{n^2}, (n\to\infty,\; x\in(0,\infty)),$$

where A_4 is a constant. Hence we have

$$n\left[L_n(f;x) - f(x)\right] \le -xf'(x) + 2f''(x)x^2 + x^2\sqrt{A_4}\sqrt{L_n\left(h^2(t-x);x\right)}.$$

The properties of h and Corollary 5.1 imply that

$$\lim_{n\to\infty} \sqrt{L_n\left(h^2(t-x);x\right)} = 0.$$

This is the desired result. □

References

[1] Becker, M. *Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces*, Indiana University Mathematics Journal **27** (1), 127–142, 1978.

[2] Chen, W. and Guo, S. *On the rate of convergence of the Gamma Operator for functions of bounded variation*, Approx. Theory Appl. **1** (5), 85–96, 1985.

[3] DeVore, R. A. and Lorentz, G. G. *Constructive Approximation* (Springer-Verlag, Berlin, 1993).

[4] Ditzian, Z. *Direct estimate for Bernstein polynomials*, J. Approx. Theory **79**, 165–166, 1994.

[5] Duman, O. and Özarslan, M. A. *Global approximation results for modified Szasz-Mirakjan operators*, Taiwanese J. Math. (accepted).

[6] Felten, M. *Local and global approximation theorems for positive linear operators*, J. Approx. Theory **94**, 396–419, 1998.

[7] Finta, Z. *Direct local and global approximation theorems for some linear positive operators*, Analysis in Theory and Applications **20** (4), 307–322, 2004.

[8] Izgi, A. *Order of approximation of functions of two variables by a new type Gamma operators*, General Mathematics **17** (1), 23–32, 2009.

[9] Izgi, A. and Büyükyazici, I. *Approximation and rate of approximation on unbounded intervals*, Kastamonu Edu. Journal Okt. **11** (2), 451–460, 2003 (in Turkish).

[10] Karsli, H. *Rate of convergence of a new Gamma Type Operators for functions with derivatives of bounded variation*, Math. Comput. Modelling **45** (5-6), 617–624, 2007.
[11] Karsli, H., Gupta, V. and Izgi, A. *Rate of pointwise convergence of a new kind of gamma operators for functions of bounded variation*, Appl. Math. Letters **22** (4), 505–510, 2009.
[12] Lupas, A. and Müller, M. *Approximationseigenschaften der Gammaoperatoren*, Math. Zeitschr. **98**, 208–226, 1967.
[13] Mazhar, S. M. *Approximation by positive operators on infinite intervals*, Mathematica Balkanica **5** (2), 99–104, 1991.
[14] Özarslan, M. A. and Duman, O. *Local approximation results for Szasz-Mirakjan type operators*, Arch. Math. (Basel) **90**, 144–149, 2008.
[15] Swiderski, T. *Global approximation theorems for the generalized modified Szasz- Mirakyan operators in polynomial weighted spaces*, Demonstratio Math. **36** (2), 383–392, 2003.
[16] Totik, V. *The Gamma operators in L_p spaces*, Publ. Math. **32**, 43–55, 1985.
[17] Xu, X.-W. and Wang, J. Y. *Approximation properties of modified Gamma operators*, J. Math. Anal. Appl. **332**, 798–813, 2007.
[18] Zeng, X.-M. *Approximation properties of gamma operators*, J. Math. Anal. Appl. **311** (2), 389–401, 2005.