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CLOSED FORM REPRESENTATIONS OF HARMONIC SUMS

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Abstract

We develop a master theorem from which we are able to represent infinite sums of harmonic numbers and binomial coefficients in both integral and closed form. The new results extend known existing results in the published literature.

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1. Introduction

There are a number of results representing harmonic number series in closed form. A result due to Euler [8] is

$$2\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^q}=(q+2)\zeta(q+1)-\sum_{r=1}^{q-2}\zeta(r+1)\zeta(q-r),$$

and recently Georghiou and Philippou [10] gave the representation

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^{2q+1}}=\frac{1}{2}\sum_{r=2}^{2q}(-1)^r\zeta(r)\zeta(2q-r+2),\ q\in\mathbb{N}:=\{1,2,3,\dots\}.$$

Further work in the summation of harmonic numbers and binomial coefficients has also been done by Flajolet and Salvy [9], and Basu [5]. The papers [1, 2, 3, 6, 11, 12, 13, 14], and [15] also investigate various representations of binomial sums, zeta functions and mathematical constants in simpler form by the use of the Beta function and other techniques. The paper of Chu and DeDonno [7], also has some innovative results associated

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with harmonic numbers. Sofo [15] and Cloitre, as reported in [16], gave results for related sums of the form

$$\sum_{n\geq 1} \frac{\alpha^n \ H_n^{(1)}}{\binom{n+k}{k}} \text{ for } \alpha = 1 \text{ and } \frac{1}{2},$$

and in this paper we obtain both integral and closed form representations for sums of the type

$$\sum_{n\geq 1} \frac{H_n^{(1)}}{n^4 \binom{n+k}{k}^2},$$

where the generalized harmonic number in power α is defined as $H_n^{(\alpha)} = \sum\limits_{r=1}^{n} \frac{1}{r^\alpha}$. The n^{th} harmonic number is

$$H_n^{(1)} = H_n = \int\limits_{t=0}^{1} \frac{1-t^n}{1-t}\, dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log(n) \right) = -\psi(1) \approx 0.57721566490153286 \cdots\cdots .$$

The Riemann Zeta function is defined by

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z}, \ \Re(z) > 1.$$

2. The main theorem

2.1. Theorem. *Take $|t| \leq 1$, m a positive integer, and a, b, c, j, k and $l \geq 0$. Then*

$$\sum_{n\geq 1} \frac{t^n Q^{(m)}(a,j)}{n^4 \binom{bn+k}{k}\binom{cn+l}{l}} = -abc \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k (1-z)^l}{xyz} \times [\ln(1-x)]^m \ln(1 - tx^a y^b z^c)\, dx\, dy\, dz \tag{2.1}$$

for $|tx^a y^b z^c| < 1$ and

$$Q^{(m)}(a,j) = \frac{d^m}{dj^m} \left(\binom{an+j}{j}^{-1} \right). \tag{2.2}$$

Proof. Consider

$$\begin{aligned}
&\sum_{n\geq 1} \frac{t^n}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} \\
&\qquad = \sum_{n\geq 1} \frac{t^n \Gamma(j+1)\Gamma(k+1)\Gamma(l+1)\Gamma(an+1)\Gamma(bn+1)\Gamma(cn+1)}{n^4 \Gamma(an+j+1)\Gamma(bn+k+1)\Gamma(cn+l+1)} \\
&\qquad = abc \sum_{n\geq 1} \frac{t^n \Gamma(j+1)\Gamma(k+1)\Gamma(l+1)\Gamma(an)\Gamma(bn)\Gamma(cn)}{n\Gamma(an+j+1)\Gamma(bn+k+1)\Gamma(cn+l+1)} \\
&\qquad = abc \sum_{n\geq 1} \frac{t^n}{n} B(bn, k+1) B(cn, l+1) \int_0^1 x^{an-1} (1-x)^j\, dx,
\end{aligned}$$

where the Beta function is given by

$$B(s,z)=\int_0^1 w^{s-1}(1-w)^{z-1}dw=\frac{\Gamma(s)\Gamma(w)}{\Gamma(s+w)}$$

for $\Re(s)>0$ and $\Re(z)>0$, and the Gamma function by

$$\Gamma(z)=\int_0^\infty w^{z-1}e^{-w}$$

for $\Re(z)>0$.

Now we can differentiate m times with respect to the parameter j, and by (2.2),

$$\begin{aligned}
&\sum_{n\geq 1}\frac{t^nQ^{(m)}(a,j)}{n^4\binom{bn+k}{k}\binom{cn+l}{l}}\\
&=abc\sum_{n\geq 1}\frac{t^n}{n}B(bn,k+1)B(cn,l+1)\int_0^1 x^{an-1}(1-x)^j[\ln(1-x)]^m\,dx\\
&=abc\int_0^1\frac{(1-x)^j[\ln(1-x)]^m}{x}\sum_{n\geq 1}\frac{t^nx^{an}}{n}\int_0^1 y^{bn-1}(1-y)^k\,dx\,dy\cdot B(cn,l+1)\\
&=abc\int_0^1\int_0^1\frac{(1-x)^j[\ln(1-x)]^m(1-y)^k}{xy}\sum_{n\geq 1}\frac{t^n(x^ay^b)^n}{n}\int_0^1 z^{cn-1}(1-z)^l\,dx\,dy\,dz\\
&=abc\int_0^1\int_0^1\int_0^1\frac{(1-x)^j(1-y)^k(1-z)^l}{xyz}[\ln(1-x)]^m\sum_{n\geq 1}\frac{(tx^ay^bz^c)^n}{n}\,dx\,dy\,dz\\
&=-abc\int_0^1\int_0^1\int_0^1\frac{(1-x)^j(1-y)^k(1-z)^l}{xyz}[\ln(1-x)]^m\ln(1-tx^ay^bz^c)\,dx\,dy\,dz
\end{aligned}$$

for $|tx^ay^bz^c|<1$. □

A number of very interesting corollaries follow which expose new sums of harmonic numbers in closed form in terms of rational multiples of Zeta functions. The new harmonic number infinite sums or integrals cannot easily be analytically evaluated by standard mathematical computer packages, for example Mathematica [17]; however our new representations make them easier to calculate.

2.2. Corollary. *Let $a=b=c=1$, $t=1$, $m=1$, $j=0=l$, and $k\geq 1$. Then*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+k}{k}}=\int_0^1\int_0^1\int_0^1\frac{(1-y)^k\ln(1-x)\ln(1-xyz)}{xyz}\,dx\,dy\,dz \tag{2.3}$$

$$\begin{aligned}
&=3\zeta(5)-\zeta(3)\zeta(2)-\frac{5}{4}H_k^{(1)}\zeta(4)+[(H_k^{(1)})^2+H_k^{(2)}]\zeta(3)\\
&\quad-6[(H_k^{(1)})^3+3H_k^{(1)}H_k^{(2)}+2H_k^{(3)}]\zeta(2)\\
&\quad-\frac{1}{2}\sum_{r=1}^{k}\frac{(-1)^{r+1}}{r^3}(\tbinom{k}{r})[(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}].
\end{aligned} \tag{2.4}$$

Proof. First note that from (2.2), $Q'(1,0)=H_n^{(1)}$ and by expansion,

$$\begin{aligned}
\sum_{n\geq 1}\frac{H_n^{(1)}k!}{n^4\prod_{r=1}^k(n+r)}&=\sum_{n\geq 1}\frac{H_n^{(1)}k!}{n^4(n+1)_{k+1}}\\
&=\sum_{n\geq 1}\frac{H_n^{(1)}k!}{n^4}\sum_{r=1}^k\frac{A_r}{n+r},
\end{aligned} \tag{2.5}$$

where

$$A_r = \lim_{n\to(-r)} \left\{ \frac{n+r}{(n+1)_{k+1}} \right\} = \frac{(-1)^{r+1}}{(k-r)!(r-1)!} = \frac{(-1)^{r+1} r}{k!} \binom{k}{r}, \tag{2.6}$$

and $(\alpha)_r$ is Pochhammer's symbol defined by

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1),$$

$r > 0$, $(\alpha)_0 = 1$. From (2.5),

$$\sum_{n\geq 1} \frac{H_n^{(1)}}{n^4} \sum_{r=1}^{k} \frac{(-1)^{r+1} r}{k!} \binom{k}{r} = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \sum_{n\geq 1} \frac{H_n^{(1)}}{n^4(n+r)}. \tag{2.7}$$

Again, by partial fraction decomposition,

$$\begin{aligned} \sum_{n\geq 1} \frac{H_n^{(1)}}{n^4(n+r)} &= \sum_{n\geq 1} H_n^{(1)} \left[\frac{1}{r^4(n+r)} - \frac{1}{r^4 n} + \frac{1}{r^3 n^2} - \frac{1}{r^2 n^3} + \frac{1}{r n^4} \right] \\ &= \frac{3\zeta(5)}{r} - \frac{\zeta(3)\zeta(2)}{r} - \frac{5\zeta(4)}{4r^2} + \frac{2\zeta(3)}{r^3} \\ &\qquad - \frac{\zeta(2)}{r^4} - \frac{1}{2r^4} \left\{ (H_{r-1}^{(1)})^2 + H_{r-1}^{(2)} \right\}. \end{aligned} \tag{2.8}$$

From (2.7) and using (2.8), we have

$$\begin{aligned} \sum_{n\geq 1} \frac{H_n^{(1)}}{n^4 \binom{n+k}{k}} = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \Bigg[& \frac{3\zeta(5)}{r} - \frac{\zeta(3)\zeta(2)}{r} - \frac{5\zeta(4)}{4r^2} \\ & + \frac{2\zeta(3)}{r^3} - \frac{\zeta(2)}{r^4} - \frac{1}{2r^4} \left\{ (H_{r-1}^{(1)})^2 + H_{r-1}^{(2)} \right\} \Bigg], \end{aligned}$$

which simplifies to (2.4) after some algebraic manipulations.

The integral (2.3) now follows from (2.1). □

The degenerate case $k = 0$ reduces to the well known result in [16]:

$$\begin{aligned} \sum_{n\geq 1} \frac{H_n^{(1)}}{n^4} &= 3\zeta(5) - \zeta(3)\zeta(2) \\ &= \int_0^1 \int_0^1 \int_0^1 \frac{\ln(1-x)\ln(1-xyz)}{xyz} \, dx\, dy\, dz \\ &= -\int_0^1 \int_0^1 \frac{\ln(1-x)\psi''(xz)}{xz} \, dx\, dz \\ &= -\int_0^1 \frac{\ln(1-x)\psi'''(x)}{x} \, dx, \end{aligned} \tag{2.9}$$

where the polygamma functions $\psi^{(k)}(z)$, $k \in \mathbb{N}$, are defined by

$$\psi^{(k)}(z) := \frac{d^{k+1}}{dz^{k+1}} \log(\Gamma(z)) = \frac{d^k}{dz^k} \left(\frac{\Gamma'(z)}{\Gamma(z)} \right) = -\int_0^1 \frac{t^{z-1}[\log(t)]^k}{1-t} \, dt$$

and $\psi^{(0)}(z) = \Psi(z)$ denotes the Psi or digamma function.

Corollary 2.3 now follows:

2.3. Corollary. *Let $a = b = 1$, $t = 1$, $m = 1$, $c = 4$, $l = 1$, $j = 0$ and $k \geq 1$. Then*

$$\sum_{n\geq 1} \frac{H_n^{(1)}}{n^4(4n+1)\binom{n+k}{k}} = 4\int_0^1\int_0^1\int_0^1 \frac{(1-y)^k \ln(1-x)\ln(1-xyz^4)}{xyz}\,dx\,dy\,dz \tag{2.10}$$

$$\begin{aligned} &= 3\zeta(5) - \zeta(3)\zeta(2) - \frac{5}{4}\left(4 + H_k^{(1)}\right)\zeta(4) \\ &\quad + \left(32 + 8H_k^{(1)} + \left(H_k^{(1)}\right)^2 + H_k^{(2)}\right)\zeta(3) \\ &\quad + \Bigg[112kB\left(\frac{3}{4},k\right) - 64 - 16H_k^{(1)} - 2\left(H_k^{(1)}\right)^2 - \frac{\left(H_k^{(1)}\right)^3}{6} \\ &\quad - 2H_k^{(2)} - \frac{H_k^{(1)}H_k^{(2)}}{2} - \frac{H_k^{(3)}}{3}\Bigg]\zeta(2) \\ &\quad + 32kB\left(\frac{3}{4},k\right)\left[8G - 3(\pi\ln 2 + 3(\ln 2)^2)\right] \\ &\quad + \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\frac{\left(H_{r-1}^{(1)}\right)^2 + H_{r-1}^{(2)}}{2r^3(4r-1)}, \end{aligned} \tag{2.11}$$

where G is the Catalan constant defined by

$$G = \frac{1}{2}\int_0^1 K(s)ds = \sum_{r=1}^{\infty}\frac{(-1)^r}{(2r+1)^2} \sim 0.915965\cdots\cdots.$$

Here $B(\cdot,\cdot)$ is the classical Beta function and $K(s)$ is the complete elliptic integral of the first kind given by

$$K(s) = \int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-s^2\sin^2 t}}.$$

Proof. Consider

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(4n+1)\binom{n+k}{k}} = \sum_{n\geq 1}\frac{H_n^{(1)}k!}{n^4(4n+1)(n+1)_{k+1}} = \sum_{n\geq 1}\frac{k!H_n^{(1)}}{n^4(4n+1)}\sum_{r=1}^{k}\frac{A_r}{n+r},$$

where A_r is given by (2.6). Hence,

$$\begin{aligned} \sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(4n+1)\binom{n+k}{k}} &= \sum_{n\geq 1}\frac{k!H_n^{(1)}}{n^4(4n+1)}\sum_{r=1}^{k}\frac{(-1)^{r+1}r\binom{k}{r}}{n+r} \\ &= \sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\sum_{n\geq 1}\frac{H_n}{n^4(4n+1)(n+r)}, \end{aligned} \tag{2.12}$$

by interchanging the summations.

From

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(4n+1)(n+r)}$$
$$= \sum_{n\geq 1} H_n^{(1)}\Big[\frac{1}{rn^4}-\frac{4r+1}{r^2\cdot n^3}+\frac{16r^2+4r+1}{r^3\cdot n^2}$$
$$-\frac{4(16r^2+4r+1)}{r^3(4n+1)(n+r)}-\frac{(4r+1)(16r^2+1)}{r^3 n(4n+1)(n+r)}\Big] \tag{2.13}$$
$$=\frac{1}{72r^4(4r-1)}\Big[216r^3(4r-1)\zeta(5)-90r^2(16r^2-1)\zeta(4)$$
$$-72r^3(4r-1)\zeta(3)\zeta(2)+144r(64r^3-1)\zeta(3)$$
$$+72(192r^4+1)\zeta(2)+73728r^4G$$
$$-9216r^4(\pi\ln 8+(\ln 8)^2)+36(H_{r-1}^{(1)})^2+36H_{r-1}^{(2)}\Big].$$

Now substituting (2.13) into (2.12), we have

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(4n+1)(n+r)}$$
$$=\sum_{r=1}^{k}\frac{(-1)^{r+1}r\binom{k}{r}}{72r^4(4r-1)}\Big[216r^3(4r-1)\zeta(5)-90r^2(16r^2-1)\zeta(4)$$
$$-72r^3(4r-1)\zeta(3)\zeta(2)+144r(64r^3-1)\zeta(3)+72(192r^4+1)\zeta(2)$$
$$+73728r^4G-9216r^4(\pi\ln 8+(\ln 8)^2)+36(H_{r-1}^{(1)})^2+36H_{r-1}^{(2)}\Big],$$

and after much algebraic simplification, and with the aid of 'Mathematica', we obtain (2.11).

The degenerate case, $k=0$, gives us the result:

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(4n+1)}$$
$$=\sum_{n\geq 1} H_n[\frac{1}{n^4}-\frac{4}{n^3}+\frac{16}{n^2}-\frac{64}{n(4n+1)}]$$
$$=3\zeta(5)-\zeta(3)\zeta(2)-5\zeta(4)+32\zeta(3)+48\zeta(2)+256G-96\ln(2)[\pi+3\ln(2)]$$
$$=4\int_0^1\int_0^1\int_0^1\frac{(1-z)\ln(1-x)\ln(1-xyz^4)}{xyz}\,dx\,dy\,dz.$$

□

The next corollary investigates the summation of terms which include the reciprocal of squared binomial coefficients.

2.4. Corollary. *Let $a=b=c=1$, $t=1$, $m=1$, $j=0$ and $l=k\geq 1$. Then:*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+k}{k}^2} \tag{2.14}$$
$$= \int_0^1\int_0^1\int_0^1 \frac{((1-y)(1-z))^k \ln(1-x)\ln(1-xyz^4)}{xyz}\,dx\,dy\,dz$$

$$= 3\zeta(5) - \zeta(3)\zeta(2) + \sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\Bigg[-\frac{5}{2r^2}\left(\frac{1}{r}+H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\zeta(4) \tag{2.15}$$
$$+\frac{1}{r^3}\left(\frac{7}{r}+4\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(3)$$
$$+\frac{1}{r^4}\left(H_{r-1}^{(1)}-\frac{4}{r}-2\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(2)$$
$$-\frac{1}{r^4}\Bigg(H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}$$
$$+\left(\frac{2}{r}+H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\left\{\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right\}\Bigg)\Bigg].$$

Proof. Consider

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+k}{k}^2} = \sum_{n\geq 1}\frac{(k!)^2H_n^{(1)}}{n^4((n+1)_{k+1})^2} = \sum_{n\geq 1}\frac{(k!)^2H_n^{(1)}}{n^4}\sum_{r=1}^{k}\left[\frac{A_r}{n+r}+\frac{B_r}{(n+r)^2}\right]$$

where

$$B_r = \lim_{n\to(-r)}\left[\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2}\right] = \left(\frac{r}{k!}\binom{k}{r}\right)^2$$

and

$$A_r = \lim_{n\to(-r)}\frac{d}{dn}\left[\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2}\right] = -2\left(\frac{r}{k!}\binom{k}{r}\right)^2\left[H_{k-r}^{(1)}-H_{r-1}^{(1)}\right];\ k\geq 1.$$

Now

$$\sum_{n\geq 1}\frac{(k!)^2H_n^{(1)}}{n^4}\sum_{r=1}^{k}\left(\frac{r}{k!}\binom{k}{r}\right)^2\left[\frac{1}{(n+r)^2}-\frac{2(H_{k-r}^{(1)}-H_{r-1}^{(1)})}{n+r}\right]$$
$$= \sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(n+r)^2} \tag{2.16}$$
$$+2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(n+r)}.$$

By further partial fraction decomposition and after many algebraic manipulations and simplifications, we can write

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4(n+r)^2} = \frac{3\zeta(5)}{r^2}-\frac{\zeta(3)\zeta(2)}{r}-\frac{5\zeta(4)}{2r^3}+\frac{7\zeta(3)}{r^4}-\frac{4\zeta(2)}{r^5}+\frac{H_{r-1}^{(1)}\zeta(2)}{r^4} \tag{2.17}$$
$$-\frac{1}{r^4}\left\{H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}\right\}-\frac{2}{r^5}\left\{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}\right\}.$$

Further, using (2.8) and (2.17) in (2.16), we may write

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+k}{k}^2}=\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\Bigg[\frac{3\zeta(5)}{r^2}-\frac{\zeta(3)\zeta(2)}{r}-\frac{5\zeta(4)}{2r^3}+\frac{7\zeta(3)}{r^4}-\frac{4\zeta(2)}{r^5}$$
$$+\frac{H_{r-1}^{(1)}\zeta(2)}{r^4}-\frac{1}{r^4}\left\{H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}\right\}$$
$$-\frac{2}{r^5}\left\{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}\right\}\Bigg]+2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\Bigg[\frac{3\zeta(5)}{r}$$
$$-\frac{\zeta(3)\zeta(2)}{r}-\frac{5\zeta(4)}{4r^2}+\frac{2\zeta(3)}{r^3}-\frac{\zeta(2)}{r^4}$$
$$-\frac{1}{2r^4}\left\{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}\right\}\Bigg]\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}.$$

Collecting the Zeta functions, we have

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+k}{k}^2}=\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\Bigg[\frac{3}{r}\left(\frac{1}{r}+2\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(5)$$
$$-\frac{1}{r}\left(\frac{1}{r}+2\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(3)\zeta(2)$$
$$-\frac{5}{2r^2}\left(\frac{1}{r}+H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\zeta(4)$$
$$+\frac{1}{r^3}\left(\frac{7}{r}+4\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(3)$$
$$+\frac{1}{r^4}\left(H_{r-1}^{(1)}-\frac{4}{r}-2\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\zeta(2)$$
$$-\frac{1}{r^4}\left(H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}+\frac{2}{r}\left\{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}\right\}\right.$$
$$\left.+\left\{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}\right\}\left\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\right\}\right)\Bigg],$$

which reduces to (2.15) upon noting that

$$\sum_{r=1}^{k}\binom{k}{r}^2[\alpha+2\alpha r\{H_{r-1}^{(1)}-H_{k-r}^{(1)}\}]=\alpha,\ \ \alpha\neq 0.$$

For the degenerate case, $k=0$, then (2.14) reduces to (2.9), and as an example for $k=5$

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^4\binom{n+5}{5}^2}=3\zeta(5)-\zeta(3)\zeta(2)-\frac{137}{24}\zeta(4)+\frac{311383}{3600}\zeta(3)$$
$$+\frac{2296919}{108000}\zeta(2)-\frac{642641}{4800}.$$

□

2.5. Remark. It may be stated that (2.4), (2.11) and (2.15) are new results, but not necessarily "elegant" representations. However, (2.3), (2.10) and (2.14) cannot be evaluated, so far, by any computer mathematical package. The integral representations are extremely useful in determining bounds of the various harmonic number sums. Bounds on harmonic numbers have recently been given in an interesting paper by Alzer [4].

3. Conclusion

We have established recursive closed form and integral representation of series involving harmonic numbers and reciprocal binomial coefficients. None of the series, in their general form, can be evaluated using standard mathematical computer packages. The closed form representation of the series are new and add value to the large existing literature on harmonic sums.

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