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A SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE HURWITZ-LERCH ZETA FUNCTION

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Abstract

Making use of a convolution operator involving the Hurwitz-Lerch Zeta function, we introduce a new class of analytic functions $\mathcal{PT}(\lambda, \alpha, \beta)$ defined in the open unit disc, and investigate its various characteristics. Further we obtained distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{PT}(\lambda, \alpha, \beta)$.

Keywords: Analytic, Univalent, Starlikeness, Convexity, Hadamard product (convolution).

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C};\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the *Hadamard product* (or *convolution*) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k,\ z \in U. \tag{1.2}$$

We now recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ (cf., e.g., [16]) defined by

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \quad (a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ s \in \mathbb{C},\ \Re(s) > 1 \text{ and } |z| = 1) \tag{1.3}$$

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where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$, $(\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, \ldots\})$; $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and López [5], Garg *et al.* [7], Lin and Srivastava [10], Lin *et al.* [11], and others.

In 2007, Srivastava and Attiya [15] (see also Riaducanu and Srivastava [13], Prajapat and Goyal [12]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b} * f(z), \tag{1.4}$$

$(z \in U;\ b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ \mu \in \mathbb{C};\ f \in \mathcal{A})$, where, for convenience,

$$\mathcal{G}_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}]\ (z \in U). \tag{1.5}$$

We recall here the following relationships (given earlier in [12, 13]) which follow easily by using (1.1), (1.4) and (1.5)

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{k=2}^{\infty} C_k(b, \mu) a_k z^k, \tag{1.6}$$

where

$$C_k(b, \mu) = \left(\frac{1+b}{k+b} \right)^{\mu}, \tag{1.7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\};\ \mu \in \mathbb{C}$.

(1) For $\mu = 0$,

$$\mathcal{J}_{0,b} f(z) := f(z). \tag{1.8}$$

(2) For $\mu = 1,\ b = 0$,

$$\mathcal{J}_{1,0} f(z) := \int_0^z \frac{f(t)}{t}\, dt := \mathcal{L}_b f(z). \tag{1.9}$$

(3) For $\mu = 1$ and $b = \nu$ $(\nu > -1)$,

$$\begin{aligned} \mathcal{J}_{1,\nu} f(z) &:= \frac{1+\nu}{z^{\nu}} \int_0^z t^{\nu-1} f(t)\, dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{1+\nu}{k+\nu} \right) a_k z^k := \mathcal{F}_{\nu} f(z). \end{aligned} \tag{1.10}$$

(4) For $\mu = \sigma (\sigma > 0)$ and $b = 1$,

$$\mathcal{J}_{\sigma,1} f(z) := z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\sigma} a_k z^k := \mathcal{I}^{\sigma} f(z), \tag{1.11}$$

where $\mathcal{L}_b(f)$ and $\mathcal{F}_{\nu}$ are the integral operators introduced by Alexander [1] and Bernardi [3], respectively, and $\mathcal{I}^{\sigma}(f)$ is the Jung-Kim-Srivastava integral operator [8] closely related to some multiplier transformations studied by Flett [6].

Making use of the operator $\mathcal{J}_{\mu,b}$, we introduce a new subclass of analytic functions with negative coefficients, and discuss some standard properties of geometric function theory in relation to this generalized function class.

For $\lambda \geq 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $\mathcal{P}(\lambda, \alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{\mathcal{J}_{\mu}^{b,\lambda} f(z) - 1}{2\gamma(\mathcal{J}_{\mu}^{b,\lambda} f(z) - \alpha) - (\mathcal{J}_{\mu}^{b,\lambda} f(z) - 1)} \right| < \beta, \tag{1.12}$$

where

$$\mathcal{J}_{\mu}^{b,\lambda} f(z) = (1-\lambda)\frac{\mathcal{J}_{\mu,b} f(z)}{z} + \lambda(\mathcal{J}_{\mu,b} f(z))', \tag{1.13}$$

$0 < \gamma \leq 1$, and $\mathcal{J}_{\mu}^{b} f(z)$ is given by (1.6). We further let

$$\mathcal{PT}(\lambda, \alpha, \beta) = \mathcal{P}(\lambda, \alpha, \beta) \cap T,$$

where

$$T := \left\{ f \in A : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \ z \in U \right\} \tag{1.14}$$

is a subclass of A introduced and studied by Silverman [14].

Furthermore, we note that by suitably specializing the values of α, β, γ and λ, the class $\mathcal{PT}(\lambda, \alpha, \beta)$ and the above subclasses reduce to the various subclasses introduced and studied in the literature, for example see [2, 9].

In the following section we obtain coefficient estimates and extreme points for the class $\mathcal{PT}(\lambda, \alpha, \beta)$.

2. Coefficient bounds

2.1. Theorem. *Let the function f be defined by* (1.14). *Then $f \in \mathcal{PT}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{k=2}^{\infty} (1 + \lambda(k-1))[1 + \beta(2\gamma - 1)] \, |C_k(b, \mu)| a_k \leq 2\beta\gamma(1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1 + \lambda(k-1))[1 + \beta(2\gamma - 1)] \, |C_k(b, \mu)|} z^k, \ k \geq 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $|z|$,

$$\begin{aligned}
&\left|\mathcal{J}_{\mu}^{b,\lambda} f(z) - 1\right| - \beta\left|2\gamma(\mathcal{J}_{\mu}^{b,\lambda} f(z) - \alpha) - (\mathcal{J}_{\mu}^{b,\lambda} f(z) - 1)\right| \\
&\quad = \left| - \sum_{k=2}^{\infty} (1 + \lambda(k-1)) C_k(b, \mu) a_k z^{k-1} \right| \\
&\qquad\qquad - \beta \left| 2\gamma(1-\alpha) - \sum_{k=2}^{\infty} (1 + \lambda(k-1))(2\gamma - 1) C_k(b, \mu) a_k z^{k-1} \right| \\
&\quad \leq \sum_{k=2}^{\infty} (1 + \lambda(k-1)) \, |C_k(b, \mu)| a_k - 2\beta\gamma(1-\alpha) \\
&\qquad\qquad + \sum_{k=2}^{\infty} (1 + \lambda(k-1))\beta(2\gamma - 1) \, |C_k(b, \mu)| a_k \\
&\quad = \sum_{k=2}^{\infty} (1 + \lambda(k-1))[1 + \beta(2\gamma - 1)] \, |C_k(b, \mu)| a_k - 2\beta\gamma(1-\alpha) \\
&\quad \leq 0, \ \text{by (2.1)}.
\end{aligned}$$

Hence, by the maximum modulus theorem and (1.12), $f \in \mathcal{PT}(\lambda, \alpha, \beta)$.

To prove the converse, assume that

$$\begin{aligned}&\left|\frac{\mathcal{J}_\mu^{b,\lambda} f(z) - 1}{2\gamma(\mathcal{J}_\mu^{b,\lambda} f(z) - \alpha) - (\mathcal{J}_\mu^{b,\lambda} f(z)) - 1)}\right| \\ &\qquad = \left|\frac{-\sum_{k=2}^{\infty}(1+\lambda(k-1))C_k(b,\mu)a_k z^{k-1}}{2\gamma(1-\alpha) - \sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)C_k(b,\mu)a_k z^{k-1}}\right| \\ &\qquad \le \beta, \ z \in U.\end{aligned}$$

Or, equivalently,

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty}(1+\lambda(k-1))|C_k(b,\mu)|a_k z^{k-1}}{2\gamma(1-\alpha) - \sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)|C_k(b,\mu)|a_k z^{k-1}}\right\} < \beta. \tag{2.3}$$

Since $\operatorname{Re}(z) \le |z|$ for all z, choose values of z on the real axis so that $\mathcal{J}_\mu^{b,\lambda} f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1$ through real values, we obtain the desired inequality (2.1). □

2.2. Corollary. *If $f(z)$ of the form* (1.14) *is in $\mathcal{PT}(\lambda, \alpha, \beta)$, then*

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}, \ k \ge 2, \tag{2.4}$$

with equality only for functions of the form (2.2).

2.3. Theorem. (Extreme Points) *Let*

$$\begin{aligned}&f_1(z) = z \text{ and,} \\ &f_k(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|} z^k, \ k \ge 2,\end{aligned} \tag{2.5}$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, $\lambda \ge 0$ and $0 < \gamma \le 1$. Then $f(z)$ is in the class $\mathcal{PT}(\lambda, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z), \tag{2.6}$$

where $\omega_k \ge 0$ and $\sum_{k=1}^{\infty} \omega_k = 1$.

Proof. Suppose $f(z)$ can be written as in (2.6). Then

$$f(z) = z - \sum_{k=2}^{\infty} \omega_k \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|} z^k.$$

Now,

$$\begin{aligned}&\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)} \omega_k \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|} \\ &\qquad = \sum_{k=2}^{\infty} \omega_k = 1 - \omega_1 \le 1.\end{aligned}$$

Thus $f \in \mathcal{PT}(\lambda, \alpha, \beta)$.

Conversely, let us have $f \in \mathcal{PT}(\lambda, \alpha, \beta)$. Then by using (2.4), we set

$$\omega_k = \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)} a_k, \;\; k \geq 2$$

and $\omega_1 = 1 - \sum_{k=2}^{\infty} \omega_k$. Then we have $f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z)$, and hence this completes the proof of Theorem 2.3. □

3. Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{PT}(\lambda, \alpha, \beta)$.

3.1. Theorem. *If $f \in \mathcal{PT}(\lambda, \alpha, \beta)$, then*

$$\begin{aligned} r - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} r^2 &\leq |f(z)| \\ &\leq r + \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]|C_2(b,\mu)|} r^2 \end{aligned} \tag{3.1}$$

holds if the sequence $\{\sigma_k(\lambda, \beta, \gamma)\}_{k=2}^{\infty}$ is non-decreasing, and

$$\begin{aligned} 1 - \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} r &\leq |f'(z)| \\ &\leq 1 + \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]|C_2(b,\mu)|} r \end{aligned} \tag{3.2}$$

holds if the sequence $\{\sigma_k(\lambda, \beta, \gamma)/k\}_{k=2}^{\infty}$ is non-decreasing, where

$$\sigma_k(\lambda, \beta, \gamma) = (1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|.$$

The bounds in (3.1) *and* (3.2) *are sharp, since the equalities are attained by the function*

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} z^2, \;\; z = \pm r. \tag{3.3}$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}. \tag{3.4}$$

Using (1.14) and (3.4), we obtain

$$\begin{aligned} |z| - |z|^2 \sum_{k=2}^{\infty} a_k &\leq |f(z)| \\ &\leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k, \\ r - r^2 \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} &\leq |f(z)| \\ &\leq r + r^2 \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}. \end{aligned} \tag{3.5}$$

Hence (3.1) follows from (3.5). Further,

$$\sum_{k=2}^{\infty} k a_k \leq \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}.$$

Hence (3.2) follows from

$$1 - r\sum_{k=2}^{\infty} k a_k \leq |f'(z)| \leq 1 + r\sum_{k=2}^{\infty} k a_k.$$

□

4. Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{PT}(\lambda,\alpha,\beta)$ are given in this section.

4.1. Theorem. *Let the function $f(z)$ defined by (1.14) belong to the class $\mathcal{PT}(\lambda,\alpha,\beta)$, Then $f(z)$ is close-to-convex of order δ, $(0 \le \delta < 1)$ in the disc $|z| < r$, where*

$$r := \inf_{k\ge 2}\left[\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2k\beta\gamma(1-\alpha)}\right]^{\frac{1}{k-1}}. \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.5).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$|f'(z)-1| < 1-\delta. \tag{4.2}$$

For the left hand side of (4.2) we have

$$|f'(z)-1| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k |z|^{k-1} < 1.$$

Using the fact that $f \in \mathcal{PT}(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)} \le 1.$$

We can say (4.2) is true if

$$\frac{k}{1-\delta}|z|^{k-1} \le \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}.$$

Or, equivalently,

$$|z|^{k-1} \le \left[\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2k\beta\gamma(1-\alpha)}\right],$$

which completes the proof. □

4.2. Theorem. *Let $f \in \mathcal{PT}(\lambda,\alpha,\beta)$. Then*

(1) *f is starlike of order δ, $(0 \le \delta < 1)$, in the disc $|z| < r$, that is,*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \ (|z| < r;\ 0 \le \delta < 1),$$

where

$$r = \inf_{k\ge 2}\left\{\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

(2) *f is convex of order δ, $(0 \le \delta < 1)$, in the disc $|z| < r$, that is*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta, \ (|z| < r;\ 0 \le \delta < 1),$$

where

$$r = \inf_{k\ge 2}\left\{\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)k(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

Each of these results is sharp for the extremal function $f(z)$ given by (2.5).

Proof. (1) Given $f \in T$ and f starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{4.3}$$

For the left hand side of (4.3) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\ |z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} a_k\ |z|^{k-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}a_k\ |z|^{k-1} < 1.$$

Using the fact that $f \in \mathcal{PT}(\lambda, \alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)} < 1,$$

we can say (4.3) is true if

$$\frac{k-\delta}{1-\delta}|z|^{k-1} < \frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}.$$

Or, equivalently,

$$|z|^{k-1} < \frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)(k-\delta)}$$

which yields the starlikeness of the family.

(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2) on lines similar to the proof of (1). □

4.3. Remark. For specific choices of the parameters $\alpha, \beta, \gamma, \mu$, various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes. The details involved in the derivations of such specializations of the results presented in this paper are fairly straightforward.

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References

[1] Alexander, J. W. *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17**, 12–22, 1915.

[2] Altintas, O. *A subclass of analytic functions with negative coefficients*, Hacettepe Univ. Bull. Nat. Sciences & Engineering **19**, 15–24, 1990.

[3] Bernardi, S. D. *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135**, 429–446, 1969.

[4] Choi, J. and Srivastava, H. M. *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput. **170**, 399–409, 2005.

[5] Ferreira, C. and López, J. L. *Asymptotic expansions of the Hurwitz-Lerch Zeta function*, J. Math. Anal. Appl. **298**, 210–224, 2004.

[6] Flett, T. M. *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38**, 746–765, 1972.

[7] Garg, M., Jain, K. and Srivastava, H. M. *Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct. **17**, 803–815, 2006.

[8] Jung, I. B., Kim, Y. C. and Srivastava, H. M. *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176**, 138–147, 1993.

[9] Owa, S. and Lee, S. K. *Certain generalized class of analytic functions with negative coefficients*, Bull. Cal. Math. Soc. **82**, 284–289, 1990.

[10] Lin, S.-D. and Srivastava, H. M. *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput. **154**, 725–733, 2004.

[11] Lin, S.-D., Srivastava, H. M. and Wang, P.-Y. *Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions*, Integral Transform. Spec. Funct. **17**, 817–827, 2006.

[12] Prajapat, J. K. and Goyal, S. P. *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal. **3**, 129–137, 2009.

[13] Riaducanu, D. and Srivastava, H. M. *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function*, Integral Transform. Spec. Funct. **18**, 933–943, 2007.

[14] Silverman, H. *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**, 109–116, 1975.

[15] Srivastava, H. M. and Attiya, A. A. *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct. **18**, 207–216, 2007.

[16] Srivastava, H. M. and Choi, J. *Series associated with the Zeta and related functions* (Kluwer Academic Publishers, Dordrecht, Boston, London, 2001).