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NOTES ON UPPER AND LOWER BOUNDS OF TWO INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract

G. D. Anderson and S. L. Qiu (*A monotonicity property of the gamma function*, Proc. Amer. Math. Soc. **125** (11) (1997), 3355–3362) obtained a double inequality for the function $\Gamma(x)$. Their result was improved by H. Alzer (*Inequalities for the gamma function*, Proc. Amer. Math. Soc. **128** (1), 141–147, 1999), and by X. Li, Ch. P. Chen (*Inequalities for the gamma function*, J. Ineq. Pure Appl. Math. **8** (1), Art.28, 2007). Li and Chen remarked that their bounds could not be compared with those of Alzer. In this note, we will show that there exist a constant γ such that, in the intervals $(1, \gamma)$ and $(\gamma, +\infty)$, the upper bounds can be compared to each other. We will also show that there exist a constant ξ such that it will be possible to compare lower bounds in the intervals $(1, \xi)$ and $(\xi, +\infty)$.

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1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt.$$

In 1997, G. D. Anderson and S. L. Qiu [2] proved the following double inequality for $\Gamma(x)$:

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}, \; (x > 1), \tag{1.1}$$

where γ is the Euler constant with value $\gamma = 0.577215\ldots$.

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Then, in 1999, H. Alzer [1] proved that for $x > 1$ the following inequalities hold:

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma}, \tag{1.2}$$

with the best possible constants

$$\alpha = \left(\frac{\pi^2}{6} - \gamma\right)/2,\ \beta = 1.$$

Relation (1.2) represents an improvement on (1.1).

In 2007, X. Li and Ch. P. Chen [4] improved inequalities (1.1) as follows:

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-\frac{1}{2}}}{e^{x-1}},\ (x > 1). \tag{1.3}$$

At the end of their paper [4], they remarked that the upper and lower bounds of (1.2) and (1.3) cannot be compared to each other.

It is natural to ask whether there exist some intervals in which the comparison of the bounds can be done? In these notes, we will show that there exist a constant γ (ξ) such that, in the intervals $(1, \gamma)$ and $(\gamma, +\infty)$ ($(1, \xi)$ and $(\xi, +\infty)$) the upper bounds (the lower bounds) can be compared to each other.

Hence, we will compare both the lower and upper bounds of (1.2) and (1.3) in both intervals, keeping in mind that with

$$\alpha = \left(\frac{\pi^2}{6} - \gamma\right)/2 \text{ and } \beta = 1$$

the relation (1.2) would be:

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{x-1-\gamma} \tag{1.4}$$

So, in fact, we will compare lower and upper bounds of (1.3) and (1.4).

2. Main results

2.1. Theorem. (Upper bounds of (1.3) and (1.4)) *Let*

$$x \in \left(1, \gamma + \frac{1}{2}\right).$$

Then:

$$x^{x-1-\gamma} < \frac{x^{x-\frac{1}{2}}}{e^{x-1}}. \tag{2.1}$$

Proof. Consider the function:

$$v(x) = x - 1 - \left(\gamma + \frac{1}{2}\right) \cdot \ln x,\ x \in \left[1, \gamma + \frac{1}{2}\right). \tag{2.2}$$

It's derivative is

$$v' = 1 - \frac{\gamma + \frac{1}{2}}{x}. \tag{2.3}$$

Clearly, for $x \in \left[1, \gamma + \frac{1}{2}\right)$, $v' < 0$. Hence, v is a decreasing function. Since $v(1) = 0$, we have $v(x) < 0$, for $x \in \left(1, \gamma + \frac{1}{2}\right)$. So, one obtains:

$$x - 1 - \left(\gamma + \frac{1}{2}\right) \cdot \ln x < 0$$

After transformations we have:

$$e^{x-1} < x^{x-\frac{1}{2}-((x-1)-\gamma)},$$

so finally

$$x^{x-1-\gamma} < \frac{x^{x-\frac{1}{2}}}{e^{x-1}},$$

completing the proof. $\square$

If we refer to (1.7) we see that for $x > \gamma + \frac{1}{2}$, we have $v' > 0$, so in that case v is an increasing function.

Clearly, for $x = \gamma + \frac{1}{2}$, the function v attains its minimum.

Now, it is easy to verify that $v\big(\gamma + \frac{1}{2}\big) < 0$, and $v(\gamma + 1) > 0$. Since, for $x > \gamma + \frac{1}{2}$, v is increasing, then in the interval $\big(\gamma + \frac{1}{2}, \gamma + 1\big)$ there is exactly one zero of the function v. Let us denote this by η. (So $v(\eta) = 0$ and $\eta \in \big(\gamma + \frac{1}{2}, \gamma + 1\big)$).

In view of the above facts (v - increasing and $v\big(\gamma + \frac{1}{2}\big) < 0$), we conclude that for $x \in \big(\gamma+\frac{1}{2},\ \eta\big)$ the function v is negative, so Theorem 2.1 can be extended to the following:

2.2. Theorem. *Let $x \in (1, \eta)$. Then* (1.5) *is true. So:*

$$x^{x-1-\gamma} < \frac{x^{x-\frac{1}{2}}}{e^{x-1}}. \quad \square$$

Theorem 2.2 says that, for $x \in (1, \eta)$, the upper bound given in [1] is better than upper bound given in [4].

Now clearly for $x \in (\eta, +\infty)$, the function v is positive so the following is true:

2.3. Theorem. *Let $x \in (\eta, +\infty)$. Then*

$$x^{x-1-\gamma} > \frac{x^{x-\frac{1}{2}}}{e^{x-1}}.$$

Proof. Similar to the proof of the Theorem 2.1. $\square$

In view of similar comments made after the proof of Theorem 2.2, now we have the opposite. The upper bound given in [4] is better than that of [1].

Clearly, for $x = \eta$ in (2.1) we have equality.

Next, we compare lower bounds of (1.3) and (1.4). First, we prove the following:

2.4. Lemma. *The function*

$$\theta(x) = x - 1 + (\alpha x - \alpha - x) \cdot \ln x,\ \ x > 1 \tag{2.4}$$

has exactly on zero, where $\alpha = \Big(\frac{\pi^2}{6} - \gamma\Big)/2 \approx 0.53385$.

Proof. We will show that the zero of the function θ is in the interval $(1, e)$. It is easy to find $x_0, x_1 > 1$ such that $\theta(x_0) > 0$ and $\theta(x_1) < 0$ ($x_0 = 1+\varepsilon$, with ε "small" and positive and $x_1 = e$ satisfy those conditions). Hence, there is at least on number $\xi \in (1, e)$ such that $\theta(\xi) = 0$.

The graphical method (see [3]) shows that in $(1, e)$ there is exactly one zero of the function θ.

By the halving method (see [3]) (or any other geometric approaching method (see [5])) we find that the root ξ is in the interval (1.5,1.51), and to be “more precise” it is approximately 1.502.

We have to show that the root is unique. We will achieve this by proving that for $x > \xi$, $\theta(x) < 0$.

We will show that for $x > \xi$, the function θ is decreasing.

It is easy to see that:

$$\theta^{'}(x) = (\alpha - 1) \cdot \ln x + \alpha - \frac{\alpha}{x}.$$

After transformations we obtain:

$$\theta^{'}(x) = (\alpha - 1) \cdot \Big(\ln x + \frac{\alpha(x-1)}{x(\alpha-1)} \Big).$$

Next we show that $\ln x + \frac{\alpha(x-1)}{x(\alpha-1)} > 0$, which is equivalent to:

$$\frac{x \ln x}{x-1} > \frac{\alpha}{1-\alpha}.$$

Let $f(x) = \dfrac{x \ln x}{x-1}$. Then $f^{'}(x) = \dfrac{x - 1 - \ln x}{(x-1)^2}$.

Using the graphical method it is easy to show that $x - 1 - \ln x > 0$ for $x > 1$ (and so, also for $x > \xi$). So for $x > \xi$, $f^{'}(x) > 0$. This means that f is increasing for $x > \xi$. So, $f(x) > f(\xi)$ or

$$\frac{x \ln x}{x-1} > \frac{\xi \ln \xi}{\xi - 1}.$$

Computations show that

$$\frac{\xi \ln \xi}{\xi - 1} > \frac{\alpha}{1-\alpha}.$$

So we have

$$\frac{x \ln x}{x-1} > \frac{\alpha}{1-\alpha}.$$

Since $\alpha - 1 < 0$, we have $\theta'(x) < 0$, so θ is a decreasing function of x. This means that for $x > \xi$, $\theta(x) < \theta(\xi) = 0$, which completes the proof. □

Now we are able to prove the following:

2.5. Theorem. *Let $x \in (\xi, +\infty)$. Then:*

$$x^{\alpha(x-1)-\gamma} < \frac{x^{x-\gamma}}{e^{x-1}}. \tag{2.5}$$

Proof. In view of Lemma 2.4, we have $\theta(x) < 0$ for $x > \xi$. So we have

$$x - 1 + (\alpha x - \alpha - x) \ln x < 0,$$

which is equivalent to

$$x - 1 < (x - \alpha(x-1)) \ln x.$$

After transformations one obtains:

$$e^{x-1} < x^{x-\gamma-\alpha(x-1)+\gamma}.$$

So finally we have:

$$x^{\alpha(x-1)-\gamma} < \frac{x^{x-\gamma}}{e^{x-1}},$$

completing the proof. □

As a comment on Theorem 2.5, we say that for $x \in (\xi, \infty)$, the upper bound given with (1.3) is better than that of (1.4).

Now, since there is exactly one zero in $(1, 2)$ and since $\theta(1 + \varepsilon) > 0$ with ε small and positive, for $x \in (1, \xi)$ we have $\theta(x) > 0$, so the following theorem holds:

2.6. Theorem. *Let $x \in (1, \xi)$. Then:*

$$x^{\alpha(x-1)-\gamma} > \frac{x^{x-\gamma}}{e^{x-1}}. \tag{2.6}$$

Proof. Similar to the proof of Theorem 2.5. □

Clearly, in this case, the upper bound given in (1.4) is better that that of (1.3). Also it is clear that for $x = \xi$ in (2.5), (2.6) we have equality.

To conclude, we pose this problem:

2.7. Problem. Find a non-computational proof for the inequality (2.5).

References

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