Hacettepe Journal of Mathematics and Statistics
Volume 39 (1) (2010), 31 – 39

RANK FUNCTIONS FOR CLOSED AND PERFECT [0, 1]-MATROIDS[‡]

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Received 01 : 05 : 2009 : Accepted 12 : 10 : 2009

Abstract

In this paper we present the notions of perfect [0, 1]-matroid and closed [0, 1]-matroid, and investigate some of their basic properties. Moreover, we prove that a closed and perfect [0, 1]-matroid can be characterized by means of its [0, 1]-fuzzy rank function.

Keywords: Matroids, L-matroids, Perfect [0, 1]-matroids, Closed [0, 1]-matroids, Fuzzy rank functions.

2000 AMS Classification: 03 E 70, 05 B 35, 52 B 40.

1. Introduction

Matroids were introduced by Whitney in 1935 as a generalization of both graphs and vector spaces. It is well-known that matroids play an important role in mathematics, especially in applied mathematics. Matroids are precisely the structures for which the very simple and efficient greedy algorithm works [1, 4]. In [6], Matroid theory was generalized to fuzzy fields by Shi, and L-fuzzy rank functions were studied. His approach to the fuzzification of matroids preserves many basic properties of crisp matroids, and L-matroids can be applied to fuzzy algebras and fuzzy graphs. Based on [6], the aim of this paper is to study the relation between a [0, 1]-matroid and its [0, 1]-fuzzy rank function.

In this paper, we obtain two results:

(1) There is a one-to-one correspondence between a closed and perfect [0, 1]-matroid and its [0, 1]-fuzzy rank function. That is, a closed and perfect [0, 1]-matroid can be characterized by means of its [0, 1]-fuzzy rank function.

(2) A [0, 1]-matroid (resp., a perfect [0, 1]-matroid, a closed [0, 1]-matroid) and its [0, 1]-fuzzy rank function are not in one-to-one correspondence in general.

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2. Preliminaries

Throughout this paper, L denotes a completely distributive lattice, and E a nonempty finite set. L^E is the set of all L-fuzzy sets (or L-sets for short) on E. The smallest element and the largest element in L are denoted by $\bot$ and $\top$ respectively. We often do not distinguish a crisp subset A of E from its characteristic function χ_A.

An element a in L is called a *prime element* if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. Dually, a in L is called *co-prime* if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [2]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$. When L is replaced by the interval $[0,1]$, it is easy to see $J(L) = (0,1]$ and $P(L) = [0,1)$.

For $\mathcal{A} \subseteq 2^E$, we define

$$\mathrm{Max}(\mathcal{A}) = \{A \in \mathcal{A} : \forall B \in \mathcal{A},\ \text{if } A \subseteq B \text{ then } A = B\}.$$

For $A \in L^E$ and $a \in L$, we define $A_{[a]} = \{e \in E : A(e) \geqslant a\}$. Some properties of the cut sets can be found in [3].

For $a \in L$ and $A \subseteq E$, define two L-fuzzy sets $a \wedge A$ and $a \vee A$ as follows:

$$(a \wedge A)(e) = \begin{cases} a, & e \in A; \\ \bot, & e \notin A. \end{cases} \qquad (a \vee A)(e) = \begin{cases} \top, & e \in A; \\ a, & e \notin A. \end{cases}$$

An L-fuzzy set $a \wedge \{e\}$ is called an *L-fuzzy point*, and denoted by e_a.

2.1. Definition. [5] Let $\mathbb{N}$ denote the set of all natural numbers. An *L-fuzzy natural number* is an antitone map $\lambda : \mathbb{N} \to L$ satisfying

$$\lambda(0) = \top, \qquad \bigwedge_{n \in \mathbb{N}} \lambda(n) = \bot.$$

The set of all L-fuzzy natural numbers is denoted by $\mathbb{N}(L)$.

For any $\lambda \in \mathbb{N}(L)$ and any $a \in J(L)$, we shall not distinguish $n \in \lambda_{[a]}$ from $n \leqslant |\lambda_{[a]}|$.

2.2. Definition. [5] For any $\lambda, \mu \in \mathbb{N}(L)$, define the sum $\lambda + \mu$ of λ and μ as follows: for any $n \in \mathbb{N}$,

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

2.3. Theorem. [5] *For any $m \in \mathbb{N}$, define $\underline{m} \in \mathbb{N}(L)$ such that*

$$\underline{m}(t) = \begin{cases} \top, & \text{if } t \leqslant m; \\ \bot, & \text{if } t \geqslant m+1. \end{cases}$$

Then for any $\lambda \in \mathbb{N}(L)$, it follows that

$$\underline{0} + \lambda = \lambda. \qquad \square$$

2.4. Definition. [6] Let A be an L-fuzzy set on a finite set E. Then the mapping $|A| : \mathbb{N} \to L$ defined $\forall\, n \in \mathbb{N}$ by,

$$|A|(n) = \bigvee \{a \in L : |A_{[a]}| \geqslant n\}$$

is called the *L-fuzzy cardinality of A*.

2.5. Lemma. [6] *For a finite set E, it holds that $|A|_{[a]} = |A_{[a]}|$ for any $A \in L^E$ and any $a \in J(L)$.* $\square$

2.6. Definition. [6] Let E be a finite set. A subfamily $\mathcal{I}$ of L^E is called *a family of independent L-fuzzy sets on E* if it satisfies the following conditions:

(LI1) $\mathcal{I}$ is nonempty;
(LI2) $A \in L^E,\ B \in \mathcal{I},\ A \leqslant B \implies A \in \mathcal{I}$;
(LI3) If $A, B \in \mathcal{I}$ and $b = |B|(n) \nleqslant |A|(n)$ for some $n \in \mathbb{N}$, then there exists $e \in F(A, B)$ such that $(b \wedge A_{[b]}) \vee e_b \in \mathcal{I}$, where

$$F(A, B) = \{e \in E : b \leqslant B(e), b \nleqslant A(e)\}.$$

If $\mathcal{I}$ is a family of independent L-fuzzy sets on E, then the pair $(E, \mathcal{I})$ is called an *L-matroid*.

2.7. Theorem. [6] *Let E be a finite set and $\mathcal{I} \subseteq L^E$. Define, $\forall\, a \in L \setminus \{\perp\}$,*

$$\mathcal{I}[a] = \{A_{[a]} : A \in \mathcal{I}\}.$$

If $(E, \mathcal{I})$ is an L-matroid, then $(E, \mathcal{I}[a])$ is a matroid for each $a \in L \setminus \{\perp\}$.

3. Closed and perfect $[0, 1]$-matroids

In the sequel we will mainly focus on the case when L is the interval $[0, 1]$.

3.1. Definition. A $[0, 1]$-matroid $(E, \mathcal{I})$ is called a *perfect* $[0, 1]$*-matroid*, if it satisfies the following condition:

(LI4) $\forall\, A \in [0, 1]^E$, if $a \wedge A_{[a]} \in \mathcal{I}$ for all $a \in (0, 1]$, then $A \in \mathcal{I}$.

3.2. Example. Let $E = \{3, 5\}$. Define $A \in [0, 1]^E$ by

$$A(x) = \begin{cases} \frac{1}{2}, & x = 3; \\ \frac{1}{3}, & x = 5, \end{cases}$$

and define

$$\mathcal{I} = \left\{B \in [0, 1]^E : B \leqslant \tfrac{1}{3} \wedge \{3, 5\}\right\} \cup \left\{B \in [0, 1]^E : B \leqslant \tfrac{1}{2} \wedge \{3\}\right\}.$$

Then we can check that $\mathcal{I}$ satisfies (LI1)–(LI3), but it does not satisfy (LI4) since $a \wedge A_{[a]} \in \mathcal{I}$ for all $a \in (0, 1]$ but $A \notin \mathcal{I}$.

3.3. Theorem. *Let $(E, \mathcal{I})$ be a $[0, 1]$-matroid. Then $(E, \mathcal{I})$ is a perfect $[0, 1]$-matroid if and only if*

$$\mathcal{I} = \{A \in [0, 1]^E : \forall\, a \in (0, 1],\ A_{[a]} \in \mathcal{I}[a]\}. \quad \square$$

3.4. Lemma. *Let $(E, \mathcal{I})$ be a $[0, 1]$-matroid. If $0 < a \leqslant b \leqslant 1$, then $\mathcal{I}[b] \subseteq \mathcal{I}[a]$.*

Proof. Let $A \in \mathcal{I}[b]$. Then $b \wedge A \in \mathcal{I}$ as $\mathcal{I}$ satisfies (LI2). Since $a \leqslant b$, $a \wedge A \leqslant b \wedge A$. Thus $a \wedge A \in \mathcal{I}$ by (LI2), hence $A = (a \wedge A)_{[a]} \in \mathcal{I}[a]$. $\square$

3.5. Theorem. *Let $\mathcal{I} \subseteq [0, 1]^E$ satisfy* (LI2) *and* (LI4)*. Then the following conditions are equivalent:*

(1) *$(E, \mathcal{I})$ is a $[0, 1]$-matroid.*
(2) *$(E,\ \mathcal{I}[a])$ is a matroid for all $a \in (0, 1]$.*

Proof. By Theorem 2.7, we only need to prove (2) $\implies$ (1).

(2) $\implies$ (1). Since $\mathcal{I}$ satisfies (LI2) and (LI4), $\mathcal{I} = \{A \in [0, 1]^E : \forall a \in (0, 1], A_{[a]} \in \mathcal{I}[a]\}$. It is easy to see that $\mathcal{I}$ satisfies (LI1). Now we prove that $\mathcal{I}$ satisfies (LI3). Suppose that $A, B \in \mathcal{I}$ and $b = |B|(n) \nleqslant |A|(n)$ for some $n \in \mathbb{N}$. Then $n \in |B|_{[b]}$ and $n \notin |A|_{[b]}$, thus $|A|_{[b]} \not\supseteq |B|_{[b]}$. By Lemma 2.5, $|A_{[b]}| \ngeqslant |B_{[b]}|$, i.e. $|A_{[b]}| < |B_{[b]}|$. Since $A_{[b]}, B_{[b]} \in \mathcal{I}[b]$, there exists $e \in B_{[b]} - A_{[b]}$ such that $A_{[b]} \cup \{e\} \in \mathcal{I}[b]$. In this case, $b \leqslant B(e)$ and $b \nleqslant A(e)$, i.e. $e \in F(A, B)$. By Lemma 3.4, it is obvious that

$$((b \wedge A_{[b]}) \vee e_b)_{[a]} = A_{[b]} \cup \{e\} \in \mathcal{I}[b] \subseteq \mathcal{I}[a]$$

for every $a \leqslant b$, and $((b \wedge A_{[b]}) \vee e_b)_{[a]} = \emptyset \in \mathfrak{I}[a]$ for every $a \nleqslant b$. This implies $(b \wedge A_{[b]}) \vee e_b \in \mathfrak{I}$. Hence $(E, \mathfrak{I})$ is a $[0,1]$-matroid. $\square$

3.6. Theorem. *Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid. Then there is a finite sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that*

(1) *If $a_i < a, b < a_{i+1}$, then $\mathfrak{I}[a] = \mathfrak{I}[b]$, $0 \leqslant i \leqslant n-1$;*
(2) *If $a_i < a < a_{i+1} < b < a_{i+2}$, then $\mathfrak{I}[a] \supset \mathfrak{I}[b]$, $0 \leqslant i \leqslant n-2$.*

The sequence $a_0, a_1, \ldots, a_n$ is called the fundamental sequence *for $(E, \mathfrak{I})$.*

Proof. We define an equivalence relation $\sim$ on $(0,1]$ by $a \sim b \Leftrightarrow \mathfrak{I}[a] = \mathfrak{I}[b]$. Since E is a finite set, the number of matroids on E is finite. Thus there exist at most finitely many equivalence classes which are respectively denoted by $I_1, I_2, \ldots, I_n$.

Each I_i $(i = 1, 2, \ldots, n)$ is an interval. We only need to show that $\forall a, b \in I_i$ with $a \leqslant b$, if $c \in [a, b]$, then $c \in I_i$. Since $a \leqslant c \leqslant b$, by Lemma 3.4, we know that $\mathfrak{I}[b] \subseteq \mathfrak{I}[c] \subseteq \mathfrak{I}[a]$. As $a, b \in I_i$, $\mathfrak{I}[a] = \mathfrak{I}[b]$. Thus $\mathfrak{I}[b] = \mathfrak{I}[c] = \mathfrak{I}[a]$, hence $c \in I_i$ by the definition of I_i. This implies that I_i is an interval.

Let $a_{i-1} = \inf I_i$ and $a_i = \sup I_i$ $(i = 1, 2, \ldots, n)$. Clearly, the sequence $a_0, a_1, \ldots, a_n$ is the fundamental sequence for $(E, \mathfrak{I})$. $\square$

3.7. Definition. A $[0,1]$-matroid $(E, \mathfrak{I})$ with the fundamental sequence $a_0, a_1, \ldots, a_n$ is called a *closed* $[0,1]$*-matroid* if whenever $a_{i-1} < a \leqslant a_i$ $(1 \leqslant i \leqslant n)$, then $\mathfrak{I}[a] = \mathfrak{I}[a_i]$.

3.8. Theorem. *Let $(E, \mathfrak{I})$ be a $[0,1]$-matroid with the fundamental sequence $a_0, a_1, \ldots, a_n$. Then $(E, \mathfrak{I})$ is a closed $[0,1]$-matroid if and only if $\mathfrak{I}$ satisfies the following condition:*

$(*)$ $\quad \forall a \in (0,1]$ *and* $A \in 2^E$, *if* $b \wedge A \in \mathfrak{I}$ *for all* $0 < b < a$, *then* $a \wedge A \in \mathfrak{I}$.

Proof. Suppose that $\mathfrak{I}$ satisfies $(*)$. Then $\forall a \in (a_{i-1}, a_i)$, $(i = 1, 2, \ldots, n)$, we have $\mathfrak{I}[a_i] \subseteq \mathfrak{I}[a]$ by Lemma 3.4. Let $A \in \mathfrak{I}[a]$ for all $a \in (a_{i-1}, a_i)$. Then $a \wedge A \in \mathfrak{I}$ for all $a \in (a_{i-1}, a_i)$, thus $b \wedge A \in \mathfrak{I}$ for all $0 < b < a_i$. Since $\mathfrak{I}$ satisfies $(*)$, $a_i \wedge A \in \mathfrak{I}$. Thus $A = (a_i \wedge A)_{[a_i]} \in \mathfrak{I}[a_i]$. This implies that $\mathfrak{I}[a] \subseteq \mathfrak{I}[a_i]$ for all $a \in (a_{i-1}, a_i)$. Therefore, $\mathfrak{I}[a] = \mathfrak{I}[a_i]$ for all $a \in (a_{i-1}, a_i]$, i.e. $(E, \mathfrak{I})$ is a closed $[0,1]$-matroid.

Conversely, assume that $(E, \mathfrak{I})$ is a closed $[0,1]$-matroid. Let $a \in (0,1], A \in 2^E$, and $b \wedge A \in \mathfrak{I}$ for all $0 < b < a$. Since $a \in (0,1]$, $a \in (a_{i-1}, a_i]$ for some $i = 1, 2, \ldots, n$. Take $b_0 \in (a_{i-1}, a) \subseteq (a_{i-1}, a_i]$, then $b_0 \wedge A \in \mathfrak{I}$, thus $A \in \mathfrak{I}[b_0] = \mathfrak{I}[a_i]$ since $(E, \mathfrak{I})$ is a closed $[0,1]$-matroid, hence $a_i \wedge A \in \mathfrak{I}$. By (LI2), $a \wedge A \in \mathfrak{I}$. This means that $\mathfrak{I}$ satisfies $(*)$. $\square$

By Example 3.2, we know that a closed $[0,1]$-matroid need not be a perfect $[0,1]$-matroid. The following example shows that a perfect $[0,1]$-matroid need not be a closed $[0,1]$-matroid either.

3.9. Example. Let E be a finite set. Define

$$\mathfrak{I} = \left\{ A \in [0,1]^E : A(x) < \frac{1}{2} \text{ for all } x \in E \right\}.$$

Then we can check that $\mathfrak{I}$ satisfies (LI1)-(LI4), but it is not closed.

4. Rank functions for closed and perfect $[0,1]$-matroids

4.1. Definition. [6] Let $(E, \mathfrak{I})$ be an L-matroid. The mapping $R_{\mathfrak{I}} : L^E \to \mathbb{N}(L)$ defined by

$$R_{\mathfrak{I}}(A) = \bigvee \{ |B| : B \leqslant A, B \in \mathfrak{I} \}$$

is called the *L-fuzzy rank function for* $(E, \mathfrak{I})$. If $A \in L^E$, $R_{\mathfrak{I}}(A)$ is called the *L-fuzzy rank of A in* $(E, \mathfrak{I})$.

4.2. Theorem. *Let $(E,\mathcal{I})$ be an L-matroid. If $R_{\mathcal{I}} : L^E \to \mathbb{N}(L)$ is the L-fuzzy rank function for $(E,\mathcal{I})$, then*

$$R_{\mathcal{I}}(A)(n) = \bigvee \{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\} \text{ for all } n \in \mathbb{N},$$

where $R_{\mathcal{I}[a]}$ is the rank function for $(E,\mathcal{I}[a])$ and

$$R_{\mathcal{I}[a]}(A_{[a]}) = \bigvee \{|B| : B \in \mathcal{I}[a],\ B \subseteq A_{[a]}\}.$$

Proof. Let $a \in \{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$, $\forall n \in \mathbb{N}$. Let $B_a \in \mathrm{Max}(\mathcal{I}[a]|_{A_{[a]}})$. Then $n \leqslant R_{\mathcal{I}[a]}(A_{[a]}) = |B_a|$.

Since $B_a \in \mathcal{I}[a]$ and $B_a \subseteq A_{[a]}$, $a \wedge B_a \in \mathcal{I}$ and $a \wedge B_a \leqslant A$. Thus

$$R_{\mathcal{I}}(A)(n) = \Big(\bigvee \{|B| : B \leqslant A, B \in \mathcal{I}\}\Big)(n) \geqslant |a \wedge B_a|(n) = a.$$

Hence $R_{\mathcal{I}}(A)(n) \geqslant \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$.

Conversely, in order to prove $R_{\mathcal{I}}(A)(n) \leqslant \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$, we only need to prove $|B|(n) \leqslant \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$ for all $B \leqslant A$ and $B \in \mathcal{I}$. Let $B \in \mathcal{I}$ and $B \leqslant A$. Then $B_{[a]} \in \mathcal{I}[a]|_{A_{[a]}}$, $\forall a \in \{a \in L \setminus \{\bot\} : |B_{[a]}| \geqslant n\}$, thus $n \leqslant |B_{[a]}| \leqslant R_{\mathcal{I}[a]}(A_{[a]})$, hence $a \in \{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$. This implies that

$$|B|(n) = \bigvee\{a \in L : |B_{[a]}| \geqslant n\} \leqslant \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\},$$

and thus $R_{\mathcal{I}}(A)(n) \leqslant \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$. Therefore,

$$R_{\mathcal{I}}(A)(n) = \bigvee\{a \in L \setminus \{\bot\} : n \leqslant R_{\mathcal{I}[a]}(A_{[a]})\}$$

for all $n \in \mathbb{N}$. □

4.3. Lemma. *Let $(E,\mathcal{I})$ be a closed $[0,1]$-matroid and $A \in [0,1]^E$. Then there is a finite sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that*

(1) *If $a_i < a, b \leqslant a_{i+1}$, then $R_{\mathcal{I}[a]}(A_{[a]}) = R_{\mathcal{I}[b]}(A_{[b]})$, $0 \leqslant i \leqslant n-1$;*
(2) *If $a_i < a \leqslant a_{i+1} < b \leqslant a_{i+2}$, then $R_{\mathcal{I}[a]}(A_{[a]}) > R_{\mathcal{I}[b]}(A_{[b]})$, $0 \leqslant i \leqslant n-2$.*

Proof. Let $(E,\mathcal{I})$ be a $[0,1]$-matroid and $A \in [0,1]^E$. We define an equivalence relation $\sim$ on $(0,1]$ by $a \sim b \iff R_{\mathcal{I}[a]}(A_{[a]}) = R_{\mathcal{I}[b]}(A_{[b]})$. Since E is a finite set, there exist at most finitely many equivalence classes which are respectively denoted by $I_1, I_2, \ldots, I_n$.

Step 1 For all $a, b \in (0,1]$, if $a \leqslant b$, $R_{\mathcal{I}[b]}(A_{[b]}) \leqslant R_{\mathcal{I}[a]}(A_{[a]})$. Since $a \leqslant b$, $\mathcal{I}[b] \subseteq \mathcal{I}[a]$ and $A_{[b]} \subseteq A_{[a]}$. Thus

$$\begin{aligned} R_{\mathcal{I}[b]}(A_{[b]}) &= \bigvee\{|B| : B \in \mathcal{I}[b], B \subseteq A_{[b]}\} \\ &\leqslant \bigvee\{|B| : B \in \mathcal{I}[a], B \subseteq A_{[a]}\} \\ &= R_{\mathcal{I}[a]}(A_{[a]}). \end{aligned}$$

Step 2 Each I_i, $(i = 1, 2, \ldots, n)$, is an interval. We only need to show that for any $a, b \in I_i$ with $a \leqslant b$, if $c \in [a,b]$, then $c \in I_i$. Since $a \leqslant c \leqslant b$, we know that $R_{\mathcal{I}_{[b]}}(A_{[b]}) \leqslant R_{\mathcal{I}[c]}(A_{[c]}) \leqslant R_{\mathcal{I}[a]}(A_{[a]})$ by Step 1. Since $a, b \in I_i$, $R_{\mathcal{I}[a]}(A_{[a]}) = R_{\mathcal{I}[b]}(A_{[b]})$, thus

$$R_{\mathcal{I}[a]}(A_{[a]}) = R_{\mathcal{I}[c]}(A_{[c]}) = R_{\mathcal{I}[b]}(A_{[b]}).$$

Hence $c \in I_i$ by the definition of I_i, and then I_i is an interval.

Step 3 Let $\inf I_i = a_{i-1}$, $\sup I_i = a_i$. Since E is a finite set, $\{\mathcal{I}[a]|_{A_{[a]}} : a \in (a_{i-1}, a_i)\}$ is a finite family. Let

$$\{\mathcal{I}[a]|_{A_{[a]}} : a \in (a_{i-1}, a_i)\} = \{\mathcal{I}[b_1]|_{A_{[b_1]}}, \mathcal{I}[b_2]|_{A_{[b_2]}}, \ldots, \mathcal{I}[b_m]|_{A_{[b_m]}}\},$$

where $a_i > b_1 > b_2 > \cdots > b_m > a_{i-1}$. Hence $\mathcal{I}[b_1]|_{A_{[b_1]}} \subset \mathcal{I}[b_2]|_{A_{[b_2]}} \subset \cdots \subset \mathcal{I}[b_m]|_{A_{[b_m]}}$. Let $B \in \mathrm{Max}(\mathcal{I}[b_1]|_{A_{[b_1]}})$, then $|B| = R_{\mathcal{I}[b_1]}(A_{[b_1]}) = R_{\mathcal{I}[b_2]}(A_{[b_2]}) = \cdots = R_{\mathcal{I}[b_m]}(A_{[b_m]})$, thus $B \in \mathrm{Max}(\mathcal{I}[a]|_{A_{[a]}})$ for all $a \in (a_{i-1}, a_i)$. Hence $a \wedge B \in \mathcal{I}$ and $A(x) \geqslant a$, $(\forall\, x \in B)$ for all $a \in (a_{i-1}, a_i)$.

Since $(E, \mathcal{I})$ is a closed $[0,1]$-matroid, $a_i \wedge B \in \mathcal{I}$ and $A(x) \geqslant a_i$, $(\forall x \in B)$, i.e. $B \in \mathcal{I}[a_i]|_{A_{[a_i]}}$. Hence $B \in \mathrm{Max}(\mathcal{I}[a_i]|_{A_{[a_i]}})$ and then $R_{\mathcal{I}[a_i]}(A_{[a_i]}) = |B| = R_{\mathcal{I}[a]}(A_{[a]})$ for all $a \in (a_{i-1}, a_i)$. This implies that $\sup I_i = a_i \in I_i$. □

4.4. Theorem. *Let $(E, \mathcal{I})$ be a closed $[0,1]$-matroid and $R_{\mathcal{I}} : [0,1]^E \to \mathbb{N}([0,1])$ the $[0,1]$-fuzzy rank function for $(E, \mathcal{I})$. Then*

$$R_{\mathcal{I}}(A)_{[a]} = R_{\mathcal{I}[a]}(A_{[a]})$$

for all $A \in [0,1]^E$, $a \in (0,1]$.

Proof. Let $A \in [0,1]^E$ and $a \in (0,1]$. By Theorem 4.2 and Lemma 4.3, $n \in R_{\mathcal{I}}(A)_{[a]}$ if and only if $n \leqslant R_{\mathcal{I}[a]}(A_{[a]})$ for all $n \in \mathbb{N}$. Hence $R_{\mathcal{I}}(A)_{[a]} = R_{\mathcal{I}[a]}(A_{[a]})$, $(\forall\, A \in [0,1]^E,\ a \in (0,1])$. □

4.5. Lemma. *Let $(E, \mathcal{I})$ be a $[0,1]$-matroid with fundamental sequence $a_0, a_1, \ldots, a_n$. For each $a \in (0,1]$, define $\bar{\mathcal{I}}_a = \mathcal{I}[\bar{a}_i]$, where $a_{i-1} < a \leqslant a_i$ and $\bar{a}_i = \frac{1}{2}(a_{i-1} + a_i)$. Let $\bar{\mathcal{I}} = \{A \in [0,1]^E : \forall a \in (0,1], A_{[a]} \in \bar{\mathcal{I}}_a\}$, then*

(1) *$(E, \bar{\mathcal{I}})$ is a closed and perfect $[0,1]$-matroid.*
(2) *$R_{\mathcal{I}} = R_{\bar{\mathcal{I}}}$.*

Proof. (1) Obviously, $\bar{\mathcal{I}}$ satisfies (LI1) and (LI2). Let $A \in [0,1]^E$. If $a \wedge A_{[a]} \in \bar{\mathcal{I}}$ for all $0 < a \leqslant 1$, then by the definition of $\bar{\mathcal{I}}$, $A_{[a]} = (a \wedge A_{[a]})_{[a]} \in \bar{\mathcal{I}}_a$ for all $0 < a \leqslant 1$, hence $A \in \bar{\mathcal{I}}$. This implies that $\bar{\mathcal{I}}$ satisfies (LI4).

For any $a \in (0,1]$, let $A \in \bar{\mathcal{I}}[a]$. Then $a \wedge A \in \bar{\mathcal{I}}$, thus $A = (a \wedge A)_{[a]} \in \bar{\mathcal{I}}_a$, hence $\bar{\mathcal{I}}[a] \subseteq \bar{\mathcal{I}}_a$. Conversely, let $A \in \bar{\mathcal{I}}_a$. It is obvious that $(a \wedge A)_{[b]} = A \in \bar{\mathcal{I}}_a \subseteq \bar{\mathcal{I}}_b$ for every $b \leqslant a$ and $(a \wedge A)_{[b]} = \emptyset \in \bar{\mathcal{I}}_b$ for every $b > a$, hence $a \wedge A \in \bar{\mathcal{I}}$, thus $A = (a \wedge A)_{[a]} \in \bar{\mathcal{I}}[a]$.

This implies that $\bar{\mathcal{I}}_a \subseteq \bar{\mathcal{I}}[a]$. Therefore, $\bar{\mathcal{I}}[a] = \bar{\mathcal{I}}_a$ for all $a \in (0,1]$.

By Theorem 3.5 and the definition of $\bar{\mathcal{I}}$, $(E, \bar{\mathcal{I}})$ is a closed and perfect $[0,1]$-matroid.

(2) We need to prove that $R_{\mathcal{I}}(A) = R_{\bar{\mathcal{I}}}(A)$, $\forall\, A \in [0,1]^E$. Let $A \in \mathcal{I}$. Then $A_{[a]} \in \mathcal{I}[a] \subseteq \bar{\mathcal{I}}_a$ for all $a \in (0,1]$, thus $A \in \bar{\mathcal{I}}$ by the definition of $\bar{\mathcal{I}}$. This implies that $\mathcal{I} \subseteq \bar{\mathcal{I}}$. Hence $R_{\mathcal{I}}(A) \leqslant R_{\bar{\mathcal{I}}}(A)$.

Conversely, we need to prove $R_{\bar{\mathcal{I}}}(A) \leqslant R_{\mathcal{I}}(A)$, i.e. $R_{\bar{\mathcal{I}}}(A)_{[a]} \subseteq R_{\mathcal{I}}(A)_{[a]}$ for all $a \in (0,1]$. Let $n \in R_{\bar{\mathcal{I}}}(A)_{[a]}$. Then $n \leqslant R_{\bar{\mathcal{I}}[a]}(A_{[a]})$ by Theorem 4.4, i.e. there exists $B \in \bar{\mathcal{I}}[a]$ such that $B \subseteq A_{[a]}$ and $n \leqslant |B|$. By the definition of $\bar{\mathcal{I}}$, $B \in \mathcal{I}[b]$ for any $0 < b < a$. Thus $b \wedge B \in \mathcal{I}$, $b \wedge B \leqslant A$ for any $0 < b < a$ and $|b \wedge B|(n) = b$. Hence $R_{\mathcal{I}}(A)(n) \geqslant \bigvee\{|b \wedge B|(n) : 0 < b < a\} = a$, i.e. $n \in R_{\mathcal{I}}(A)_{[a]}$.

This implies that $R_{\bar{\mathcal{I}}}(A)_{[a]} \subseteq R_{\mathcal{I}}(A)_{[a]}$. Therefore, we have $R_{\mathcal{I}} = R_{\bar{\mathcal{I}}}$. □

By Lemma 4.5, Example 3.2 and Example 3.9, we obtain the following result:

4.6. Remark. In general, a $[0,1]$-matroid (resp., a perfect $[0,1]$-matroid, a closed $[0,1]$-matroid) is not in one-to-one correspondence with its $[0,1]$-fuzzy rank function.

By Lemma 4.5, we can limit our study of fuzzy rank functions for $[0,1]$-matroids to closed and perfect $[0,1]$-matroids.

4.7. Theorem. *Let $(E, \mathcal{I})$ be a $[0,1]$-matroid and $R_{\mathcal{I}} : [0,1]^E \to \mathbb{N}([0,1])$ the $[0,1]$-fuzzy rank function for $(E, \mathcal{I})$. Then $R_{\mathcal{I}}$ satisfies the following conditions:*

(LR1) *For any* $A \in [0,1]^E$, $\underline{0} \leqslant R_{\mathfrak{I}}(A) \leqslant |A|$;
(LR2) *If* $A, B \in [0,1]^E$ *and* $A \leqslant B$, *then* $R_{\mathfrak{I}}(A) \leqslant R_{\mathfrak{I}}(B)$;
(LR3) *For any* $A, B \in [0,1]^E$, $R_{\mathfrak{I}}(A) + R_{\mathfrak{I}}(B) \geqslant R_{\mathfrak{I}}(A \vee B) + R_{\mathfrak{I}}(A \wedge B)$;
(LR4) *For any* $A \in [0,1]^E$ *and any* $a \in (0,1]$, $R_{\mathfrak{I}}(a \wedge A_{[a]})_{[a]} = R_{\mathfrak{I}}(A)_{[a]}$.

Proof. By [6, Theorem 3.14], $R_{\mathfrak{I}}$ satisfies (LR1)-(LR3). We only need to check that $R_{\mathfrak{I}}$ satisfies (LR4). For any $A \in \mathfrak{I}$ and $a \in (0,1]$, by Theorem 4.4 and Lemma 4.5,

$$R_{\mathfrak{I}}(A)_{[a]} = R_{\bar{\mathfrak{I}}}(A)_{[a]} = R_{\bar{\mathfrak{I}}[a]}(A_{[a]}) = R_{\bar{\mathfrak{I}}[a]}((a \wedge A_{[a]})_{[a]}) = R_{\bar{\mathfrak{I}}}(a \wedge A_{[a]})_{[a]}$$
$$= R_{\mathfrak{I}}(a \wedge A_{[a]})_{[a]}.$$
□

4.8. Lemma. *Let* $(E, \mathfrak{I})$ *be a closed and perfect* $[0,1]$*-matroid and* $R_{\mathfrak{I}} : [0,1]^E \to \mathbb{N}([0,1])$ *the* $[0,1]$*-fuzzy rank function for* $(E, \mathfrak{I})$*. Then* $A \in \mathfrak{I} \iff R_{\mathfrak{I}}(A) = |A|$.

Proof. Let $A \in \mathfrak{I}$, then $A_{[a]} \in \mathfrak{I}[a]$ for all $a \in (0,1]$, hence

$$R_{\mathfrak{I}}(A)(n) = \bigvee \left\{a \in (0,1] : n \leqslant R_{\mathfrak{I}[a]}(A_{[a]})\right\} = \bigvee \left\{a \in (0,1] : n \leqslant |A_{[a]}|\right\} = |A|(n),$$

i.e. $R_{\mathfrak{I}}(A) = |A|$.

Conversely, let $A \in [0,1]^E$ and $R_{\mathfrak{I}}(A) = |A|$, then $R_{\mathfrak{I}[a]}(A_{[a]}) = R_{\mathfrak{I}}(A)_{[a]} = |A|_{[a]} = |A_{[a]}|$ for all $a \in (0,1]$ by Theorem 4.4 and Lemma 2.5, i.e. $A_{[a]} \in \mathfrak{I}[a]$ for all $a \in (0,1]$, hence $a \wedge A_{[a]} \in \mathfrak{I}$ for all $a \in (0,1]$ by (LI2). Since $(E, \mathfrak{I})$ is a perfect $[0,1]$-matroid, $A \in \mathfrak{I}$. □

4.9. Lemma. *For any* $\lambda, \mu \in \mathbb{N}(L)$ *and for any* $a \in J(L)$*, it follows that*

$$(\lambda + \mu)_{[a]} = \lambda_{[a]} + \mu_{[a]}.$$

Proof. By [5, Theorem 19], we only need to prove that $(\lambda + \mu)_{[a]} \subseteq \lambda_{[a]} + \mu_{[a]}$. Suppose that $n \in (\lambda + \mu)_{[a]}$. Then

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)) \geqslant a.$$

By $a \in J(L)$, we know that there exist $k, l \in \mathbb{N}$ with $n = k + l$ such that $\lambda(k) \wedge \mu(l) \geqslant a$. This implies that $k \in \lambda_{[a]}$ and $l \in \mu_{[a]}$, i.e. $n \in \lambda_{[a]} + \mu_{[a]}$. Hence $(\lambda + \mu)_{[a]} \subseteq \lambda_{[a]} + \mu_{[a]}$. □

By Lemma 4.9, we can easily obtain the following lemma.

4.10. Lemma. *Let* E *be a finite set and* $R : [0,1]^E \to \mathbb{N}([0,1])$ *a mapping satisfying* (LR1)-(LR4)*. Define* $R_a : 2^E \to \mathbb{N}$ *for each* $a \in (0,1]$ *by* $R_a(A) = R(a \wedge A)_{[a]}$*. Then* R_a *satisfies the following conditions* (R1), (R2) *and* (R3)*. Hence there exists a crisp matroid* $(E, \mathfrak{I}_{R_a})$ *such that* R_a *is the rank function for* $(E, \mathfrak{I}_{R_a})$*:*

(R1) *For any* $A \in 2^E$, $0 \leqslant R_a(A) \leqslant |A|$;
(R2) *For each* $A, B \in 2^E$ *and* $A \subseteq B$, *then* $R_a(A) \leqslant R_a(B)$;
(R3) *For each* $A, B \in 2^E$, $R_a(A) + R_a(B) \geqslant R_a(A \cup B) + R_a(A \cap B)$, *where*

$$\mathfrak{I}_{R_a} = \{A \in 2^E : R_a(A) = |A|\}.$$
□

4.11. Lemma. *If* $a \geqslant b$*, then* $\mathfrak{I}_{R_a} \subseteq \mathfrak{I}_{R_b}$.

Proof. Let $A \in \mathfrak{I}_{R_a}$. Then $|A| = R_a(A) = R(a \wedge A)_{[a]} \leqslant R(a \wedge A)_{[b]}$. By (LR4), $R(a \wedge A)_{[b]} = R(b \wedge A)_{[b]} = R_b(A)$. Hence $|A| \leqslant R_b(A)$, thus $|A| = R_b(A)$ by (R1), i.e. $A \in \mathfrak{I}_{R_b}$. This implies that $\mathfrak{I}_{R_a} \subseteq \mathfrak{I}_{R_b}$. □

4.12. Theorem. *Let* E *be a finite set and* $R : [0,1]^E \to \mathbb{N}([0,1])$ *a mapping satisfying* (LR1)-(LR4)*. Define* $\mathfrak{I}_R = \{A \in [0,1]^E : \forall a \in (0,1], A_{[a]} \in \mathfrak{I}_{R_a}\}$*, then*

(1) $\mathfrak{I}_R = \{A \in [0,1]^E : R(A) = |A|\}$.

(2) $(E, \mathcal{I}_R)$ *is a closed and perfect* $[0,1]$*-matroid.*
(3) R *is the* $[0,1]$*-fuzzy rank function for* $(E, \mathcal{I}_R)$.

Proof. (1) Let $A \in \mathcal{I}_R$. Then $|A|_{[a]} = |A_{[a]}| = R(a \wedge A_{[a]})_{[a]} = R(A)_{[a]}$ for all $a \in (0,1]$ by Lemma 2.5 and (LR4), hence $|A| = R(A)$.

Conversely, let $A \in [0,1]^E$ with $|A| = R(A)$. Then $|A_{[a]}| = |A|_{[a]} = R(A)_{[a]} = R(a \wedge A_{[a]})_{[a]} = R_a(A_{[a]})$ for all $a \in (0,1]$ by (LR4) and the definition of $R_a(A_{[a]})$, hence $A_{[a]} \in \mathcal{I}_{R_a}$ for all $a \in (0,1]$ by the definition of $\mathcal{I}_{R_a}$. By the definition of $\mathcal{I}_R$, $A \in \mathcal{I}_R$.

(2) **Step 1** Obviously, $\mathcal{I}_R$ satisfies (LI1) and (LI2). Let $A \in [0,1]^E$. If $a \wedge A_{[a]} \in \mathcal{I}_R$ for all $0 < a \leqslant 1$, then by the definition of $\mathcal{I}_R$, $A_{[a]} = (a \wedge A_{[a]})_{[a]} \in \mathcal{I}_{R_a}$ for all $0 < a \leqslant 1$, hence $A \in \mathcal{I}_R$. This implies that $\mathcal{I}_R$ satisfies (LI4).

Step 2 We prove $\mathcal{I}_R[a] = \mathcal{I}_{R_a}$ $(\forall a \in (0,1])$. For each $a \in (0,1]$, let $A_{[a]} \in \mathcal{I}_R[a]$, where $A \in \mathcal{I}_R$. By the definition of $\mathcal{I}_R$, $A_{[a]} \in \mathcal{I}_{R_a}$. This implies that $\mathcal{I}_R[a] \subseteq \mathcal{I}_{R_a}$.

Conversely, let $A \in \mathcal{I}_{R_a}$, then $|A| = R(a \wedge A)_{[a]}$. For all $b \in (0,1]$, $(a \wedge A)_{[b]} = A \in \mathcal{I}_{R_a} \subseteq \mathcal{I}_{R_b}$ for each $b \leqslant a$ by Lemma 4.11, $(a \wedge A)_{[b]} = \emptyset \in \mathcal{I}_{R_b}$ for each $b > a$. Thus $a \wedge A \in \mathcal{I}_R$, hence $A = (a \wedge A)_{[a]} \in \mathcal{I}_R[a]$. This implies that $\mathcal{I}_{R_a} \subseteq \mathcal{I}_R[a]$.

Step 3 Let $A \in 2^E$ and $a \in (0,1]$. If $b \wedge A \in \mathcal{I}_R$ for all $0 < b < a$, then

$$R(a \wedge A) \geqslant \bigvee_{0<b<a} R(b \wedge A) = \bigvee_{0<b<a} |b \wedge A| = |a \wedge A|$$

by (LR2) and (1), hence $R(a \wedge A) = |a \wedge A|$ by (LR1). By (1), $a \wedge A \in \mathcal{I}_R$.

Step 4 By Step 1, Step 2 and Theorem 3.5, $(E, \mathcal{I}_R)$ is a perfect $[0,1]$-matroid. Thus $(E, \mathcal{I}_R)$ is a closed and perfect $[0,1]$-matroid by Step 3 and Theorem 3.8.

(3) $\forall A \in [0,1]^E, a \in (0,1]$. By Theorem 4.4, Step 2 and (LR4),

$$R_{\mathcal{I}_R}(A)_{[a]} = R_{\mathcal{I}_R[a]}(A_{[a]}) = R_{\mathcal{I}_{R_a}}(A_{[a]}) = R_a(A_{[a]}) = R(a \wedge A_{[a]})_{[a]} = R(A)_{[a]}.$$

Hence, $R_{\mathcal{I}_R}(A) = R(A)$, $(\forall A \in [0,1]^E)$, thus $R_{\mathcal{I}_R} = R$, i.e. R is the $[0,1]$-fuzzy rank function for $(E, \mathcal{I}_R)$. □

4.13. Theorem. *Let* $(E, \mathcal{I})$ *be a closed and perfect* $[0,1]$*-matroid, then* $\mathcal{I}_{R_\mathcal{I}} = \mathcal{I}$.

Proof. By Lemma 4.8 and Theorem 4.12 (1), we have $A \in \mathcal{I} \iff R_\mathcal{I}(A) = |A| \iff A \in \mathcal{I}_{R_\mathcal{I}}$. This implies that $\mathcal{I}_{R_\mathcal{I}} = \mathcal{I}$. □

4.14. Remark. For any $[0,1]$-matroid $(E, \mathcal{I})$, we have $\mathcal{I}_{R_\mathcal{I}} = \bar{\mathcal{I}}$. Indeed, by Lemma 4.5, $(E, \bar{\mathcal{I}})$ is a closed and perfect $[0,1]$-matroid and $R_\mathcal{I} = R_{\bar{\mathcal{I}}}$. Hence $\mathcal{I}_{R_\mathcal{I}} = \mathcal{I}_{R_{\bar{\mathcal{I}}}} = \bar{\mathcal{I}}$ by Theorem 4.13.

By Theorem 4.12 and Theorem 4.13, the following theorem is obvious.

4.15. Theorem. *A closed and perfect* $[0,1]$*-matroid is in one-to-one correspondence with its* $[0,1]$*-fuzzy rank function. That is, a closed and perfect* $[0,1]$*-matroid can be characterized by means of its* $[0,1]$*-fuzzy rank function.* □

5. Conclusions

The notions of closed $[0,1]$-matroid and perfect $[0,1]$-matroid are presented, and some of their basic properties are studied. For any $[0,1]$-matroid $(E, \mathcal{I})$, we can find a closed and perfect $[0,1]$-matroid $(E, \bar{\mathcal{I}})$ such that $\bar{\mathcal{I}} = \mathcal{I}_{R_\mathcal{I}}$ and $R_\mathcal{I} = R_{\bar{\mathcal{I}}}$. Therefore, a $[0,1]$-matroid (resp., a perfect $[0,1]$-matroid, a closed $[0,1]$-matroid) and its $[0,1]$-fuzzy rank function are not in one-to-one correspondence in general. Also, we can limit our study of fuzzy rank functions of $[0,1]$-matroids to closed and perfect $[0,1]$-matroids. A closed and perfect $[0, 1]$-matroid can be characterized by means of its $[0, 1]$-fuzzy rank function. That is, a closed and perfect $[0, 1]$-matroid and its $[0, 1]$-fuzzy rank function are in one-to-one correspondence.

Acknowledgments The authors would like to thank the unknown reviewers for their valuable comments and suggestions.

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