COMPLETION OF CONE METRIC SPACES

Thabet Abdeljawad*

Abstract

In this paper a completion theorem for cone metric spaces and a completion theorem for cone normed spaces are proved. The completion spaces are defined by means of an equivalence relation obtained by convergence via the scalar norm of the Banach space E.

1. Introduction and Preliminaries

Fixed point theory occupies a prominent place in the study of metric spaces. One of the important questions that may arise in this connection is whether metric spaces really provide enough space for this theory or not. Recently, in [6] the authors rather implied that the answer is no. Actually, they introduced the notion of cone metric space, and gave an example of a function which is a contraction in the category of cone metric spaces but not a contraction if considered over metric spaces and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. After that a series of articles about cone metric spaces started to appear. Some of those articles dealt with fixed point theorems in those spaces, specially in complete ones, and others dealt with the structure of the spaces themselves. For example we name [15, 7, 8, 12, 13, 11, 1, 9, 4] and [14]. Motivated by this, we shall prove two completion theorems for cone metric spaces and cone normed spaces in this article. Since we use the scalar norm of the Banach space containing the cone in defining these completions, we prefer to call them scalar completions, as in the titles of sections 2 and 3.

A subset P of a real Banach space E is called a *cone* if and only if

P1) P is closed, nonempty and $P \neq \{0\}$.
P2) If $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$.
P3) If both $x \in P$ and $-x \in P$ then $x = 0$.

---

*Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey. E-mail: thabet@cankaya.edu.tr

Given a cone P in E, a partial ordering $\leq$ on E via P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(P)$.

It is worthwhile to mention here, that there are certain real Banach spaces whose positive cones have empty interior, such as the sequences spaces l^p, $1 \leq p < \infty$ and the Lebesgue integrable spaces L_p, $1 \leq p < \infty$ [5]. On the other hand the positive cone in the Euclidean space $\mathbb{R}^n$ does not have empty interior. For example in $\mathbb{R}^2$ the interior of the positive cone $P = \{(x,y) : x \geq 0, y \geq 0\}$ is $\{(x,y) : x > 0, y > 0\}$, which is non-empty. For the infinite dimensional case the positive cones of $AM - spaces$ can have non-empty interiors [2]. For more details about cones we refer also to [3].

In the sequel, one also has to note that by using the properties of the cone and the definition of the interior that $\text{int}(P) + \text{int}(P) \subseteq \text{int}(P)$ and $\alpha \text{int}(P) \subseteq \text{int}(P)$, $\alpha > 0$. For the purposes in defining convergence [6] and other topological concepts in cone metric spaces [13], the cones under consideration are always assumed to have non-empty interiors.

The cone P is called *normal* if there exists a constant $K > 0$ such that for all $a, b \in E$, $0 \leq a \leq b$ implies $\|a\| \leq K\|b\|$. The cone $[0, \infty)$ in $(\mathbb{R}, |\cdot|)$, and the cone $P = \{(x,y) : x \geq 0, y \geq 0\}$ in $\mathbb{R}^2$ are normal cones with constant $K = 1$. However, there are examples of non-normal cones.

1.1. Example. [10] Let $E = C^1[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone $P = \{f \in E : f \geq 0\}$.

For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then, $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$. Since $k\|f\| < \|g\|$, k is not a normal constant of P and hence P is a non-normal cone.

There are no normal cones with normal constant $K < 1$. Indeed, if P were a normal cone with normal constant $K < 1$, we could choose a non-zero element $x \in P$ and $0 < \epsilon < 1$ such that $K < 1 - \epsilon$. Then, $(1 - \epsilon)x \leq x$, but $(1 - \epsilon)\|x\| > K\|x\|$, see [10].

For each $k > 1$, consider the real vector space

$$E = \left\{ f(x) = ax + b : a, b \in \mathbf{R}; x \in \left[1 - \tfrac{1}{k}, 1\right] \right\},$$

with the supremum norm and the cone $P = \{f(x) = ax + b : a \leq 0, b \geq 0\}$. Then, P is regular and hence normal. Moreover, it can be shown that the normal constant for this cone is bigger than k, for details see [10]. This shows that we can construct cones with different normal constants $K > 1$.

1.2. Definition. A *cone metric space* is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:

    d1) $0 < d(x,y)$ for all $x, y \in X$.
    d2) $d(x,y) = 0$ if and only if $x = y$.
    d3) $d(x,y) = d(y,x)$ for all $x, y \in X$.
    d4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

1.3. Definition. A sequence (x_n) in a cone metric space (X, d) is said to *converge* to an element $x \in X$ if for any $c \in E$ with $c \gg 0$ there exists a natural number n_0 such that

$$d(x_n, x) \ll c \text{ for all } n \geq n_0.$$

In this case we write $\lim_{n \to \infty} x_n = x$.

1.4. Definition. A sequence (x_n) in a cone metric space (X, d) is said to be *Cauchy* if for any $c \in E$ with $c \gg 0$ there exists a natural number n_0 such that

$$d(x_n, x_m) \ll c \text{ for all } m, n \geq n_0.$$

Cone metric spaces in which every Cauchy sequence is convergent are called *complete cone metric spaces.*

The following lemma, which characterizes convergence and Cauchyness by means of the scalar norm of the Banach space E under the normality assumption for the cone P, makes it possible to obtain a scalar norm completion for cone metric spaces.

1.5. Lemma. [6] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let (x_n) be a sequence in X. Then,*

  (i) *(x_n) converges to x if and only if $\lim_{n \to \infty} \|d(x_n, x)\| = 0$.*
  (ii) *(x_n) is Cauchy if and only if $\lim_{m,n \to \infty} \|d(x_n, x_m)\| = 0$.* $\square$

Regarding the above Lemma, it is worthwhile mentioning that normality is only used in proving the necessity (for example see [10]). In connection to this, we state and prove the following Lemma:

1.6. Lemma. *If (x_n) is a sequence in a cone metric space such that for some $c_0 \gg 0$ we have*

$$(1.1) \qquad d(x_n, x_m) \ll \frac{c_0}{n} \text{ for all } m > n,$$

then (x_n) is Cauchy.

Proof. Let $c \gg 0$ be arbitrary, hence find $\delta > 0$ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{b \in E : \|b\| < \delta\}$. Noting that $\frac{c_0}{n}$ is convergent to zero in $(E, \| \cdot \|)$, choose n_0 such that $-\frac{c_0}{n} \in N_\delta(0)$, for all $n \geq n_0$. Then, $\frac{c_0}{n} \ll c$ for all $n \geq n_0$ and hence by (1.1)

$$(1.2) \qquad d(x_n, x_m) \ll \frac{c_0}{n} \ll c \text{ for all } m > n \geq n_0.$$

1.7. Lemma. [6] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let (x_n) and (y_n) be two sequences in X such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. Then*

$$\lim_{n \to \infty} \|d(x_n, y_n) - d(x, y)\| = 0. \quad \square$$

1.8. Definition. A cone normed space is an ordered pair $(X, \| \cdot \|_c)$, where X is a vector space over $\mathbb{R}$ and $\| \cdot \|_c : X \to (E, \| \cdot \|)$ is a function satisfying:

  C1) $0 < \|x\|_c$, for all $x \in X$.
  C2) $\|x\|_c = 0$ if and only if $x = 0$.
  C3) $\|\alpha x\|_c = |\alpha| \|x\|_c$, for each $x \in X$ and $\alpha \in \mathbb{R}$.
  C4) $\|x + y\|_c \leq \|x\|_c + \|y\|_c$, for all $x, y \in X$.

It is clear that each cone normed space is a cone metric space. In fact, the cone metric is given by $d(x, y) = \|x - y\|_c$. Complete cone normed spaces are called *cone Banach spaces.*

According to the definition of convergence in cone metric spaces and Lemma 1.5, we see that $x_n \to x$ in $(X, \| \cdot \|_c)$ if and only if for all $c \gg 0$ in E there exists n_0 such that $\|x_n - x\|_c \ll c$ for all $n \geq n_0$ and, if the cone is normal, if and only if $\lim_{n \to \infty} \|\|x_n - x\|_c\| = 0$. Also, $x_n \in (X, \| \cdot \|_c)$ will be Cauchy if and only if for all $c \gg 0$ in E there exits n_0 such that $\|x_n - x_m\|_c \ll c$ for all $m, n \geq n_0$ and, if the cone is normal, if and only if $\lim_{m,n \to \infty} \|\|x_n - x_m\|_c\| = 0$.

1.9. Example. Let $X = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ and $\|(x, y)\|_c = (\alpha|x|, \beta|y|)$, $\alpha > 0$, $\beta > 0$. Then, $(X, \| \cdot \|_c)$ is a cone normed space over $\mathbb{R}^2$.

2. The scalar norm completion of cone metric spaces

Before proceeding to prove a scalar norm completion theorem, we first give the meaning of isometries of cone metric spaces.

2.1. Definition. Let (X, d) and (Y, ρ) be cone metric spaces. A mapping T of X into Y is said to be an *isometry* if it preserves cone distances, that is, if for all $x_1, x_2 \in X$,

$$(2.1) \qquad \rho(Tx_1, Tx_2) = d(x_1, x_2).$$

It is clear that if T is bijective and an isometry, then it is, together with its inverse, (sequentially) continuous and hence (X, d) and (Y, ρ) become topologically isomorphic [13]. Throughout, we shall say that the cone metric space X is isometric with the cone metric space Y if there exists a bijective isometry of X onto Y. In the sequel, one has to note that every cone isometry is one to one.

The following lemma will be useful in proving a scalar-norm completion theorem for cone metric spaces.

2.2. Lemma. *Let (x_n) and (y_n) be two Cauchy sequences in a cone metric space (X, d) over a normal cone with constant K. Then, $\lim_{n \to \infty} d(x_n, y_n)$ exists in $(E, \| \cdot \|)$.*

Proof. Since $(E, \| \cdot \|)$ is a Banach space, it will be enough to show that the sequence $\{d(x_n, y_n)\}$ is Cauchy in $(E, \| \cdot \|)$. To this end, let $\epsilon > 0$ and choose $c \in E$ with $c \gg 0$ and $\|c\| < \frac{\epsilon}{4K+2}$. Since (x_n) and (y_n) are Cauchy sequences, there exists a natural number n_0 such that

$$(2.2) \qquad d(x_i, x_j) \ll c \text{ and } d(y_i, y_j) \ll c$$

for all $i, j > n_0$. Then, we have

$$(2.3) \qquad d(x_i, y_i) \le d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i) \le d(x_j, y_j) + 2c$$

and

$$(2.4) \qquad d(x_j, y_j) \le d(x_j, x_i) + d(x_i, y_i) + d(y_i, y_j) \le d(x_i, y_i) + 2c,$$

and hence, (2.3) and (2.4) lead to

$$(2.5) \qquad 0 \le d(x_j, y_j) + 2c - d(x_i, y_i) \le d(x_i, y_i) + 2c + 2c - d(x_i, y_i) = 4c.$$

Since the cone is normal, then (2.5) implies that

$$(2.6) \qquad \|d(x_j, y_j) + 2c - d(x_i, y_i)\| \le K\|4c\|.$$

Finally, by the triangle inequality of the norm $\| \cdot \|$ and (2.6) we have

$$(2.7) \qquad \|d(x_j, y_j) - d(x_i, y_i)\| \le \|d(x_j, y_j) + 2c - d(x_i, y_i)\| + \|2c\| \le \|c\|(4K+2) < \epsilon.$$

Therefore, $\{d(x_i, y_i)\}$ is Cauchy in $(E, \| \cdot \|)$ and hence convergent. $\qquad \square$

2.3. Lemma. *Let (x_n), (x_n'), (y_n), (y_n') be sequences in a cone metric space (X, d) over a normal cone P with normal constant K. If*

$$\lim_{n \to \infty} d(x_n, x_n') = 0 \ \text{ and } \ \lim_{n \to \infty} d(y_n, y_n') = 0$$

in $(E, \| \cdot \|)$, then

$$(2.8) \qquad \lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x_n', y_n') \ \text{ in } (E, \| \cdot \|).$$

Proof. Let $\epsilon > 0$ and choose $c \in E$ such that $c \gg 0$ and $\|c\| < \frac{\epsilon}{4K+2}$. For this $c \gg 0$ find $\delta > 0$ such that

(2.9) $\|x\| < \delta$ implies $c - x \in \text{int}(P)$.

By assumption, for the above $\delta > 0$ find n_0 such that for all $n \geq n_0$ we have

(2.10) $\|d(x_n, x'_n)\| < \delta$ and $\|d(y_n, y'_n)\| < \delta$.

But then (2.9) and (2.10) imply that

(2.11) $d(x_n, x'_n) \ll c$ and $d(y_n, y'_n) \ll c$,

for all $n > n_0$. Now, by the triangle inequality and (2.11), for all $n \geq n_0$ we have

(2.12) $d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \leq d(x'_n, y'_n) + 2c$

and

(2.13) $d(x'_n, y'_n) \leq d(x_n, x'_n) + d(x_n, y_n) + d(y'_n, y_n) \leq d(x_n, y_n) + 2c$,

and hence, (2.12) and (2.13) lead to

(2.14) $0 \leq d(x'_n, y'_n) + 2c - d(x_n, y_n) \leq d(x_n, y_n) + 2c + 2c - d(x_n, y_n) = 4c$.

Since the cone is normal, then (2.14) together with the choice of $c \gg 0$ imply that

(2.15) $\|d(x_n, y_n) - d(x'_n, y'_n)\| \leq \|d(x'_n, y'_n) + 2c - d(x_n, y_n)\| + \|2c\| < \epsilon$

for all $n > n_0$, which completes the proof. $\square$

We can now state and prove the theorem that every cone metric space can be completed. The space X_s appearing in this theorem is called the *norm scalar completion* of the given space X.

2.4. Theorem. *For a cone metric space (X, d) over a normal cone there exists a complete cone metric space (X_s, d_s) which has a subspace W that is isometric with X and dense in X_s. The space (X^s, d_s) is unique up to isometry, that is, if Z is any complete cone metric space having a dense subspace U isometric with X, then Z and X^s are isometric.*

Proof. The proof will be divided into four steps. We construct:

(a) (X^s, d_s),
(b) An isometry T of X onto W, where W is dense in X^s.

Then, we prove

(c) The completeness of X^s,
(d) Uniqueness of X^s except for isometries.

Here are the details of these steps:

(a) Let (x_n) and (x'_n) be Cauchy sequences in (X, d). Define (x_n) to be equivalent to (x'_n), written $(x_n) \sim (x'_n)$, if

(2.16) $\lim_{n \to \infty} d(x_n, x'_n) = 0$ in $(E, \|\cdot\|)$.

Let X^s be the set of all equivalence classes $x^s, y^s, \ldots$ of Cauchy sequences. We write $(x_n) \in x^s$ to mean that (x_n) is a member x^s (a representative of the class x^s). We now set

(2.17) $d_s(x^s, y^s) = \lim_{n \to \infty} d(x_n, y_n)$.

By Lemma 2.2, the limit in (2.17) exists and by Lemma 2.3 it is independent of the particular choice of the representatives.

The proof that d_s satisfies the first three axioms of cone metrics is straightforward. The triangle inequality follows from the closeness of the cone P and from

$$(2.18) \quad d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n),$$

by letting $n \to \infty$.

(b) Define $T : X \to X^s$ as follows. For each $b \in X$ associate the class $b^s \in X^s$ which contains the constant Cauchy sequence $(b, b, \ldots)$. Then, let $T(b) = b^s$, where $(b, b, \ldots) \in b^s$. Let $W = T(X)$. We see that T is an isometry because (2.17) becomes simply

$$d_s(b^s, a^s) = d(b, a),$$

where a^s is the class of the sequence (y_n) with $y_n = a$ for all n. Any isometry is one to one, and $T : X \to W$ is onto. Hence W and X are isometric.

To show that W is dense in X^s, take $x^s \in X^s$. Let $(x_n) \in x^s$. Since (x_n) is Cauchy, for every $c \gg 0$ there is a natural number n_0 such that

$$(2.19) \quad d(x_n, x_{n_0}) \ll \frac{c}{2} \text{ for all } n \geq n_0.$$

Let $(x_{n_0}, x_{n_0}, \ldots) \in x_{n_0}^s$. Then $x_{n_0}^s \in W$. By (2.17),

$$(2.20) \quad ds(x^s, x_{n_0}^s) = \lim_{n \to \infty} d(x_n, x_{n_0}) \leq \frac{c}{2} \ll c.$$

This shows that every c-neighborhood of $x^s \in X^s$ contains an element of W. Hence, W is dense in X^s.

(c) Completeness of (X^s, d_s): Let x_n^s be any Cauchy sequence in (X^s, d_s). Since W is dense in (X^s, d_s), then for each fixed $c \gg 0$ and every x_n^s there is $z_n^s \in W$ such that

$$(2.21) \quad d_s(x_n^s, z_n^s) \ll \frac{c}{n}.$$

Hence by the triangle inequality,

$$(2.22) \quad d_s(z_m^s, z_n^s) \leq d_s(z_m^s, x_m^s) + d_s(x_m^s, x_n^s) + d_s(x_n^s, z_n^s) \ll \frac{c}{m} + d_s(x_m^s, x_n^s) + \frac{c}{n}.$$

Apply the norm $\| \cdot \|$ for (2.22), use normality of P, take the limit as $m, n \to \infty$ of both sides and make use of Lemma 1.5 to conclude that (z_m^s) is a Cauchy sequence in (X^s, d_s), or alternatively, Cauchyness can be obtained from Lemma 1.6 without using normality. Since $T : X \to W$ is cone isometric, then clearly the sequence $z_m = T^{-1} z_m^s$ is also Cauchy in (X, d). Let x^s be the equivalence class to which (z_m) belongs. We show that x^s is the limit of x_n^s in (X^s, d_s).

By (2.21),

$$(2.23) \quad d_s(x_n^s, x^s) \leq d_s(x_n^s, z_n^s) + d_s(z_n^s, x^s) \ll \frac{c}{n} + d_s(z_n^s, x^s).$$

Since $z_m \in x^s$ and $z_n^s \in W$, so that $(z_n, z_n, z_n, \ldots) \in z_n^s$, the above inequality (2.23) becomes

$$(2.24) \quad d_s(x_n^s, x^s) \leq d_s(x_n^s, z_n^s) + d_s(z_n^s, x^s) \ll \frac{c}{n} + \lim_{m \to \infty} d(z_n, z_m).$$

Then with the help of the normality of the cone P and the fact that z_n is Cauchy, (2.24) (or alternatively, by an idea similar to Lemma 1.6 without using normality) shows that $x_n^s \to x^s$ in (X^s, d_s).

(d) Uniqueness of (X^s, d_s) up to isometry: If (Y^s, ρ_s) is another complete cone metric space with U dense in (Y^s, ρ_s) and isometric with X, then for any $x^s, y^s \in X^s$ we have the

sequences (x_n^s), (y_n^s) in W such that $x_n^s \to x^s$ and $y_n^s \to y_s$ in (X^s, d_s); hence Lemma 1.7 implies that

$$(2.25) \quad d_s(x^s, y^s) = \lim_{n \to \infty} d_s(x_n^s, y_n^s).$$

That U is isometric with W and U is dense in (Y^s, ρ_s) lead to the cone distances on X^s and Y^s being the same. Hence, X^s and Y^s are isometric. $\qquad\square$

3. The scalar norm completion of cone normed spaces

As every cone normed space is a cone metric space, and cone metric spaces can be completed, as we have done in the previous section, this suggests a method for completing a cone normed spaces. Before stating and proving this result, we define the meaning of isometry of cone normed spaces.

3.1. Definition. Two cone normed spaces $(X, \| \cdot \|_{c_1})$ and $(Y, \| \cdot \|_{c_2})$ are said to be *isometric* if there exists a bijective linear operator $T : X \to Y$ such that

$$\|Tx\|_{c_2} = \|x\|_{c_1}, \text{ for all } x \in X.$$

3.2. Theorem. *Let $(X, \| \cdot \|_c)$ be a cone normed space over a normal cone. Then there is a cone Banach space $(X^s, \| \cdot \|_s)$ and an isometry T from X onto a subspace W of X^s which is dense in X^s. The space X^s is unique up to isometry.*

Proof. Since cone normed spaces are cone metric spaces, then Theorem2.4 implies the existence of a complete cone metric space (X^s, d_s) and an isometry $T : X \to W = T(X)$, where W is dense in X^s and X^s is unique up to isometry. Consequently, to prove the present theorem, we must make X^s into a vector space, then provide X^s with a suitable cone norm.

To define on X^s the two algebraic operations of a vector space, we consider any $x^s, y^s \in X^s$ and any representatives $(x_n) \in x^s$ and $(y_n) \in y^s$, where to recall that x^s and y^s are equivalence classes of Cauchy sequences in $(X, \| \cdot \|_c)$. Set $z_n = x_n + y_n$. Then (z_n) is Cauchy sequence in $(X, \| \cdot \|_c)$ since

$$\|z_n - z_m\|_c = \|x_n + y_n - (x_n + x_m)\|_c \leq \|x_n - x_m\|_c + \|y_n - y_m\|_c.$$

Define the sum $z^s = x^s + y^s$ of x^s and y^s to be the equivalence class for which (z_n) is a representative; thus $(z_n) \in z^s$. This definition is independent of the particular choice of Cauchy sequences belonging to x^s and y^s. In fact, if $(x_n) \sim (x_n')$ (i.e., $\lim_{n\to\infty} \|x_n - x_n'\|_c = 0$ in $(E, \| \cdot \|)$) and $(y_n) \sim (y_n')$ (i.e., $\lim_{n\to\infty} \|y_n - y_n'\|_c = 0$ in $(E, \| \cdot \|)$), then normality of the cone P and

$$\|x_n + y_n - (x_n' + y_n')\|_c \leq \|x_n - x_n'\|_c + \|y_n - y_n'\|_c$$

imply that $(x_n + y_n) \sim (x_n' + y_n')$.

Similarly, the product $\alpha x^s \in X^s$ of a scalar α and x^s is defined as the equivalence class for which the sequence (αx_n) is a representative. Again, this definition is independent of the particular choice of a representative of x^s. The zero element of X^s is the equivalence class containing all Cauchy sequences which converge to zero in the sense of the cone norm. Showing that the two defined algebraic operations satisfy all the required properties for a vector space is straightforward. Also, from the definition it follows that on W the vector space operations induced from X^s agree with those induced from X by means of T.

Furthermore, T induces on W a cone norm $\| \cdot \|_1$ whose value at every $y^s = Tx^s \in W$ is $\|y^s\|_1 = \|x\|_c$. The corresponding cone metric on W is the restriction of d_s to W, since T is an isometry. The cone norm $\| \cdot \|_1$ can be extended to X^s by setting $\|x^s\|_s = d_s(0^s, x^s)$

for every $x^s \in X^s$. In fact, it is obvious that $\|\cdot\|_s$ satisfies N_1 and N_2, and the other two axioms follow from those for $\|\cdot\|_1$ by a limit process. $\qquad\qquad\qquad\square$

References

[1] Abbas, M. and Rhoades, B. E. *Fixed and periodic point results in cone metric spaces*, Applied Mathematics Letters **22** (4), 511–515, 2009.

[2] Abramovich, Y. A. and Aliprantis, C. D. *An invitation to Operator Theory* (American Mathematical Society, 2002).

[3] Aliprantis, C. D. and Tourky, R. *Cones and Duality* (American Mathematical Society, 2007).

[4] Altun, İ. and Durmaz, G. *Some fixed point theorems on ordered cone metric spaces*, Rendiconti del Circolo Mathematico di Palermo **58**, 319–325, 2009.

[5] Deimling, K. *Nonlinear Functional Analysis* (Springer-Verlage, 1985).

[6] Huang, L.-G. and Zhang, X. *Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications **323**, 1468–1476, 2007.

[7] Iliç, D. and Rakocevic, V. *Quasi-contraction on a cone metric space*, Applied Mathematics Letters **22**, 728–731, 2009.

[8] Iliç, D. and Rakocevic, V. *Common fixed points for maps on cone metric space*, Journal of Mathematical Analysis and Applications **341**, 876–882, 2008.

[9] Radenovìc, S. and Rhoades, B. E. *Fixed point theorem for two non-self mappings in cone metric spaces*, Computers and Mathematics with Applications **57** (10), 1701–1707, 2009.

[10] Rezapour, Sh. and Hamlbarani Haghi, R. *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"* Journal of Mathematical Analysis and Applications **345**, 719–724, 2008.

[11] Rezapour, Sh. and Hamlbarani Haghi, R. *Fixed points of multifunctions on regular cone metric spaces*, Expositiones Mathematicae **28** (1), 71–77, 2010.

[12] Şahin, İ. and Telci, M. *Fixed points on contractive mappings on complete cone metric spaces*, Hacettepe J. Math. Stat. **38** (1), 59–67, 2009.

[13] Turkoglu, D. and Abuloha, M. *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta Mathematica Sinica (English Series) **26** (3), 489–496, 2010.

[14] Turkoglu, D., Abuloha, M. and Abdeljawad, T. *KKM mappings in cone metric spaces and some fixed point theorems*, Nonlinear Analysis: Theory, Methods & Applications **72** (1), 348–353, 2010.

[15] Wardowski, D. *Endpoints and fixed points of set-valued contractions in cone metric spaces*, Nonlinear Analysis, Theory, Methods and Applications **71**, 512–516, 2009.