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FINITE LENGTH MODULES OVER HNPs

Irawati*

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Abstract

We characterize a finite length module over HNP that is annihilated by an invertible ideal. We also characterize finite length modules over HNP that have no composition factors annihilated by an invertible ideal. The two characterizations are used to prove Levy's Theorem about the decomposition of finite length modules over HNPs. We also prove that the ring of matrices over a uniserial ring is serial by generalizing the technique of proving that the ring of matrices over a simple ring is simple. This is done by exploring the form of a one sided ideal of a matrix ring. We also characterize a uniserial Artinian ring as a local, principle ideal, Artinian ring. We use the two results to prove that the component that is annihilated by an invertible ideal in the Levy decomposition is a serial module.

Keywords: HNP ring, Uniserial, Serial, Invertible ideal.

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1. Results

The Theorem of decomposing a finite length module over HNPs [4], will be proved in another way. Let R be a HNP ring. First, we state Lemma 4.3(i) in [4].

1.1. Lemma. *Let S, T be simple R-modules. If there is an invertible ideal annihilating S but not annihilating T, then $Ext^1(S,T) = 0$.*

In the next Theorem, we characterize a finite length R-module that is annihilated by an invertible ideal.

1.2. Theorem. *Let M be a finite length R-module. Then M is annihilated by an invertible ideal if and only if each of its simple submodules is annihilated by an invertible ideal.*

*Algebra research group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Ganesha no 10, Bandung, Indonesia. E-mail: `irawati@math.itb.ac.id`

Proof. We prove by induction on the composition length of M. Let S be a simple module of M annihilated by an invertible ideal I, and let T be a simple submodule of M/S. If $Ext^1(S,T) = 0$, then T is isomorphic to a submodule of M and hence is annihilated by an invertible ideal. If $Ext^1(S,T)$ is non zero, then by Lemma 1.1, T is annihilated by the same invertible ideal I as is S. Either way, the hypotheses of the Theorem are satisfied by M/S, so by induction it is annihilated by some invertible ideal J, and then M is annihilated by JI. □

As a consequence we have the following:

1.3. Corollary. *Let M be a finite length R-module not annihilated by invertible ideals. Then M has a simple submodule not annihilated by invertible ideals.*

Next we will see a theorem related to the extension of a finite length R-module by a finite length R-module.

1.4. Theorem. *Let T, M be finite length R-modules. Let $Ext^1(T,X) = 0$ for every composition factor X of M. Then $Ext^1(T,M) = 0$.*

Proof. The proof is by induction on the composition length of M, the case of length 1 being trivial. Let us consider the following exact sequence:

$$Ext^1(T,S) \longrightarrow Ext^1(T,M) \longrightarrow Ext^1(T,M/S).$$

Note that $Ext^1(T,M/S) = 0$ by induction, and $Ext^1(T,S) = 0$ by assumption, so $Ext^1(M,S) = 0$. □

Similarly we have the next Theorem:

1.5. Theorem. *Let T, M be finite length R-modules. Let $Ext^1(X,T) = 0$ for every composition factor X of M. Then $Ext^1(M,T) = 0$.*

In the next Theorem we characterize a finite length module that has no composition factors annihilated by invertible ideals.

1.6. Theorem. *Let M be a finite length R-module. Then M has no composition factors annihilated by invertible ideals if and only if each of its simple submodules is not annihilated by invertible ideals.*

Proof. Suppose that each of the simple submodules of M is not annihilated by invertible ideals. The theorem will be proved by induction on the length. Let the proposition be true for modules with length $< n$. It will be proved that it is true for modules with length n. Let M_1 be a submodule of M with length $n-1$. Then M_1 only has simple submodules that are not annihilated by invertible ideals. For the next composition series

$$0 \subseteq S = M_{n-1} \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M,$$

the factor module M_1/S has no composition factors annihilated by invertible ideals. It will be proved that M/M_1 is not annihilated by invertible ideals. We can see that

$$M/M_1 \approx (M/S)/(M_1/S).$$

Let us consider the next exact sequence

$$0 \longrightarrow M_1/S \longrightarrow M/S \longrightarrow (M/S)/(M_1/S).$$

Suppose $(M/S)/(M_1/S)$ is annihilated by an invertible ideal. By Lemma 1.1,

$$Ext((M/S)/(M_1/S), T) = 0$$

for every composition factor T of M_1/S. By Theorem 1.4, then

$$Ext((M/S)/(M_1/S), M_1/S) = 0.$$

This means that the exact sequence above is split. So we have $(M/S)/(M_1/S)$ is isomorphic with a simple submodule of M/S. Let the simple submodule be K/S. Then K/S is a composition factor of a submodule of M with length $n-1$. So K/S is not annihilated by an invertible ideal. This is in contradiction with the fact that $(M/S)/(M_1/S)$ is annihilated by an invertible ideal. So we have that M/M_1 is not annihilated by invertible ideals.

The converse is obviously true. □

The next Theorem is about the decomposition of a finite length module over HNPs. This Theorem was proved as Theorem 4.6 in [4], inductively on the length of the module. In this paper, it will be proved by separating it into simple submodules. The idea of this method came from the method used in [1], about the decomposition of a module over a Dedekind prime ring.

1.7. Theorem. *Let M be a finite length R-module. Then $M = M_u \oplus M_h$, where M_u is annihilated by an invertible ideal and M_h has no composition factors annihilated by invertible ideals.*

Proof. Let $S_1, \ldots, S_k$ be all of the different simple submodules annihilated by invertible ideals, and $S_{k+1}, \ldots, S_n$ all of the different simple submodules not annihilated by invertible ideals. Let

$$\Phi = \{T \mid T \text{ is a submodule of } M \text{ that has no composition factors annihilated by an invertible ideal}\}.$$

By the Noetherian property of M, then Φ has a maximal element. Let N be the maximal element of Φ. Firstly it will be showed that M/N is annihilated by an invertible ideal. Suppose it is not true. By Corollary 1.3, there is a simple submodule of M/N that is not annihilated by invertible ideals. Let M'/N be the simple submodule. Then the length of M' is the length of $N+1$. If M' has a simple submodule S_i that is annihilated by an invertible ideal, but N does not contain S_i, then there is a submodule $N' \supseteq S_i$, with the length of N' equal to the length of N and $N \cap N' = 0$. So we have $M' \cong N \oplus N'$. This cannot happen. So M' has no simple submodules annihilated by invertible ideals. According to Theorem 1.6, then M' is a submodule that has no composition factors annihilated by invertible ideals, with $M' \supseteq N$. This is a contradiction with the fact that N is a maximal submodule that has no composition factors annihilated by invertible ideals. So M/N must be annihilated by an invertible ideal.

Next it will be shown that $M = N \oplus M/N$. All of the composition factors of M/N are annihilated by an invertible ideal, because M/N is annihilated by an invertible ideal. By Lemma 1.1, we have $Ext(X, Y) = 0$ for all composition factors X of M/N and Y of N. By Theorem 1.5, then $Ext(M/N, Y) = 0$ for every composition factor Y of N. By Theorem 1.4, then $Ext(M/N, N) = 0$. And we have $M = N \oplus M/N$, where N has no composition factors annihilated by invertible ideals, and M/N is annihilated by an invertible ideal. □

It can be shown that we can obtain the maximal element of Φ by taking the sum of all the elements of Φ.

Next it will be shown that the submodule M_u in Theorem 1.7 is a uniserial module. Let M_u be annihilated by an invertible ideal I. Then we can see M_u as an R/I-module. Generally it will be proved that for every invertible ideal I of a HNP ring R, then R/I is a serial, Artinian ring. So M_u is a module over a serial, Artinian ring R/I. Then M_u is a serial R/I-module, that is also a serial R-module.

In [3], the theorem on the decomposition of an Artinian principal ideal ring was stated as follows:

1.8. Theorem. *An Artinian principal ideal ring can be decomposed into the direct sum of the ring of matrices over an Artinian, local, principal ideal ring.*

Next we will look at the structure of matrices over an Artinian, local, principal ideal ring. We start by showing that the structure of an Artinian, local, principal ideal ring is uniserial, and then we show that the matrices over a uniserial ring become a serial ring.

Let R be an Artinian ring. Then the Jacobson radical J is nilpotent. Consequently, R has finite length and there is a natural number t such that

$$0 \subseteq J^t \subseteq \cdots \subseteq J \subseteq R,$$

where $J^s = 0$ for every number $s \geq t+1$, is a chain of R. If we add the local property in R, then J is the only right/left maximal ideal. In the next theorem we will show that the ring will be uniserial, Artinian if we add the property that J is a principal ideal. We will also show the converse. To begin with we will first look at a Lemma.

1.9. Lemma. *Let M be a right module over an Artinian ring R. Then M is uniserial if and only if the following chain is a composition series.*

$$0 \subseteq MJ^n \subseteq \cdots \subseteq MJ \subseteq M$$

Proof. $\Longleftarrow$. Let N_1 be a maximal right submodule of M. If $MJ \neq N_1$, then $MJ + N_1 = M$. Following Nakayama [3], because R is Artinian, then $MJ <<< M$. So $M = N_1$. This is a contradiction with the fact that N_1 is a maximal submodule. So it must be $MJ = N_1$. Now let N_2 be a maximal submodule of MJ. If $MJ^2 \neq N_2$, then $MJ^2 + N_2 = MJ$. Because of $MJ << M$, then $MJ^2 = (MJ)J << MJ$. So $MJ = N_2$. This is a contradiction with N_2 being a maximal submodule of MJ. So it must be $MJ^2 = N_2$. The process is continued giving that M is uniserial.

$\Longrightarrow$. Let M/JM be a module over R/J. By the way we have R/J is semi simple because R is an Artinian ring. Then M/JM is a semi simple module. So $M/JM = \bigoplus M_i/JM$ with M_i/JM simple modules. Then we can make two different composition series as follow

$$JM \subseteq M_1 \subseteq M_1 \oplus M_2 \subseteq \cdots \subseteq M$$

and

$$JM \subseteq M_2 \subseteq M_1 \oplus M_2 \subseteq \cdots \subseteq M.$$

This cannot happen because M is uniserial. So M/JM must be simple. Next we have that JM/J^2M is an R/J-module, so we also have JM/J^2M semi-simple. So there will be more than one composition series between J^2M and JM. And this is impossible because JM is uniserial. So J^2M must be simple. □

1.10. Theorem. *Let R be a ring. Then the following are equivalent:*

1. *The ring R is uniserial, Artinian.*
2. *The ring R is Artinian, local and principal ideal ring.*
3. *The ring R is Artinian, local and the Jacobson radical J is a principal ideal.*

Proof. $1 \Longrightarrow 2$ The ring R is uniserial, so $0 \subseteq J^t \subseteq \cdots \subseteq J \subseteq R$ is the only composition series. Obviously R is an Artinian, local ring. Let I be a non-trivial two sided ideal in R. Then $I = J^s$ for a natural number s, where $1 \leq s \leq t$. Let $a \in J^s - J^{s+1}$. So $J^{s+1} \subset J^{s+1} + aR \subseteq J^s$. By the way $J^sJ << J^sR$ ($<<$ means small), because R is an Artinian ring. So we have $aR = J^sR = J^s$. Similarly we have $J^s = Rb$ for an element $b \in J^s$. So R is a principal ideal ring.

$2 \Longrightarrow 3$. Obvious.

$3 \Longrightarrow 1$. We will show that the following chain is a composition series.

$$0 \subseteq J^t \subseteq \cdots \subseteq J \subseteq R.$$

Because J is a principal ideal, there exists a and b in R such that $J = aR = Rb$. Let $aras \in J^2$. Because J is a two sided ideal and $a \in J$, then $ra \in J$, that means there exists $r' \in R \ni ra = ar'$. So $aras = a^2r's \in a^2R$. We have

$$J^2 \subseteq a^2R.$$

The other containment is obvious. So $J^2 = a^2R$.

Similarly we can also show that

$$J^n = a^nR \text{ for every natural number } n.$$

Now let us consider the homomorphism

$$\varphi : R \to aR/a^2R \text{ with } \varphi(r) = aR + a^2R \forall\, r \in R.$$

We can see that aR is contained in the kernel. But $aR = J$ is the unique maximal right ideal, so R/aR is simple, and hence aR/a^2R is either simple or 0. If it is 0, then $aR = 0$ by Nakayama's Lemma. The proof for $a^kR/a^{k+1}R$ is similar. So $0 \subseteq J^t \subseteq \cdots \subseteq J \subseteq R$ is a composition series. According to Lemma 1.9, we can conclude that the ring R is uniserial. □

Next we will see the structure of matrices over a uniserial ring. The next theorem, shown in [5], talks about a two sided ideal in the ring of matrices. The first lemma gives the form of the right ideal of matrices over a ring R.

1.11. Lemma. *Let R be a ring and I a right ideal in $M_n(R)$. Then $I = (U \cdots U)$ for a right submodule U of R^n.*

1.12. Lemma. *Let R be a ring. Let $\begin{pmatrix} 0 \\ \vdots \\ R_i \\ \vdots \\ 0 \end{pmatrix} \subseteq R^n$, where $R_i = R$ is in the i^{th} row. Then I is a right submodule of $\begin{pmatrix} 0 \\ \vdots \\ R_i \\ \vdots \\ 0 \end{pmatrix}$ if and only if $I = \begin{pmatrix} 0 \\ \vdots \\ J_i \\ \vdots \\ 0 \end{pmatrix}$, where J_i is a right ideal of R.*

1.13. Theorem. *Let R be a uniserial ring. Then $M_n(R)$ is a serial ring.*

Proof. Because R is a uniserial ring, then the composition series of R is unique, that is

$$0 \subseteq J^k \subseteq \cdots \subseteq J \subseteq R.$$

We have the following fact,

$$M_n(R) = \begin{pmatrix} R & \cdots & R \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \cdots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ R & \cdots & R \end{pmatrix}.$$

Consider the right ideal $I = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ R & & R \\ 0 & \cdots & 0 \end{pmatrix}$. Because each right ideal in R is just J^t for some positive integer t, using the form of a right ideal in $M_n(R)$ from Lemma 1.11, and that of the right submodule of $\begin{pmatrix} 0 \\ \vdots \\ R_i \\ \vdots \\ 0 \end{pmatrix}$ from Lemma 1.12, then the right ideal $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ R & & R \\ 0 & \cdots & 0 \end{pmatrix}$ is a uniserial right module with composition series as follows:

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ J^k & & J^k \\ 0 & \cdots & 0 \end{pmatrix} \subseteq \cdots \subseteq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ J & & J \\ 0 & \cdots & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ R & & R \\ 0 & \cdots & 0 \end{pmatrix}.$$

So $M_n(R)$ is a direct sum of right uniserial modules. So $M_n(R)$ is a serial ring. □

By Theorems 1.8, 1.10 and 1.13, we have the next corollary.

1.14. Corollary. *An Artinian principal ideal ring is serial.*

The next two theorems are proved in [7].

1.15. Theorem. *Let I be an ideal in R. Then there is an ideal $H \subseteq I$ such that R/H is a serial ring.*

By the way, we have that R/I is isomorphic with $(R/H)/(I/H)$. So we have that R/I is a serial ring. The component M_u that is annihilated by an invertible ideal I in Theorem 1.7 is also an R/I-module. So M_u is a serial module.

References

[1] Eisenbud, D. and Robson, J. C. *Modules over Dedekind prime rings*, Journal of Algebra **16**, 67–85, 1970

[2] Goodearl, K. R. and Warfield R. B. *Simple modules over hereditary Noetherian prime rings*, Journal of Algebra **57**, 82–100, 1979.

[3] Jacobson, N. *The Theory of Rings* (AMS, 1943).

[4] Klingler, L. and Levy, L. S. *Wild torsion modules over Weyl algebras and general torsion modules over HNPs*, Journal of Algebra **172**, 273–300, 1995.

[5] Lam, T. Y. *A First Course in Noncommutative Rings* (Springer Verlag, 1991).

[6] Passman, D. S. *A Course in Ring Theory* (Brooks/Cole Publishing Co., 1991).

[7] Robson, J. C. *Idealizers and hereditary Noetherian prime rings*, Journal of Algebra **22**, 45–81, 1972.