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SOME MATRIX TRANSFORMATIONS ON SEQUENCE SPACES OF INVARIANT MEANS

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Abstract

In this paper we define new sequence spaces $V_\sigma(\theta)$ and $V_\sigma^\infty(\theta)$ which are related to the concept of σ-mean and lacunary sequence $\theta = (k_r)$, and characterize the matrix classes $(l_1, V_\sigma^\infty(\theta))$ and $(l_\infty, V_\sigma^\infty(\theta))$.

Keywords: Lacunary sequence, Matrix transformation, Invariant mean, Almost lacunary convergence.

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1. Introduction and preliminaries

We shall write w for the set of all complex sequences $x = (x_k)_{k=0}^\infty$. Let φ, l_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences respectively. We write $l_p := \{x \in w : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \ne n)$. For any sequence $x = (x_k)_{k=0}^\infty$, let $x^{[n]} = \sum_{k=0}^n x_k e^{(k)}$ be its n-section.

Note that c_0, c, and l_∞ are Banach spaces with the sup-norm $\| x \|_\infty = \sup_k |x_k|$, and l^p $(1 \le p < \infty)$ are Banach spaces with the norm $\| x \|_p = (\sum |x_k|^p)^{1/p}$ while φ is not a Banach space with respect to any norm.

A sequence $(b^{(n)})_{n=0}^\infty$ in a linear metric space X is called a *Schauder basis* if for every $x \in X$ there is a unique sequence $(\beta_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \beta_n b^{(n)}$. A sequence space X with a linear topology is called a *K-space* if each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an *FK-space* if X is a complete linear metric space, and a *BK-space* is a normed *FK*-space. An *FK*-space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$, that is, $x = \lim_{n\to\infty} x^{[n]}$. We use here standard notations as in [7].

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Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional ϕ on the space l_∞ is said to be an *invariant mean* or a *σ-mean* if and only if

(i) $\phi(x) \geq 0$, when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
(ii) $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and
(iii) $\phi(x) = \phi((x_{\sigma(k)}))$ for all $x \in \ell_\infty$.

Throughout this paper we assume the mapping σ has no finite orbits, that is, $\sigma^p(k) \neq k$ for all integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ denotes the p^{th} iterate of σ at k. Note that, a σ-mean extends the limit functional on the space c in the sense that $\phi(x) = \lim x$ for all $x \in c$, (cf. [6]). Consequently $c \subset V_\sigma$, the set of bounded sequences all of whose σ-means are equal. We say that a sequence $x = (x_k)$ is *σ-convergent* if and only if $x \in V_\sigma$, where

$$V_\sigma := \{x \in l_\infty : \lim_{p\to\infty} t_{pn}(x) = L \text{ uniformly in } n;\ L = \sigma\text{-}\lim x\}, \text{ where}$$

$$t_{pn}(x) = \frac{1}{p+1} \sum_{m=0}^{p} x_{\sigma^m(n)}.$$

Using this concept, Schaefer [8] defined and characterized the σ-conservative, σ-regular and σ-coercive matrices. If σ is translation then the σ-mean is often called a Banach limit [2] and the set V_σ reduces to the set f of almost convergent sequences studied by Lorentz [5].

By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of integers such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r (see Fredman *et al* [4]). Recently, Aydin [1] defined the concept of almost lacunary convergence as follows: A bounded sequence $x = (x_k)$ is said to be *almost lacunary convergent* to the number l if and only if

$$\lim_r \frac{1}{h_r} \sum_{j\in I_r} x_{j+n} = l, \text{ uniformly in } n.$$

Quite recently, this idea has been studied for double sequences by Çakan *et al* [3]. In this paper, we define new sequence spaces $V_\sigma(\theta)$ and $V_\sigma^\infty(\theta)$, which are related to the concept of σ-mean and the lacunary sequence $\theta = (k_r)$, and characterize the matrix classes $(l_1, V_\sigma^\infty(\theta))$ and $(l_\infty, V_\sigma^\infty(\theta))$.

2. σ-lacunary convergent sequences

We define the following:

2.1. Definition. A bounded sequence $x = (x_k)$ is said to be *σ-lacunary convergent* to the number l if and only if $\lim_r \frac{1}{h_r} \sum_{j\in I_r} x_{\sigma^j(n)} = l$, uniformly in n, and we let $V_\sigma(\theta)$ denote the set of all such sequences, i.e.

$$V_\sigma(\theta) := \{x \in l_\infty : \lim_r \frac{1}{h_r} \sum_{j\in I_r} x_{\sigma^j(n)} = l, \text{ uniformly in } n\}.$$

Note that for $\sigma(n) = n + 1$, σ-lacunary convergence is reduced to almost lacunary convergence. Results similar to that of Aydin [1] can easily be proved for the space $V_\sigma(\theta)$.

2.2. Definition. A bounded sequence $x = (x_k)$ is said to be *σ-lacunary bounded* if and only if $\sup_{r,n} |\frac{1}{h_r} \sum_{j\in I_r} x_{\sigma^j(n)}| < \infty$, and we let $V_\sigma^\infty(\theta)$ denote the set of all such sequences, i.e.

$$V_\sigma^\infty(\theta) := \{x \in l_\infty : \sup_{r,n} |\tau_{rn}(x)| < \infty\},$$

where

$$\tau_{rn}(x) =: \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(n)}.$$

Note that $c \subset V_\sigma(\theta) \subset V_\sigma^\infty(\theta) \subset l_\infty$.

2.3. Theorem. *The spaces $V_\sigma(\theta)$ and $V_\sigma^\infty(\theta)$ are both BK spaces with the norm*

$$\| x \| = \sup_{r,n} |\tau_{rn}(x)|. \tag{2.1}$$

Proof. We consider the space $V_\sigma(\theta)$. The case $V_\sigma^\infty(\theta)$ can be proved similarly. Let $(x^{(i)}) = ((x_k^{(i)})_{k=0}^\infty)$ be a Cauchy sequence in $V_\sigma(\theta)$, i.e. for $\varepsilon > 0$, there is an $N > 0$ such that $\| x^{(i)} - x^{(m)} \| = \sup_{r,n} |\tau_{rn}(x^{(i)} - x^{(m)})| < \varepsilon$ for all $i, m \geq N$. Since $|x_k^{(i)}| \leq \| x^{(i)} \|$ for each i, and $V_\sigma(\theta) \subset l_\infty$, we have $|x^{(i)} - x^{(m)}| < \varepsilon$ for all $i, m \geq N$. So $(x^{(i)})$ is a Cauchy sequence in $\mathbb{R}$, and hence convergent in $\mathbb{R}$ (since $\mathbb{R}$ is complete). That is, for each k, $x_k^{(i)} \to x_k$, say, as $i \to \infty$. Let $x = (x_k)_{k=0}^\infty$. Then by the definition of $V_\sigma(\theta)$, we have $\| x^{(i)} - x \| = \sup_{m,n} |\tau_{mn}(x^{(i)} - x)| \to 0$, $(i \to \infty)$, since $x_n^{(i)} \to x_n$ and $\tau_{rn}(x^{(i)} - x) = \frac{1}{h_r} \sum_{j \in I_r} T^j (x_n^{(i)} - x_n) \to 0$, where $T^j x_n$ means $x_{\sigma^j(n)}$.

Now, we have to show that $x \in V_\sigma(\theta)$. Since $(x^{(i)})$ is a Cauchy sequence in $V_\sigma(\theta)$, we have that for a given $\varepsilon > 0$ there is a positive integer N depending upon ε such that, for all $i, m \geq N$,

$$\| x^{(i)} - x^{(m)} \| < \varepsilon.$$

Hence by (2.1) we have

$$\sup_{r,n} |\tau_{rn}(x^{(i)} - x^{(m)})| < \varepsilon.$$

This implies that

$$|\tau_{rn}(x^{(i)} - x^{(m)})| < \varepsilon, \text{ for each } r, n; \tag{2.2}$$

or

$$|L^{(i)} - L^{(m)}| < \varepsilon, \tag{2.3}$$

where $L^{(i)} = \sigma\text{-}\lim x^{(i)}$. Let $L = \lim_{m \to \infty} L^{(m)}$. Then the σ-mean of x is $\phi(x) = \lim_i \phi(x^{(i)})$ (since $x = \lim_i x^{(i)}$ and ϕ is continuous and linear). Further $\lim_i \phi(x^{(i)}) = \lim_i L^{(i)} = L$ (since $\phi(x^{(i)})$ means $\sigma\text{-}\lim x^{(i)}$). Now letting $m \to \infty$ in (2.2) and (2.3), we get

$$|\tau_{rn}(x^{(i)} - x)| < \varepsilon, \text{ for each } r, n; \text{ (since } x = \lim_m x^{(m)}) \tag{2.4}$$

and

$$|L^{(i)} - L| < \varepsilon, \text{ (since } \lim_m L^{(m)} = L) \tag{2.5}$$

for $i > N$. Now fix i in the above inequalities. Since $x^{(i)} \in V_\sigma(\theta)$ for fixed i, we obtain

$$\lim_r \tau_{rn}(x^{(i)}) = L^{(i)}, \text{ uniformly in } n$$

(since $L^{(i)} = \sigma\text{-}\lim x^{(i)} = \lim_r \tau_{rn}(x^{(i)})$ uniformly in n). Hence, for a given ε, there exists a positive integer r_0 (depending upon i and ε but not on n) such that

$$|\tau_{rn}(x^{(i)}) - L^{(i)}| < \varepsilon, \text{ (since } x = \lim_m x^{(m)}) \tag{2.6}$$

for $r \geq r_0$ and for all n. Now by (2.4), (2.5) and (2.6), we get

$$\begin{aligned} |\tau_{rn}(x) - L| &\leq |\tau_{rn}(x) - \tau_{rn}(x^{(i)}) + \tau_{rn}(x^{(i)}) - L^{(i)} + L^{(i)} - L| \\ &\leq |\tau_{rn}(x) - \tau_{rn}(x^{(i)})| + |\tau_{rn}(x^{(i)}) - L^{(i)}| + |L^{(i)} - L| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon, \end{aligned}$$

for $r \geq r_0$ and for all n. Then $x \in V_\sigma(\theta)$, which proves the completeness of $V_\sigma(\theta)$.

Now, let $\| x^{(m)} - x \| \to 0$ as $m \to \infty$. Then, for given $\varepsilon > 0$, there is $m_0 \in \mathbb{N}$ such that

$$\| x^{(m)} - x \| < \varepsilon \text{ for all } m \geq m_0,$$

which implies

$$\sup_{r,n} |\tau_{rn}(x^{(m)} - x)| < \varepsilon \text{ for all } m \geq m_0,$$

and so that

$$|L^{(m)} - L| < \varepsilon \text{ for all } m \geq m_0, \text{ as above in (2.5).}$$

Hence we easily get

$$|x_k^{(m)} - x_k| < \varepsilon \text{ for all } m \geq m_0, \text{ and for all } k,$$

that is $|x_k^{(m)} - x_k| \to 0$ as $m \to \infty$, and this proves the continuity of the coordinate projection. Hence $V_\sigma(\theta)$ is a BK space.

This completes the proof of the theorem. □

3. Matrix transformations into $V_\sigma^\infty(\theta)$

Let X and Y be two sequence spaces and $A = (a_{nk})_{n;k=1}^\infty$ an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$, $A_n(x) = \sum_k a_{nk} x_k$ provided that the series on the right converges for each n. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and we denote the class of such matrices by (X, Y).

In this section, we characterize the matrix classes $(l_1, V_\sigma^\infty(\theta))$ and $(l_\infty, V_\sigma^\infty(\theta))$.

Let Ax be defined. Then, for all r, n, we write

$$\tau_{rn}(Ax) = \sum_{k=1}^\infty t(n,k,r) x_k,$$

where

$$t(n,k,r) = \frac{1}{h_r} \sum_{j \in I_r} a(\sigma^j(n), k),$$

and $a(n,k)$ denotes the element a_{nk} of the matrix A.

3.1. Theorem. $A \in (l_1, V_\sigma^\infty(\theta))$ *if and only if*

$$\sup_{n,k,r} |t(n,k,r)| < \infty. \tag{3.1}$$

Proof. Sufficiency. Suppose that $x = (x_k) \in l_1$. We have

$$\begin{aligned} |\tau_{rn}(Ax)| &\leq \sum_k |t(n,k,r) x_k| \\ &\leq (\sup_k |t(n,k,r)|) \Big(\sum_k |x_k| \Big). \end{aligned}$$

Taking the supremum over n, r on both sides and using (3.1), we get $Ax \in V_\sigma^\infty(\theta)$ for $x \in l_1$.

Necessity. Let us define a continuous linear functional Q_{rn} on l_1 by

$$Q_{rn}(x) = \tau_{rn}(Ax) = \sum_k t(n,k,r)x_k.$$

Now

$$|Q_{rn}(x)| \le \sup_k |t(n,k,r)| ||x||_1, \tag{3.2}$$

$$\| Q_{rn} \| = \sup_{||x||_1=1} \frac{|Q_{rn}(x)|}{||x||_1}$$

and hence

$$\| Q_{rn} \| \le \sup_k |t(n,k,r)|, \tag{3.3}$$

by (3.2). For any fixed r and $n \in \mathbb{N}$, define $x = (x_i)$ by

$$x_i = \begin{cases} \operatorname{sgn} t(n,k,r); & \text{for } i = k \\ 0; & \text{for } i \ne k; \end{cases} \tag{3.4}$$

Then $\| x \|_1 = 1$, and

$$\begin{aligned} |Q_{rn}(x)| &= |t(n,k,r)x_k| \\ &= |t(n,k,r)|. \end{aligned}$$

Further,

$$\begin{aligned} \| Q_{rn} \| &= \sup_{|||x||_1=1} \frac{||Q_{rn}(x)||}{||x||_1} \\ &= \| Q_{rn}(x) \|, \text{ since } \| x \|_1 = 1 \\ &= \sup_{r,n} |Q_{rn}(x)| \ge |Q_{rn}(x)| \\ &= \Big| \sum_i t(n,i,r)x_i \Big| \\ &= |t(n,k,r)|, \end{aligned}$$

for x_i as defined in (3.4), hence

$$\| Q_{rn} \| \ge \sup_k |t(n,k,r)|. \tag{3.5}$$

Now, by (3.3) and (3.5),

$$\| Q_{rn} \| = \sup_k |t(n,k,r)|.$$

Therefore, by the Banach-Steinhauss Theorem

$$\sup_{r,n} \| Q_{rn} \| = \sup_{r,n,k} |t(n,k,r)| < \infty,$$

since $A \in (l_1, V_\sigma^\infty(\theta))$ gives

$$\sup_{r,n} |Q_{rn}(x)| = \sup_{r,n} \Big| \sum_k t(n,k,r)x_k \Big| < \infty.$$

This completes the proof of the theorem. □

3.2. Theorem. $A \in (l_\infty, V_\sigma^\infty(\theta))$ *if and only if*

$$\sup_{n,r} \sum_k |t(n,k,r)| < \infty. \tag{3.6}$$

Proof. *Sufficiency.* Suppose that (3.6) holds and $x=(x_k)\in l_\infty$. We have

$$\begin{aligned}|\tau_{rn}(Ax)| &\le \sum_k \left|t(n,k,r)x_k\right|\\ &\le \left(\sum_k |t(n,k,r)|\right)(\sup_k |x_k|).\end{aligned}$$

Taking the supremum over n,r on both sides and using (3.6), we get $Ax\in V_\sigma^\infty(\theta)$ for $x\in l_\infty$.

Necessity. Let $A\in(l_\infty,V_\sigma^\infty(\theta))$. Write $q_n(x)=\sup_r|\tau_{rn}(Ax)|$. It is easy to see that q_n is a continuous seminorm on l_∞, since for $x\in l_\infty$

$$|q_n(x)|\le M\,\|x\|,\ M>0.$$

Suppose (3.6) is not true. Then there exists $x\in l_\infty$ with $\sup_n q_n(x)=\infty$. By the principle of condensation of singularities (cf. [9]), the set $\{x\in l_\infty:\sup_n q_n(x)=\infty\}$ is of the second category in l_∞, and hence non-empty, that is, there is $x\in l_\infty$ with $\sup_n q_n(x)=\infty$. But this contradicts the fact that q_n is pointwise bounded on l_∞. Now by the Banach-Steinhauss Theorem, there is a constant M such that

$$q_n(x)\le M\,\|x\|_1\ . \tag{3.7}$$

Now define $x=(x_k)$ by

$$x_k=\begin{cases}\operatorname{sgn} t(n,k,r); & \text{for each } r,n\ (1\le k\le k_0),\\ 0; & \text{for } k>k_0.\end{cases}$$

Then $x\in l_\infty$. Applying this sequence to (3.7), we get (3.6).

This completes the proof of the theorem. □

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