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ON QUADRAPELL NUMBERS AND QUADRAPELL POLYNOMIALS

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Abstract

In this paper we define and deal with the quadrapell numbers, $D(n)$, in terms of a linear recurrence relation of order 4, and define the quadrapell polynomials in x. Further we give the generating function, a Binet-like formula and formulae for sums of these numbers.

Keywords: Quadrapell numbers, Quadrapell polynomials.

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1. Introduction

Recurrence relations and their generalizations have many interesting properties, and they have many applications to every field of science. Koshy's book [8] is a good source for these applications.

One application of second order linear recurrences occurs in graph theory. In [10] Minc give the relationship between the permanent of a certain $(0, 1)$ superdiagonal matrix and the k-generalized Fibonacci numbers.

Second order linear recurrences, especially the Fibonacci and Lucas numbers and their generalizations, are studied in [6, 7, 9, 11]

In [6] the authors consider the relationship between the generalized order k Fibonacci, Lucas numbers and their sums of bipartite graphs.

In [7] the authors defined two tridiagonal matrices and they gave a relationship between the permanents and determinants of these matrices and second order linear recurrences.

In [12] the authors presented a sequence of progressively better upper bounds for the Perron root of a nonnegative matrix. They showed that each element in the sequence is a function of the Perron root of the arithmetic symmetrization of a power of the matrix.

The authors in [2] and [3] defined and studied various fourth order linear recurrences and their polynomial analogous.

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In this paper we define and deal with the quadrapell numbers, $D(n)$, using a linear recurrence relation of order 4, and define the quadrapell polynomials in x. Then we give some identities and formulae among the terms of the Fibonacci sequence, the Lucas sequence, the sequence $D(n)$ and the sequence $C(n)$ all of whose elements are $-1, 0$ or 1. Also various summation formulae including the terms $D(n)$ are given. Moreover a generating matrix (a companion matrix) of order 4 is given for generating the terms $D(n)$. The ratios of consecutive Fibonacci numbers and Lucas numbers converge to the Golden ratio $\alpha = 1.61803$, and we give evidence to support the conjecture that the ratio of consecutive terms of the Quadrapell numbers is also the Golden ratio.

2. Main results

2.1. Quadrapell Numbers.

2.1. Definition. The Quadrapell numbers $D(n)$ are defined by the recurrence relation

$$D(n) = D(n-2) + 2D(n-3) + D(n-4), \; n \geq 4,$$

where $D(0) = D(1) = D(2) = 1$, $D(3) = 2$.

The first few Quadrapell numbers $D(n)$ are

$$1, \; 1, \; 1, \; 2, \; 4, \; 5, \; 9, \; 15, \; 23, \; 38, \; 62, \; 99, \; 161, \; 261, \; 421, \; 682, \; 1104, \; 1785,$$
$$2889, \; 4675, \; 7563, \; 12238, \ldots$$

Just as ratios of consecutive Fibonacci numbers and Lucas numbers converge to the the Golden ratio $\alpha = 1.61803$, so we conjecture that the Quadrapell ratios $\frac{D(n)}{D(n-1)}$ also converge to $\alpha = 1.61803$ (Golden ratio).

The characteristic equation of the Quadrapell recurrence relation is given by

$$x^4 - x^2 - 2x - 1 = 0.$$

It can be shown that this equation has two real roots and two imaginary roots. The real roots are given by

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ (Golden ratio)}, \; \beta = \frac{1-\sqrt{5}}{2}$$

and the imaginary roots are given by

$$\gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Using the recursive definition, we can develop a generating function for the Quadrapell numbers. We require a real-valued function $g(x)$ whose formal power series expansion at $x = 0$ has the form

$$\sum_{n=0}^{\infty} D(n)x^n = D(0) + D(1)x + D(2)x^2 + \cdots + D(n)x^n + \cdots .$$

If this is the case, the formal power series expansions of $x^2 g(x), 2x^3 g(x)$ and $x^4 g(x)$ are respectively

$$D(0)x^2 + D(1)x^3 + D(2)x^4 + \cdots + D(n-2)x^n + \cdots ,$$
$$2D(0)x^3 + 2D(1)x^4 + 2D(2)x^5 + \cdots + 2D(n-3)x^n + \cdots , \text{and}$$
$$D(0)x^4 + D(1)x^5 + D(2)x^6 + \cdots + D(n-4)x^n + \cdots ,$$

so the expansion for $g(x) - x^2 g(x) - 2x^3 g(x) - x^4 g(x)$ is

$$\begin{aligned} &D(0) + D(1)x + (D(2) - D(0))x^2 \\ &\qquad + (D(3) - D(2) - 2D(0))x^3 \\ &\qquad\quad + (D(4) - D(2) - 2D(1) - D(0))x^4 + \cdots \\ &\qquad\qquad + (D(n) - D(n-2) - 2D(n-3) - D(n-4))x^n + \cdots \\ &= 1 + x - x^3, \end{aligned}$$

since

$$D(0) = D(1) = D(2) = 1, D(3) = 2$$

and

$$D(n) = D(n-2) + 2D(n-3) + D(n-4).$$

However, $1 + x - x^3$ is a finite series so we may write

$$(1 - x^2 - 2x^3 - x^4)g(x) = 1 + x - x^3,$$

that is,

$$g(x) = \frac{1 + x - x^3}{1 - x^2 - 2x^3 - x^4}.$$

Now we present a theorem giving a Binet-like formula for Quadrapell numbers.

2.2. Theorem.

$$D(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - \frac{1}{2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + \frac{1}{2}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right),$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}\ ,\ \ \beta = \frac{1 - \sqrt{5}}{2}\ ,\ \gamma = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\ ,\ \delta = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

are the roots of the equation

$$x^4 - x^2 - 2x - 1 = 0.$$

Proof. Expressing the generating function of the Quadrapell sequence as a sum of partial fractions we obtain:

$$\begin{aligned} g(x) &= \frac{\frac{1}{2}x - 1}{x^2 + x - 1} + \frac{1}{2}\frac{x}{x^2 + x + 1} \\ &= -\frac{1}{2}\frac{x}{1 - x - x^2} + \frac{1}{1 - x - x^2} + \frac{1}{2}\frac{x}{x^2 + x + 1} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)x^n + \sum_{n=0}^{\infty}\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)x^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)x^n \\ &= \sum_{n=0}^{\infty}\left[-\frac{1}{2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + \frac{1}{2}\left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)\right]x^n \end{aligned}$$

We note from [1] that

$$\frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)x^n$$

and

$$\frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty}\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right).$$

Thus, by the equality of generating functions, we get the stated Binet-like formula for the Quadrapell sequence. □

2.3. Corollary.

i) $2D(n) - C(n) = F(n+1) + F(n-1)$,

ii) $2D(n) - C(n) = L(n)$,

where $F(n)$, $L(n)$ and $C(n)$ denote the Fibonacci sequence, the Lucas sequence, and

$$C(n) = \frac{\gamma^n - \delta^n}{\gamma - \delta},$$

respectively.

Now we present an important conjecture.

It is of interest to consider the ratio of successive terms in this sequence. Initial values are given in Table 1. This ratio appears to converge to the Golden ratio, $\alpha = 1.61803$. For large n, this value is found to be

$$\frac{D(n+1)}{D(n)} \approx 1.61803 \text{ (Golden Ratio)}$$

Table 1

n	0	1	2	3	4	5	6	7
$D(n)$	1	1	1	2	4	5	9	15
$\frac{D(n+1)}{D(n)}$	1	1	1	2	2	1.25	1.8	1.66666
n	8	9	10	11	12	13	14	15
$D(n)$	23	38	62	99	161	261	421	682
$\frac{D(n+1)}{D(n)}$	1.53333	1.65217	1.63157	1.59677	1.62626	1.62111	1.61302	1.61995
n	16	17	18	19	20	21	22	
$D(n)$	1104	1785	2889	4675	7563	12238	19802	
$\frac{D(n+1)}{D(n)}$	1.61876	1.61684	1.61848	1.61820	1.61775	1.61814	1.61807	

It appears from Table 1 that $\frac{D(n+1)}{D(n)}$ approaches $\alpha = 1.6180338\ldots$, the magic number, as $n \to \infty$. This encourages us to make the following conjecture:

2.4. Conjecture. $\frac{D(n+1)}{D(n)}$ *converges to the limit α, the positive root of*

$$x^4 - x^2 - 2x - 1 = 0.$$

We can extended the definition of $D(n)$ to negative subscripts. If we apply the Quadrapell recurrence relation to the negative side, we obtain

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$D(-n)$	-1	2	-2	3	-5	9	-15	24	-38	61	-99	161	-261	422	-682

We can generalize Table 2, for $k \in \mathbb{N}$, $n \in \mathbb{Z}^+$:

$$D(-n) = \begin{cases} D(n) + 1, & \text{if } n = 6k+2 \\ D(n) - 1, & \text{if } n = 6k+4 \\ -D(n), & \text{if } n = 2k+1 \end{cases}$$

Now we give some theorems related to sums of Quadrapell numbers.

2.5. Theorem. *For* $k \in \mathbb{N}$

$$\sum_{j=0}^{3k+1} D(j) = D(3k+3).$$

Proof. We use the principle of mathematical induction. Since

$$D(0) + D(1) = 2 = D(3),$$

clearly the result is true for $k = 0$.

Now assume it is true for an arbitrary positive integer $k > 1$,

$$\sum_{j=0}^{3k+1} D(j) = D(3k+3).$$

Then, we have

$$\begin{aligned}\sum_{j=0}^{3k+4} D(j) &= \sum_{j=0}^{3k+1} D(j) + D(3k+2) + D(3k+3) + D(3k+4)\\ &= D(3k+3) + D(3k+2)\\ &\qquad + D(3k+3) + D(3k+4) \text{ (by the induction hypothesis)}\\ &= D(3k+4) + 2D(3k+3) + D(3k+2)\\ &= D(3k+6) \text{ (by the Quadrapell recurrence relation)}\end{aligned}$$

So the formula works for $k+1$. Thus, by the principle of mathematical induction the formula holds for every integer $k \geq 0$. □

2.6. Theorem. *For* $k \in \mathbb{N}$*,*

$$\sum_{j=0}^{3k+2} D(j) = D(3k+4) - 1.$$

Proof. We proceed by induction on k. Since

$$D(0) + D(1) + D(2) = 3 = D(4) - 1,$$

the statement is true for $k = 0$.

Assume it is true for $k > 1$,

$$\sum_{j=0}^{3k+2} D(j) = D(3k+4) - 1.$$

Then, we show that the formula holds for $k+1$. Indeed,

$$\begin{aligned}\sum_{j=0}^{3k+5} D(j) &= \sum_{j=0}^{3k+2} D(j) + D(3k+3) + D(3k+4) + D(3k+5)\\ &= D(3k+4) - 1 + D(3k+3)\\ &\qquad + D(3k+4) + D(3k+5) \text{ (by the induction hypothesis)}\\ &= D(3k+5) + 2D(3k+4) + D(3k+3) - 1\\ &= D(3k+7) - 1 \text{ (by the Quadrapell recurrence relation)}\end{aligned}$$

Thus the formula works for $k+1$. □

2.7. Lemma.

$$D(n) + D(n+1) + D(n+3) + D(n+5) = D(n+6).$$

Proof. By the Quadrapell recurrence relation, we have

$$D(n+5) = D(n+3) + 2D(n+2) + D(n+1).$$

Thus we obtain

$$\begin{aligned} D(n) + D(n+1) + D(n+3) + D(n+5) \\ &= D(n) + 2D(n+1) + 2D(n+2) + 2D(n+3) \\ &= D(n+2) + 2D(n+1) + D(n) + 2D(n+3) + D(n+2) \\ &= D(n+4) + 2D(n+3) + D(n+2) \text{ (by the recurrence relation)} \\ &= D(n+6) \text{ (by the recurrence relation).} \end{aligned}$$

So the lemma is proved. □

2.8. Theorem. *For $k \in \mathbb{N}$,*

$$\sum_{j=0}^{3k+1} D(2j) = D(6k+3).$$

Proof. We use induction on k. Since

$$D(0) + D(2) = 2 = D(3),$$

the formula works for $k = 0$.

Now we assume that it is true for $k > 1$. Then

$$\begin{aligned} \sum_{j=0}^{3k+4} D(2j) &= \sum_{j=0}^{3k+1} D(2j) + D(6k+4) + D(6k+6) + D(6k+8) \\ &= D(6k+3) + D(6k+4) + D(6k+6) \\ &\qquad + D(6k+8) \text{ (by the induction hypothesis)} \\ &= D(6k+9) \text{ (by Lemma 2.7).} \end{aligned}$$

Note that if we take $n = 6k + 3$ in Lemma 2.7, then we obtain

$$D(6k+3) + D(6k+4) + D(6k+6) + D(6k+8) = D(6k+9).$$

Thus the formula is true for $k + 1$. So, the proof is complete. □

2.9. Theorem. *For $k \in \mathbb{N}$,*

$$\sum_{j=0}^{3k+2} D(2j) = D(6k+5) + 1.$$

Proof. We will use proof by induction. Since

$$D(0) + D(2) + D(4) = 6 = D(5) + 1,$$

clearly it is true for $k = 0$.

We now suppose that the statement holds for $k > 1$, i.e.

$$\sum_{j=0}^{3k+2} D(2j) = D(6k+5) + 1.$$

Then

$$\begin{aligned}\sum_{j=0}^{3k+5} D(2j) &= \sum_{j=0}^{3k+2} D(2j) + D(6k+6) + D(6k+8) + D(6k+10)\\ &= D(6k+5) + 1 + D(6k+6) + D(6k+8) + D(6k+10)\\ &= D(6k+11) + 1 \text{ (by Lemma 2.7)}.\end{aligned}$$

Thus the formula works for $k+1$. So, by the principle of mathematical induction, the given statement is true for every integer $k \geq 0$. ☐

2.10. Theorem. *For* $k \in \mathbb{N}$,

$$\sum_{j=0}^{3k} D(2j+1) = D(6k+2).$$

Proof. We use the principle of mathematical induction on k. Since

$$D(1) = 1 = D(2),$$

the formula is true for $k = 0$.

Now we assume that it is true for $k > 1$, that is,

$$\sum_{j=0}^{3k} D(2j+1) = D(6k+2).$$

Then we have

$$\begin{aligned}\sum_{j=0}^{3k+3} D(2j+1) &= \sum_{j=0}^{3k} D(2j+1) + D(6k+3) + D(6k+5) + D(6k+7)\\ &= D(6k+2) + D(6k+3) + D(6k+5) + D(6k+7)\\ &= D(6k+8) \text{ (by Lemma 2.7)}.\end{aligned}$$

Thus, the formula works for $k+1$. Consequently the given statement is true for every integer $k \geq 0$. ☐

2.11. Theorem. *For* $k \in \mathbb{N}$,

$$\sum_{j=0}^{3k+1} D(2j+1) = D(6k+4) - 1.$$

Proof. We proceed by induction on k, Since

$$D(1) + D(3) = 3 = D(4) - 1,$$

the formula is true for $k = 0$.

Assume it is true for $k > 1$, that is

$$\sum_{j=0}^{3k+1} D(2j+1) = D(6k+4) - 1.$$

Then we have

$$\begin{aligned}\sum_{j=0}^{3k+4} D(2j+1) &= \sum_{j=0}^{3k+1} D(2j+1) + D(6k+5) + D(6k+7) + D(6k+9)\\ &= D(6k+4) - 1 + D(6k+5) + D(6k+7) + D(6k+9)\\ &= D(6k+10) - 1 \text{ (by Lemma 2.7)}.\end{aligned}$$

Therefore the formula works for $k+1$. So, by the principle of mathematical induction the formula holds for every integer $k \geq 0$. $\square$

2.12. Definition. The matrix of the Quadrapell recurrence relation is defined by

$$T = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix T is

$$\lambda^4 - \lambda^2 - 2\lambda - 1.$$

We note that the characteristic equation of the matrix T is same as that of the Quadrapell equation. Furthermore, the eigenvalues of the matrix T are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \alpha,\ \lambda_2 = \frac{1-\sqrt{5}}{2} = \beta,\ \lambda_3 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i = \gamma,\ \lambda_4 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i = \delta.$$

Also the eigenvectors of T are

$$X_1 = \begin{bmatrix} 2+\sqrt{5} \\ \frac{3+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2\alpha \\ \alpha^2 \\ \alpha \\ 1 \end{bmatrix}, \text{ for } \lambda_1 = \frac{1+\sqrt{5}}{2} = \alpha \text{ (Golden ratio)},$$

$$X_2 = \begin{bmatrix} 2-\sqrt{5} \\ \frac{3-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2\beta \\ \beta^2 \\ \beta \\ 1 \end{bmatrix}, \text{ for } \lambda_2 = \frac{1-\sqrt{5}}{2} = \beta,$$

$$X_3 = \begin{bmatrix} 1 \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \delta \\ \gamma \\ 1 \end{bmatrix}, \text{ for } \lambda_3 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i = \gamma,$$

$$X_4 = \begin{bmatrix} 1 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \gamma \\ \delta \\ 1 \end{bmatrix}, \text{ for } \lambda_4 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i = \delta.$$

We note that $\det(T) = -1$.

2.13. Definition. [8] Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the (real or complex) eigenvalues of a matrix A. Then its spectral radius $\rho(A)$ is defined as:

$$\rho(A) = \max_{1 \leq i \leq n} (|\lambda_i|).$$

2.14. Theorem. [12] *For any matrix norm* $\|\cdot\|$,

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

2.15. Lemma. *Let T be the matrix of the Quadrapell recurrence relation. Then for every integer $n \geq 1$ we have*

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \frac{1+\sqrt{5}}{2} = \alpha \text{ (Golden ratio)}.$$

Proof. Since the spectral radius of the matrix T is

$$\rho(T) = \frac{1+\sqrt{5}}{2},$$

the desired result follows by Theorem 2.14. $\square$

2.16. Theorem.

$$\begin{vmatrix} D(n+3) & D(n+2) & D(n+1) & D(n) \\ D(n+2) & D(n+1) & D(n) & D(n-1) \\ D(n+1) & D(n) & D(n-1) & D(n-2) \\ D(n) & D(n-1) & D(n-2) & D(n-3) \end{vmatrix} = (-1)^n 5$$

Proof. We use the principle of mathematical induction on n. Since

$$\begin{vmatrix} D(4) & D(3) & D(2) & D(1) \\ D(3) & D(2) & D(1) & D(0) \\ D(2) & D(1) & D(0) & D(-1) \\ D(1) & D(0) & D(-1) & D(-2) \end{vmatrix} = \begin{vmatrix} 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 2 \end{vmatrix} = -5$$

the result is true for $n = 1$.

Now we assume that it is true for an arbitrary positive integer $n > 1$, that is

$$\begin{vmatrix} D(n+3) & D(n+2) & D(n+1) & D(n) \\ D(n+2) & D(n+1) & D(n) & D(n-1) \\ D(n+1) & D(n) & D(n-1) & D(n-2) \\ D(n) & D(n-1) & D(n-2) & D(n-3) \end{vmatrix} = (-1)^n 5$$

Then using the Quadrapell recurrence relation and properties of determinants, we have

$$\begin{aligned} &\begin{vmatrix} D(n+4) & D(n+3) & D(n+2) & D(n+1) \\ D(n+3) & D(n+2) & D(n+1) & D(n) \\ D(n+2) & D(n+1) & D(n) & D(n-1) \\ D(n+1) & D(n) & D(n-1) & D(n-2) \end{vmatrix} \\ &\quad = \begin{vmatrix} D(n+2)+2D(n+1)+D(n) & D(n+3) & D(n+2) & D(n+1) \\ D(n+1)+2D(n)+D(n-1) & D(n+2) & D(n+1) & D(n) \\ D(n)+2D(n-1)+D(n-2) & D(n+1) & D(n) & D(n-1) \\ D(n-1)+2D(n-2)+D(n-3) & D(n) & D(n-1) & D(n-2) \end{vmatrix} \end{aligned}$$

$$= \underbrace{\begin{vmatrix} D(n+2) & D(n+3) & D(n+2) & D(n+1) \\ D(n+1) & D(n+2) & D(n+1) & D(n) \\ D(n) & D(n+1) & D(n) & D(n-1) \\ D(n-1) & D(n) & D(n-1) & D(n-2) \end{vmatrix}}_{0}$$

$$+ \underbrace{\begin{vmatrix} 2D(n+1) & D(n+3) & D(n+2) & D(n+1) \\ 2D(n) & D(n+2) & D(n+1) & D(n) \\ 2D(n-1) & D(n+1) & D(n) & D(n-1) \\ 2D(n-2) & D(n) & D(n-1) & D(n-2) \end{vmatrix}}_{0}$$

$$+ \begin{vmatrix} D(n) & D(n+3) & D(n+2) & D(n+1) \\ D(n-1) & D(n+2) & D(n+1) & D(n) \\ D(n-2) & D(n+1) & D(n) & D(n-1) \\ D(n-3) & D(n) & D(n-1) & D(n-2) \end{vmatrix}$$

$$= (-1) \begin{vmatrix} D(n+3) & D(n+2) & D(n+1) & D(n) \\ D(n+2) & D(n+1) & D(n) & D(n-1) \\ D(n+1) & D(n) & D(n-1) & D(n-2) \\ D(n) & D(n-1) & D(n-2) & D(n-3) \end{vmatrix}$$

$$= (-1)(-1)^n 5 \text{ (by the induction hypothesis)}$$

$$= (-1)^{n+1} 5$$

Thus the statement works for $n+1$. So, the equality is true for every integer $n \geq 1$. $\square$

2.2. Quadrapell Polynomials. Quadrapell numbers can be generalized to a set of polynomials, called the Quadrapell polynomials.

2.17. Definition. The Quadrapell polynomials are defined by

$$D_n(x) = \begin{cases} x, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ x^2 & \text{if } n = 2, \\ 2x^3, & \text{if } n = 3, \\ x^2 D_{n-2}(x) + 2x D_{n-3}(x) + D_{n-4}(x), & \text{if } n \geq 4. \end{cases}$$

The first eight Quadrapell polynomials are:

$$D_0(x) = x$$
$$D_1(x) = 1$$
$$D_2(x) = x^2$$
$$D_3(x) = 2x^3$$
$$D_4(x) = x^4 + 3x$$
$$D_5(x) = 2x^5 + 2x^3 + 1$$
$$D_6(x) = x^6 + 4x^4 + 3x^3 + x^2$$
$$D_7(x) = 2x^7 + 4x^5 + 2x^3 + 7x^2$$
$$D_8(x) = x^8 + 8x^6 + 3x^5 + 6x^4 + 5x.$$

Here we make an interesting observation:

$$D_i(1) = D(i), \ i = 1, 2, 3, \ldots, n.$$

In fact,

$$D_n(1) = D(n).$$

2.18. Theorem.

$$(x^2+2x)\sum_{i=1}^{n} D_i(x) = D_n(x)+(1-x^2)D_{n+1}(x)+D_{n+2}(x)+D_{n+3}(x)-2x^3-x^2-x.$$

Proof. Considering Definition 16, we have

$$\sum_{i=1}^{n} D_{i+3}(x) = x^2\sum_{i=1}^{n} D_{i+1}(x) + 2x\sum_{i=1}^{n} D_i(x) + \sum_{i=1}^{n} D_{i-1}(x)$$

or

$$D_n(x) + (1-x^2)D_{n+1}(x) + D_{n+2}(x) + D_{n+3}(x)$$
$$= (x^2+2x)\sum_{i=1}^{n} D_i(x) + D_0(x) + D_2(x) + D_3(x).$$

On the other hand, since $D_0(x) = x$, $D_2(x) = x^2$ and $D_3(x) = 2x^3$, it follows that

$$(x^2+2x)\sum_{i=1}^{n} D_i(x) = D_n(x)+(1-x^2)D_{n+1}(x)+D_{n+2}(x)+D_{n+3}(x)-2x^3-x^2-x.$$

So the theorem is proved. □

2.19. Corollary.

$$\sum_{i=1}^{n} D_i = \frac{1}{3}(D_n + D_{n+2} + D_{n+3} - 4).$$

Proof. Since $D_i(1) = D_i$, the proof is easily seen by Theorem 2.18. □

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