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NEW COMMON FIXED POINT THEOREMS OF GREGUŠ TYPE FOR R-WEAKLY COMMUTING MAPPINGS IN 2-METRIC SPACES

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Abstract

In this paper we extend and generalize a theorem of M. R. Singh, L. S. Singh and P. P. Murthy (*Common fixed points of set valued mappings*, Int. J. Math. Sci., **25** (6), 411–415, 2001) in a 2-metric space with a Greguš type condition, and give some common fixed point theorems of set-valued maps in 2-metric spaces.

Keywords: Contraction, Fixed point, Greguš condition, 2-metric space.

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1. Introduction

The concept of 2-metric spaces was introduced and studied initially by Gahler [7, 8, 9]. After Gahler there was a flood of new results obtained by many authors in these spaces [3, 11, 12, 13, 15]. Military applications of fixed point theory in 2-metric spaces can be found, as well as applications in Medicine and Economics [1, 2, 18].

Dhage [4] introduced the concept of D-metric space as follows:

Let X be a non-empty set and $\mathbb{R}^+$ the set of non-negative real numbers. If the real-valued mapping $D : X \times X \times X \to \mathbb{R}^+$ satisfies the following properties:

(D_1) $D(x_1, x_2, x_3) \geq 0$ for every $x_1, x_2, x_3 \in X$ and $D(x_1, x_2, x_3) = 0$ if and only if $x_1 = x_2 = x_3$;

(D_2) $D(x_1, x_2, x_3) = D(x_1, x_3, x_2) = D(x_3, x_2, x_1) = D(x_2, x_1, x_3) = D(x_3, x_1, x_2) = D(x_2, x_3, x_1)$ (*symmetric*) for all $x_1, x_2, x_3 \in X$;

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(D_3) $D(x_1, x_2, x_3) \leq d(x_1, x_2, u) + d(x_1, u, x_3) + d(u, x_2, x_3)$ for all $x_1, x_2, x_3, u \in X$ (*rectangle inequality*),

then the pair (X, D) is called a *D-metric space*.

Gahler defined a 2-metric space as follows:

A 2-metric on a set X with at least three points is a non-negative real-valued mapping d: $X \times X \times X \to \mathbb{R}^+$ satisfying the following properties:

(G_1) To each pair of points a, b with $a \neq b$ in X there is a point $c \in X$ such that $d(a, b, c) \neq 0$;
(G_2) $d(a, b, c) = 0$, if at least two of the points are equal;
(G_3) $d(a, b, c) = d(b, c, a) = d(a, c, b)$;
(G_4) $d(a, b, c) \leq d(a, b, u) + d(a, u, c) + d(u, b, c)$ for all $a, b, c, u \in X$.

The pair (X, d) is then called a *2-metric space*.

Geometrically the value of a 2-metric $d(x, y, c)$ represents the area of a triangle with vertices x, y and c, whereas, the value of a D-metric $D(x, y, c)$ represents the perimeter of the triangle with vertices x, y and c.

Throughout this note (X, D) stands for a D-metric space, (X, d) is a 2-metric space and $B(X)$ the class of all non-empty bounded subsets of X.

Let A, B, C be non-empty sets in $B(X)$. We define

$$\delta(A, B, C) = \sup\{d(a, b, c) : a \in A, b \in B, c \in C\}$$
$$D(A, B, C) = \inf\{d(a, b, c) : a \in A, b \in B, c \in C\}.$$

If A is a singleton set, then $\delta(A, B, C) = \delta(a, B, C)$. In case B and C are also singleton sets, then

$$\delta(A, B, C) = D(A, B, C) = d(a, b, c)$$

for every $A = \{a\}, B = \{b\}, C = \{c\}$. From the definition of δ we can say that,

$$\delta(A, B, C) = \delta(A, C, B) = \delta(C, A, B) = \delta(B, C, A) = \delta(C, B, A) = \delta(B, A, C) \geq 0.$$

Also,

$$\delta(A, B, C) \leq \delta(A, B, E) = \delta(A, E, C) = \delta(E, B, C);$$

for all $A, B, C, E \in B(X)$. Let us note that $\delta(A, B, C) = 0$ if at least two of A, B and C are equal singleton sets.

We need the following definitions and lemmas for our main theorems:

1.1. Definition. A sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X is said to be convergent to a subset A of X if;

i. Given $a \in A$, there is a sequence $\{a_n\}$ of X such that $a_n \in A_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} d(a_n, a, c) = 0$.
ii. Given $\epsilon > 0$, there exists a positive integer n_0 such that $A_n \subseteq A_\epsilon$ for every $n > n_0$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

1.2. Definition. [1] Let $G : X \to X$ and $F : X \to B(X)$. Then the pair $\{G, F\}$ is said to be *weakly commuting* if $GFx \in B(X)$ and

$$\delta(FGx, GFx, C) \leq \max\{\delta(Gx, Fx, C),\ \delta(GFx, GFx, C)\}$$

for every $x \in X$ and $C \in B(X)$.

1.3. Definition. [1] Let $G : X \to X$ and $F : X \to B(X)$. Then the pair $\{G, F\}$ is said to be *R-weakly commuting* if

$$\delta(FGx, GFx, C) \leq R \cdot \max\{\delta(Gx, Fx, C), \delta(GFx, GFx, C)\}$$

for every $x \in X$, $C \in B(X)$ and $R > 0$.

1.4. Remark. If F is a single valued function, then Definitions 1.2 and 1.3 reduce to the following:

$$\delta(FGx, GFx, C) = d(FGx, GFx, C) \leq d(Gx, Fx, C) = \delta(Gx, Fx, C))$$

and

$$\delta(FGx, GFx, C) = d(FGx, GFx, C) \leq R.d(Gx, Fx, C) = R \cdot \delta(Gx, Fx, C),$$

respectively.

In recent years, common fixed points of Greguš [10] type have been proved by Diviccaro, Fisher and Sessa [5], Fisher and Sessa [6], Mukherjee and Verma [14], Murthy, Cho and Fisher [16], M. R. Singh, L. S. Singh and P. P. Murthy [19] under weaker conditions.

In this paper, we have extended and generalized a theorem of M. R. Singh, L. S. Singh and P. P. Murthy [19] in a 2-metric space.

2. Main results

Let S and T be mappings of 2-metric space (X, d) into itself and $A, B : X \to B(X)$ are two set valued mappings satisfying the following conditions:

(2.1) $\quad \bigcup A(X) \subset T(X)$ and $\bigcup B(X) \subset S(X)$;

(2.2) $\quad$ For every $x, y \in X$, $C \in B(X)$ and $p > 0$,

$$\delta^p(Ax, By, C) \leq \varphi(a \cdot \delta^p(Sx, Ty, C) + (1-a)\max\{\delta^p(Ax, Sx, C), \delta^p(By, Ty, C), b \cdot D^p(Sx, By, C) + c \cdot D^p(Ty, Ax, C)\})$$

where $a \in (0, 1)$ and $\varphi : [0, \infty) \to [0, \infty)$ is

(i) non-increasing;
(ii) upper-semi continuous,
(iii) satisfies $\varphi(t) < t$ for every $t > 0$.

Let x_0 be an arbitrary point of X. Since $\bigcup A(X) \subset T(X)$, then there exists a point $x_1 \in X$ such that $Tx_1 \in Ax_0 = y_0$. Again, since $\bigcup B(X) \subset S(X)$, for the point $x_1 \in X$ we can find a point $x_2 \in X$ such that $Sx_1 \in Bx_0 = y_1$, and so on. Inductively, we can define a sequence $\{x_n\}$ in X such that

$$(2.3) \qquad \begin{cases} Tx_{n+1} \in Ax_n = y_n, & \text{when } n \text{ is even} \\ Sx_{n+1} \in Bx_n = y_n, & \text{when } n \text{ is odd} \end{cases}$$

Now we are ready to prove the following lemma for our theorem:

2.1. Lemma. *Let (X, d) be a 2-metric space. Let S, T be self maps of X and $A, B : X \to B(X)$ satisfying the conditions* (2.1) *and* (2.2). *Then for every $n \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \delta(y_n, y_{n+1}, y_{n+2}) = 0.$$

Proof. Since

$$\delta(y_{2n+2}, y_{2n+1}, y_{2n}) = \delta(Ax_{2n+2}, Bx_{2n+1}, y_{2n})$$

we have

$$\begin{aligned}
&\delta(y_{2n+2}, y_{2n+1}, y_{2n}) \\
&\quad \leq [\varphi(a \cdot \delta^p(Sx_{2n+2}, Tx_{2n+1}, y_{2n}) \\
&\qquad + (1-a) \cdot \max\{\delta^p(Sx_{2n+2}, Ax_{2n+2}, y_{2n}), \delta^p(Tx_{2n+2}, Bx_{2n+1}, y_{2n}), \\
&\qquad\quad b \cdot D^p(Sx_{2n+2}, Bx_{2n+1}, y_{2n}) + c.D^p(Tx_{2n+1}, Ax_{2n+2}, y_{2n})\})]^{\frac{1}{p}}, \\
&\quad \leq [\varphi(a \cdot \delta^p(y_{2n+1}, y_{2n}, y_{2n}) + (1-a) \cdot \max\{\delta^p(y_{2n+1}, y_{2n+2}, y_{2n}), \\
&\qquad \delta^p(y_{2n+1}, y_{2n+1}, y_{2n}), b \cdot \delta^p(y_{2n+1}, y_{2n+1}, y_{2n}) \\
&\qquad\quad + c \cdot \delta^p(y_{2n}, y_{2n+2}, y_{2n})\})]^{\frac{1}{p}}, \\
&\quad = [\varphi((1-a) \cdot \max\{\delta^p(y_{2n+1}, y_{2n+2}, y_{2n})\})]^{\frac{1}{p}} \\
&\quad < [\varphi(\delta^p(y_{2n+1}, y_{2n+2}, y_{2n}))]^{\frac{1}{p}}, \text{ if } [\varphi(\delta^p(y_{2n+1}, y_{2n+2}, y_{2n}))]^{\frac{1}{p}} \neq 0.
\end{aligned} \tag{2.4}$$

Again we consider,

$$\begin{aligned}
&\delta(y_{2n+3}, y_{2n+2}, y_{2n+1}) \\
&\quad = \delta(Bx_{2n+3}, Ax_{2n+2}, y_{2n+1}) \\
&\quad \leq [\,\varphi(a \cdot \delta^p(Sx_{2n+2}, Tx_{2n+3}, y_{2n+1}) \\
&\qquad + (1-a) \cdot \max\{\delta^p(Sx_{2n+2}, Ax_{2n+3}, y_{2n+1}), \delta^p(Tx_{2n+3}, Bx_{2n+3}, y_{2n+1}), \\
&\qquad\quad b \cdot D^p(Sx_{2n+2}, Bx_{2n+1}, y_{2n}) + c \cdot D^p(Tx_{2n+1}, Ax_{2n+2}, y_{2n})\})]^{\frac{1}{p}}, \\
&\quad \leq [\,\varphi(a \cdot \delta^p(y_{2n+1}, y_{2n+2}, y_{2n+1}) + (1-a) \cdot \max\{\delta^p(y_{2n+1}, y_{2n+2}, y_{2n+1}), \\
&\qquad \delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}), b \cdot D^p(y_{2n+1}, y_{2n+2}, y_{2n+1}) \\
&\qquad\quad + c \cdot D^p(y_{2n+2}, y_{2n+3}, y_{2n+1})\})]^{\frac{1}{p}} \\
&\quad \leq [\varphi((1-a) \cdot \max\{\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}), c \cdot D^p(y_{2n+2}, y_{2n+3}, y_{2n+1})\})]^{\frac{1}{p}} \\
&\quad = [\varphi((1-a)(\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1})))]^{\frac{1}{p}} \\
&\quad \leq [\varphi(\delta^p(y_{2n+2}, y_{2n+3}, y_{2n+1}))]^{\frac{1}{p}},
\end{aligned}$$

(since $0 < a < 1$). By the definition of φ, this implies

$$\delta(y_{2n+1}, y_{2n+2}, y_{2n}) \to 0.$$

Hence we conclude that

$$\lim_{n\to\infty} \delta(y_n, y_{n+1}, y_{n+2}) = 0. \tag{2.5}$$

□

2.2. Lemma. [1] *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$ respectively, then the sequence $\{\delta(A_n, B_n, C)\}$ converges to $\{\delta(A, B, C)\}$.*

2.3. Theorem. *Let S and T be mappings of a 2-metric space (X, d) into itself, and $A, B : X \to B(X)$ two set-valued mappings satisfying the conditions (2.1), (2.2), (2.3), and the following:*

(2.6) *$S(X)$ or $T(X)$ is a complete subspace of X;*

(2.7) *The pairs $\{A, S\}$ and $\{B, T\}$ are R-weakly commuting,*

then A, B, S and T have a unique common fixed point in X.

Proof. From Lemma 2.1, the sequence $\{y_n\}$ is a Cauchy sequence. Assume $T(X)$ is a complete subspace of X. Since the sequence $\{x_n\}$ defined by (2.3) is a subsequence, then $\{Tx_{2n+1}\}$ is Cauchy and converges to a point z in $T(X)$. Since $T(X)$ is a complete subspace of X, for some $u \in X$, $Tx_{2n+1} \to z = T(u)$. By using (2.2), we have

$$\delta(Sx_{2n+2}, Tx_{2n+1}, C) \leq \delta(y_{2n+1}, y_{2n}, C).$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} \delta(Sx_{2n+2}, Tx_{2n+1}, C) \leq \lim_{n\to\infty} \delta(y_{2n+1}, y_{2n}, C) = 0.$$

The above implies

$$\lim_{n\to\infty} \delta(Sx_{2n+2}, Tx_{2n+1}, C) = 0.$$

Therefore, we get

$$\lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z.$$

We can also show that

$$\lim_{n\to\infty} \delta(Ax_{2n+2}, z, C) = 0.$$

Now, we shall show that u is a coincidence point of B and T.

For $n = 0, 1, 2, \dots$ and using (2.2) we have

$$\delta^p(Ax_{2n}, Bu, C) \leq \varphi(a \cdot \delta^p(Sx_{2n}, Tu, C) + (1-a)\max\{\delta^p(Sx_{2n}, Ax_{2n}, C), \delta^p(Tu, Bu, C), b \cdot D^p(Sx_{2n}, Tu, C) + c \cdot D^p(Tu, Ax_{2n}, C)\}).$$

Now letting $n \to \infty$, the above inequality implies that

$$\lim_{n\to\infty} \delta^p(Ax_{2n}, Bu, C) \leq \varphi((1-a)\max\{\delta^p(Sx_{2n}, Ax_{2n}, C), \delta^p(Tu, Bu, C), b.D^p(z, Bu, C)\})$$

and so

$$\lim_{n\to\infty} \delta^p(Ax_{2n}, Bu, C) \leq \varphi(\delta^p(Tu, Bu, C)) < \delta^p(z, Bu, C),$$

which is a contradiction. Thus $\{z\} = Bu = \{Tu\}$.

Since $\bigcup B(X) \subset S(X)$, for some $v \in X$ we have $\{Sv\} = Bu = \{Tu\}$.

If $Av \neq Bu$, then we have from (2.2),

$$\delta^p(Av, Bu, C) \leq \varphi(a \cdot \delta^p(Sv, Tu, C) + (1-a)\max\{\delta^p(Av, Sv, C), \delta^p(Bu, Tu, C), b \cdot D^p(Sv, Bu, C) + c \cdot D^p(Tu, Av, C)\}),$$

which implies

$$\delta^p(Av, Bu, C) \leq \varphi(a \cdot \delta^p(Sv, Tu, C) + (1-a)\max\{\delta^p(Av, Sv, C), \delta^p(Bu, Tu, C), b \cdot \delta^p(Sv, Bu, C) + c \cdot \delta^p(Tu, Av, C)\}),$$

or equivalently

$$\delta^p(Av, Bu, C) \leq \varphi((1-a)\max\{\delta^p(Av, Sv, C), c \cdot \delta^p(Tu, Av, C)\}).$$

Since $0 \leq b + c \leq \frac{1}{2},\ 0 < \alpha < 1,\ b, c \geq 0$, we have

$$\delta^p(Av, Bu, C) < \varphi((1-a) \cdot \delta^p(Av, Sv, C)),$$

and

$$\delta^p(Av, Bu, C) < \delta^p(Av, Sv, C))$$

which implies $\{Sv\} = Av$. Therefore, $Av = \{Sv\} = \{z\} = \{Tu\} = Bu$.

Since $Av = \{Sv\} = \{z\}$ and $\{A, S\}$ are R-weakly commuting maps, then

$$\delta(ASv, SAv, C) < R \cdot \max\{d(Av, Sv, C), \delta(SAv, SAv, C)),$$

which implies that

$$ASv = SAv \implies Az = \{Sz\}.$$

Again, using (2.2),

$$\begin{aligned}\delta^p(Az, z, C) &\le \delta^p(Az, Bu, C)\\ &\le \varphi(a \cdot \delta^p(Sz, Tu, C) + (1-a)\max\{\delta^p(Az, Sz, C), \delta^p(Bu, Tu, C),\\ &\qquad b \cdot D^p(Sz, Bu, C) + c \cdot D^p(Tu, Az, C)\}),\end{aligned}$$

or equivalently

$$\begin{aligned}\delta^p(Az, z, C) &\le \varphi(a \cdot \delta^p(Sz, Tu, C) + (1-a)\max\{\delta^p(Az, Sz, C), \delta^p(Bu, Tu, C),\\ &\qquad b \cdot D^p(Sz, Bu, C) + c \cdot D^p(Tu, Az, C)\})\end{aligned}$$

or equivalently

$$\begin{aligned}\delta^p(Az, z, C) &\le \varphi(a \cdot \delta^p(Az, z, C) + (1-a)\max\{0, 0, b \cdot \delta^p(Az, z, C)\\ &\qquad + c \cdot \delta^p(z, Az, C)\})\\ &\le \varphi(\delta^p(Az, z, C))\\ &\le \delta^p(Az, z, C),\end{aligned}$$

which is a contradiction. Thus $Az = \{Sz\} = \{z\}$, and z is a common fixed point of A and S.

Similarly, we can show that $\{z\}$ is a common fixed point of B and T by assuming $\{B, T\}$ is a pair of R-weakly commuting maps. Hence, $Az = Bz = \{z\} = \{Sz\} = \{Tz\}$.

Now we shall prove that $\{z\}$ is a unique fixed point of A, B, S, T.

Let z^* be a second fixed point of A, B, S and T. Then from (2.2) we have,

$$\begin{aligned}d^p(z, z^*, C) &\le \delta^p(Az, Bz^*, C)\\ &\le \varphi(a \cdot \delta^p(Sz, Tz^*, C) + (1-a)\max\{\delta^p(Az, Sz, C)),\\ &\qquad \delta^p(Bz^*, Tz^*, C), b \cdot D^p(Sz, Bz^*, C) + c \cdot D^p(Tz^*, Az, C)\})\\ &\le \varphi(a \cdot \delta^p(Sz, Tz^*, C) + (1-a) \cdot \max\{\delta^p(Az, Sz, C),\\ &\qquad \delta^p(Bz^*, Tz^*, C), b \cdot \delta^p(Sz, Bz^*, C) + c \cdot \delta^p(Tz^*, Az, C)\})\\ &\le \varphi(a \cdot \delta^p(z, z^*, C) + (1-a) \cdot \delta^p(z, z^*, C))\\ &\le \varphi(\delta^p(z, z^*, C))\\ &< d^p(z, z^*, C),\end{aligned}$$

which is a contradiction. Hence we get $z = z^*$.

That means that z is a unique common fixed point of A, B, S and T in X, which completes the proof. □

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