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BEST SUBORDINANTS OF THE STRONG DIFFERENTIAL SUPERORDINATION

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Abstract

S. S. Miller and P. T. Mocanu in (*Subordinants of differential superordinations*, Complex Variables **48** (10), 815–826, 2003) introduced the notion of differential superordination as a dual concept of differential subordination (S. S. Miller and P. T. Mocanu, *Differential subordinations. Theory and applications* (Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000)). The notion of strong differential subordination was introduced by J. A. Antonino and S. Romaguera in (*Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations **114**, 101–105, 1994). This notion was developed in (Georgia I. Oros and Gheorghe Oros, *Strong differential subordination*, Turkish Journal of Mathematics **33**, 249–257, 2009).

In (*Strong differential superordination*, Acta Universitatis Apulensis **19**, 110–106, 2009), Georgia I. Oros introduces the dual concept of strong differential superordinations. The aim of this paper is to obtain the best subordinants of the strong differential superordinations.

Keywords: Differential subordination, Differential superordination, Strong differential subordination, Strong differential superordination, Best subordinant, Univalent function, Analytic function.

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1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \le 1\}.$$

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \cdots, \ z \in U\}$$

with $A_1 = A$, and

$$S = \{f \in A; \ f \text{ is univalent in } U\},$$

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots, \ z \in U\}.$$

Let Ω and Δ be any sets in the complex plane $\mathbb{C}$, let p be analytic in the unit disc U and $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$.

In a series of articles such as [4, 6, 7, 8] the authors have determined properties of functions p that satisfy the strong differential subordination

(i) $\{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U, \ \xi \in \overline{U}\} \subset \Omega \ \Rightarrow \ p(U) \subset \Delta.$

In [5] the author considers the dual problem of determining properties of functions p that satisfy the strong differential superordination

(ii) $\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U, \ \xi \in \overline{U}\} \ \Rightarrow \ \Delta \subset p(U).$

1.1. Definition. [5] Let $H(z, \xi)$ be analytic in $U \times \overline{U}$ and $f(z)$ analytic and univalent in U. The function $f(z)$ is called *strongly subordinate to* $H(z, \xi)$, or $H(z, \xi)$ is said to be *strongly superordinate to* $f(z)$, written $f(z) \prec\prec H(z, \xi)$, if $f(z)$ is subordinate to $H(z, \xi)$ as a function of z, for all $\xi \in \overline{U}$.

If $H(z, \xi)$ is univalent in U for all $\xi \in \overline{U}$, then $f(z) \prec\prec H(z, \xi)$ if and only if $f(0) = H(0, \xi)$ for all $\xi \in \overline{U}$ and $f(U) \subset H(U \times \overline{U})$.

If Ω or Δ in (ii) is a simply connected domain, then it may be possible to rephrase (ii) in terms of strong differential superordination.

If p is univalent in U, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = p(0)$. In this case, (ii) be rewritten as

(iii) $\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U, \ \xi \in \overline{U}\}$ implies $q(z) \prec p(z), \ z \in U.$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(p(0), 0, 0; 0, \xi)$. If, in addition, the function $\psi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in U for all $\xi \in \overline{U}$, then (iii) can be rewritten as

(iv) $h(z) \prec\prec \psi(p(z), zp'(z), z^2p''(z); z, \xi)$ implies $q(z) \prec p(z), \ z \in U.$

In the implication (iv), the functions h and q can be analytic and not necessarily univalent.

This last result leads us to some of the important definitions that will be used in this article.

1.2. Definition. [5] Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ are univalent in U for all $\xi \in \overline{U}$ and satisfy the (second-order) strong differential superordination

(j) $h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi)$

then p is called a *solution* of the strong differential superordination.

An analytic function q is called a *subordinant of the solutions of the strong differential superordination*, or more simply a *subordinant*, if $q \prec p$ for all p satisfying (j).

A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (j) is said to be the *best subordinant*.

Note that the best subordinant is unique up to a rotation of U.

1.3. Definition. [2, Definition 2.2.b, p. 21] We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ f \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

1.4. Definition. [5] Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q(z) \neq 0$. The class of *admissible functions* $\phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

(A) $\qquad \varphi(r, s, t; z, \xi) \in \Omega$

whenever $r = q(z)$, $s = \dfrac{zq'(z)}{m}$ and $\text{Re}\left[\dfrac{t}{s} + 1\right] \leq \dfrac{1}{m}\text{Re}\left[\dfrac{zq''(z)}{q'(z)} + 1\right]$, where $z \in U$, $z \in \partial U$, $\xi \in \overline{U}$ and $m \geq n \geq 1$.

When $n = 1$ we write $\phi_1[\Omega, q]$ as $\phi[\Omega, q]$.

In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$ we denote this class $\phi_n[h(U), q]$ by $\phi_n[h, q]$.

In order to prove the main results, we need the following lemma.

1.5. Lemma. [5, Theorem 2] *Take $q \in \mathcal{H}[a, n]$, let h be analytic in U and $\varphi \in \phi_n[h, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in U for all $\xi \in \overline{U}$, then*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \; z \in U, \; \xi \in \overline{U}$$

implies

$$q(z) \prec p(z), \; z \in U.$$

1.6. Remark. The conclusion of Lemma 1.5 can be written in the generalized form:

$$h(w(z)) \prec\prec \varphi(p(w(z)), w(z)p'(w(z)), (w^2(z)p''(w(z)); w(z); \xi)),$$

$z \in U, \; \xi \in \overline{U}$, where $w : U \to U$.

2. Main results

Using the following theorem, the result from Lemma 1.5 can be extended to those cases in which the behavior of q on the boundary of U is unknown.

2.1. Theorem. *Let h and q be univalent in U, with $q(0) = a$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ satisfy one of the following conditions:*

(i) *$\varphi \in \phi_n[h, q_\rho]$, for some $\rho \in (0, 1)$, or*
(ii) *There exists $\rho_0 \in (0, 1)$ such that $\varphi \in \phi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.*

If $p \in \mathcal{H}[a,n]$, $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$ *and*

$$h(z) \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi),\ z \in U,\ \xi \in \overline{U}, \tag{2.1}$$

then

$$q(z) \prec p(z),\ z \in U.$$

Proof. Case (i). By applying Lemma 1.5 we obtain

$$q_\rho(z) \prec p(z),\ z \in U.$$

Since $q(z) \prec q_\rho(z)$ we deduce

$$q(z) \prec p(z), \quad z \in U.$$

Case (ii). If we let $p_\rho(z) = p(\rho z)$, then

$$\begin{aligned}\varphi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); z, \xi) &= \varphi(p(\rho z), \rho z p'(\rho z), \rho^2 z^2 p''(\rho z); \rho z, \xi)\\ &\supset h_\rho(U).\end{aligned}$$

By using Remark 1.6 and Lemma 1.5 with $w(z) = \rho z$, we obtain

$$q_\rho(z) \prec p_\rho(z),\ \text{for}\ \rho \in (\rho_0, 1).$$

By letting $\rho \to 1$ we obtain

$$q(z) \prec p(z),\ z \in U.$$

□

The next two theorems yield best subordinants of the differential superordination (1).

The following theorems provide the existence of best subordinants of (1) for certain φ and also provide a method for finding the best subordinant for the cases $n = 1$ and $n > 1$.

2.2. Theorem. *Let* h *be univalent in* U *and* $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\varphi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{2.2}$$

has a solution $q \in Q(a)$. *If* $\varphi \in \phi[h,q]$, $p \in Q(a)$ *and* $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *is univalent in* U, *for all* $\xi \in \overline{U}$ *then*

$$h(z) \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \tag{2.3}$$

implies $q(z) \prec p(z)$ *and* q *is the best subordinant.*

Proof. Since $\varphi \in \phi[h,q]$, by applying Lemma 1.5 we deduce that q is a subordinant of (2.3). Since q also satisfies (2.2), it is also a solution of the strong differential superordination (2.3) and therefore all subordinants of (2.3) will be subordinate to q. Hence, q will be the best subordinant of (2.3). □

From this theorem we see that the problem of finding the best subordinant of (2.3) essentially reduces to showing that the differential equation (2.2) has a univalent solution and checking that $\varphi \in \phi[h,q]$.

The conclusion of the theorem can be written in the symmetric form

$$\varphi(q(z), zq'(z), z^2q''(z); z, \xi) \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi)$$

implies

$$q(z) \prec p(z), \quad z \in U,\ \xi \in \overline{U}.$$

This result can be extended to those cases in which the behavior of q on the boundary of U is unknown, by the following theorem.

2.3. Theorem. *Let h be univalent in U and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{2.4}$$

has a solution q with $q(0) = a$, and that one of the following conditions is satisfied:

(i) *$q \in Q$ and $\varphi \in \phi[h, q]$, or*
(ii) *q is univalent in U and $\varphi \in \phi[h, q_\rho]$, for some $\rho \in (0, 1)$, or*
(iii) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that*

$$\varphi \in \phi[h_\rho, q_\rho] \text{ for all } \rho \in (\rho_0, 1).$$

If $p \in \mathcal{H}[a, 1]$ and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in U, for all $\xi \in \overline{U}$ and if p satisfies

$$h(z) \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi),\ z \in U,\ \xi \in \overline{U} \tag{2.5}$$

then

$$q(z) \prec p(z), \quad z \in U,$$

and q is the best subordinant.

Proof. By applying Lemma 1.5 and Theorem 2.1 we deduce that q is a subordinant of (2.5). Since q satisfies (2.4), it is a solution of (2.5) and therefore q will be subordinated by all subordinants of (2.5). Hence q will be the best subordinant of (2.5). □

2.4. Example. Let $q(z) = 1 + z$, $h(z) = q(z) + zq'(z) + z^2q''(z) = 1 + 2z$, $p \in \mathcal{H}[1, n]$ and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$, with

$$\text{Re } \varphi(p(z), zp'(z), z^2p''(z); z, \xi) > 0,\ z \in U,\ \xi \in \overline{U}.$$

If

$$1 + 2z \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi),\ z \in U,\ \xi \in \overline{U}$$

then from Theorem 2.2 we have

$$1 + z \prec p(z),\ z \in U,$$

and $q(z) = 1 + z$ is the best subordinant.

2.5. Theorem. *Let h be univalent in U and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^{2n}q''(z)) = h(z) \tag{2.6}$$

has a solution q, with $q(0) = a$, and that one of the following conditions is satisfied:

(i) *$q \in Q$ and $\varphi \in \phi_n[h, q]$, or*
(ii) *q is univalent in U and $\varphi \in \phi_n[h, q_\rho]$, for some $\rho \in (0, 1)$, or*
(iii) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \phi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $p \in \mathcal{H}[a, n]$, $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in U for all $\xi \in \overline{U}$, and p satisfies

$$h(z) \prec\!\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi),\ z \in U,\ \xi \in \overline{U}, \tag{2.7}$$

then

$$q(z) \prec p(z),$$

and q is the best subordinant.

Proof. By applying Lemma 1.5 and Theorem 2.1 we deduce that q is a subordinant of (2.7). If we let $p(z) = q(z^n)$, then

$$zp'(z) = nz^n q'(z^n)$$

and

$$z^2 p''(z) = n(n-1)z^n q'(z^n) + n^2 z^{2n} q''(z^n).$$

Therefore, from (6) we obtain

$$\begin{aligned}\varphi(p(z), zp'(z), z^2p''(z); z, \xi) &= \varphi(q(z^n), nz^n q'(z^n), n(n-1)z^n q'(z^n) + n^2 z^{2n} q''(z^n); z, \xi)\\ &= h(z^n)\\ &\prec\!\prec h(z)\end{aligned}$$

$$\begin{aligned}\varphi(q(z^n), nz^n q'(z^n), n(n-1)z^n q'(z^n) + n^2 z^{2n} q''(z^n); z, \xi)\\ \prec\!\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi).\end{aligned}$$

Since $q(U) = p(U)$, we conclude that q is the best subordinant. □

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