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SOME RESULTS ON SUBTRACTION ALGEBRAS

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Abstract

In this paper, some additional concepts relating to subtraction algebras, the so called subalgebra, bounded subtraction algebra and unions of subtraction algebras, are introduced, and some properties are investigated.

Keywords: Subtraction algebra, Subalgebra, Bounded subtraction algebra.

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1. Introduction

B. M. Schein [7] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup), and set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [8] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called atomic subtraction algebras. Y. B. Jun, H. S. Kim and E. H. Roh [2] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In [4], Y. B. Jun, Y. H. Kim and K. A. Oh introduced the notion of complicated subtraction algebras and investigated some related properties. In [6], K. J. Lee, Y. B. Jun, and Y. H. Kim introduced the notion of weak subtraction algebras and provided a method to make a weak subtraction algebra from a quasi-ordered set.

In this paper, some additional concepts concerning subtraction algebras, so called subalgebras, bounded subtraction algebras and unions of subtraction algebras, are introduced, and some properties are investigated.

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2. Preliminaries

An algebra $(X;-)$ with a single binary operation "$-$" is called a *subtraction algebra* if for all $x, y, z \in X$ the following conditions hold:

(S1) $x-(y-x)=x$,
(S2) $x-(x-y)=y-(y-x)$,
(S3) $(x-y)-z=(x-z)-y$.

The subtraction determines an order relation on X: $a \leq b \iff a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$.

The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a-(a-b)$ and the complement of an element $b \in [0,a]$ is $a-b$.

In a subtraction algebra, the following are true [2, 5]:

(a1) $(x-y)-y=x-y$,
(a2) $x-0=x$ and $0-x=0$,
(a3) $(x-y)-x=0$,
(a4) $x-(x-y)\leq y$,
(a5) $(x-y)-(y-x)=x-y$,
(a6) $x-(x-(x-y))=x-y$,
(a7) $(x-y)-(z-y)\leq x-z$,
(a8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$,
(a9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$,
(a10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$,
(a11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$,
(a12) $(x-y)-z=(x-z)-(y-z)$.

2.1. Definition. [2] A nonempty subset A of a subtraction algebra X is called an *ideal of X* if it satisfies

(1) $0\in A$,
(2) $(\forall x\in X)(\forall y\in A)(x-y\in A \implies x\in A)$.

2.2. Lemma. [5] *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x\in X)(\forall y\in A)(x\leq y \implies x\in A).$$

2.3. Definition. [4] Let X be a subtraction algebra. For any $a,b\in X$, let $G(a,b)=\{x\in X : x-a\leq b\}$. Then X is said to be *complicated* if for any $a,b\in X$ the set $G(a,b)$ has a greatest element.

Note that $0,a,b\in G(a,b)$. The greatest element of $G(a,b)$ is denoted by $a+b$.

3. Results

3.1. Proposition. *Let X be a subtraction algebra and I a subset of X. Then I is an ideal of X if and only if $G(x,y)\subseteq I$ for all $x,y\in I$.*

Proof. $\Longrightarrow$. Let I be an ideal and x,y any elements of I. For any $z\in G(x,y)$, we have $z-x\leq y$. Hence $z-x\in I$ from Lemma 2.2. Then we obtain $z\in I$ since I is an ideal.

$\Longleftarrow$. If $G(x,y)\subseteq I$, for all $x,y\in I$ we have $0\in I$ since $0\in G(x,y)$. For any $b\in I$ and $a\in X$, let $a-b\in I$. Then $G(a-b,b)\subseteq I$. Hence, since $a-(a-b)\leq b$ from (a4), we obtain $a\in G(a-b,b)\subseteq I$. Hence, $a\in I$. □

3.2. Definition. Let X be a subtraction algebra and Y a nonempty subset of X. Then Y is called a *subalgebra of* X if $x - y \in Y$ whenever $x, y \in Y$.

3.3. Theorem. *Let X be a subtraction algebra and Y a subalgebra of X. Then the following conditions hold:*

(a) $0 \in Y$,
(b) *Y is a subtraction algebra,*
(c) *$\{0\}$ is a subalgebra of X,*
(d) *X is a subalgebra of X,*
(e) *For any x, y in X, $G(x, y)$ is a subalgebra of X.*
(f) *Any ideal I of X is a subtraction algebra.*

Proof. (a)-(d) follow easily from the definition.

(e) For $a, b \in G(x, y)$, we have $a - x \leq y$ and $b - x \leq y$. Then, from (a12) and (a10) and the fact that $u \wedge v = v \wedge u \leq u, v$, we obtain

$$\begin{aligned}(a - b) - x &= (a - x) - (b - x)\\ &= (a - x) \wedge (y - (b - x))\\ &\leq y - (b - x)\\ &\leq y.\end{aligned}$$

Hence $a - b \in G(x, y)$.

(f) For any $x, y \in I$, from (a3) we have $(x - y) - x = 0 \in I$, then $x - y \in I$. □

3.4. Definition. Let X be a subtraction algebra and $x \in X$. Then, the set $A(x) = \{y \in X : y \leq x\}$ is called the *initial section of* x.

3.5. Theorem. *In a subtraction algebra X, $A(x) \cap A(y) = A(x \wedge y)$ for all $x, y \in X$.*

Proof. Let $z \in A(x) \cap A(y)$. Then we have $z \leq x$ and $z \leq y$. From (a9) we obtain

(3.1) $\quad z - (x - y) \leq x - (x - y)$

and since $x - y \leq x - z$,

(3.2) $\quad z - (x - z) \leq z - (x - y)$.

From (3.1), (3.2) and (S1), we get $z = z - (x - z) \leq z - (x - y) \leq x - (x - y) = x \wedge y$. Hence $z \in A(x \wedge y)$.

Now let $z \in A(x \wedge y)$. We have $z \leq x - (x - y) \leq y$ from (a4), and we get $z \in A(y)$. Using (S2) and (a4), we obtain $z \leq x - (x - y) = y - (y - x) \leq x$. Hence $z \in A(y)$. So $z \in A(x) \cap A(y)$. □

3.6. Definition. Let X be a subtraction algebra. If there is an element 1 of X satisfying $x \leq 1$ for all x in X, then X is called a *bounded subtraction algebra*.

In a bounded subtraction algebra X, we denote $1 - x$ by x'.

3.7. Example. [4] Let $X = \{0, a, b, c\}$ be a subtraction algebra with the following Cayley table:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then for all $x \in X$ we have $x - c = 0$. Hence X is a bounded subtraction algebra.

3.8. Theorem. *In a bounded subtraction algebra, the following properties hold:*

(i) $1' = 0,\ 0' = 1$,
(ii) $(x')' \leq x$,
(iii) $x' - y' \leq y - x$,
(iv) $x \leq y$ *implies* $y' \leq x'$,
(v) $x' - y = y' - x$,
(vi) $x - x' = x,\ x' - x = x'$,
(vii) $x \wedge x' = 0$,
(vii) $((x')')' = x'$.

Proof. (i) $1' = 1 - 1 = 0,\ 0' = 1 - 0 = 1$,

(ii) $(x')' = 1 - x' = 1 - (1 - x) \leq x$,

(iii) From (a7) and (S3), we have

$$0 = ((x - y) - (z - y)) - (x - z) = ((x - y) - (x - z)) - (z - y).$$

Hence we have $((x-y)-(x-z)) \leq (z-y)$. So, we obtain $x' - y' = (1-x)-(1-y) \leq y-x$.

(iv) If $x \leq y$ then, with (a9), we get $1 - y \leq 1 - x$.

(v) $x' - y = (1 - x) - y = (1 - y) - x = y' - x$, (from (S3)).

(vi) $x - x' = x - (1 - x) = x$, and $x' - x = (1 - x) - x = 1 - x = x'$, (from (S1)).

(vii) $x \wedge x' = x - (x - x') = x - x = 0$.

(viii) $((x')')' = 1 - (1 - (1 - x)) = 1 - x = x'$, (from (a6)). □

3.9. Proposition. *If X is a bounded subtraction algebra, then for all $x \in X$ the following hold:*

$$x + 1 = 1 + x = 1 \text{ and } x + x' = 1.$$

Proof. Since $x + 1 \in X$ and X is bounded, we have $x + 1 \leq 1$. Also, for all $y \in X$, since $y - x \leq 1$ we have $y \leq x + 1$. Then, we obtain $1 \leq x + 1$. Hence $x + 1 = 1$. Furthermore, since $y \leq 1$, we get $y - x \leq 1 - x = x'$, or $y \leq x + x'$ for all $y \in X$. So we have $1 \leq x + x'$, and therefore $x + x' = 1$. □

Let $S(X) = \{x \in X : (x')' = x\}$, where X is a bounded subtraction algebra. Since $(1')' = 0' = 1$ and $(0')' = 1' = 0$, we have $0, 1 \in S(X)$.

3.10. Theorem. *If X is a bounded subtraction algebra, then*

$$x - y' = y - x'$$

for all $x, y \in S(X)$.

Proof. Using Theorem 3.3 (v), we have $x - y' = (x')' - y = (y')' - x' = y - x'$. □

3.11. Theorem. *If X is a bounded subtraction algebra, then $S(X)$ is a bounded subalgebra of X.*

Proof. We know that $1 \in S(X)$. Let $x, y \in S(X)$. We need to show that $x - y \in S(X)$, that is, $((x-y)')' = x - y$. From Theorem 3.8 (ii), we have $((x-y)')' \leq x - y$. Also we get

$$\begin{aligned}(x-y) - ((x-y)')' &= (x - ((x-y)')') - y, \text{ (using (S3))} \\ &= ((x-y)' - x') - y, \text{ (from Theorem 3.10)} \\ &= ((x-y)' - y) - x', \text{ (using (S3))} \\ &= (y' - (x-y)) - x', \text{ (from Theorem 3.8 (v))} \\ &= (y' - x') - (x-y), \text{ (using (S3))} \\ &= ((x')' - y) - (x-y), \text{ (from Theorem 3.10)} \\ &= (x-y) - (x-y), \text{ (since } x \in S(X)) \\ &= 0.\end{aligned}$$

Then we obtain $x - y \in S(X)$, and hence $S(X)$ is a bounded subalgebra of X. □

3.12. Theorem. *Suppose X is a bounded complicated subtraction algebra. Then $S(X)$ is a complicated subalgebra.*

Proof. For all $a, b \in S(X)$, it suffices to show that $a + b \in S(X)$. We know from [4, Proposition 3.4] that $a, b \leq a + b$. From Theorem 3.8 (vi), we have $(a+b)' \leq a', b'$. Hence $a = (a')' \leq ((a+b)')'$ and $b = (b')' \leq ((a+b)')'$. Then from [4, Proposition 3.4 and Proposition 3.5], and the property $x + x = x$, we obtain

$$a \leq ((a+b)')' \Longrightarrow a+b \leq ((a+b)')' + b \leq ((a+b)')' + ((a+b)')' = ((a+b)')' \leq a+b.$$

Then it follows that $a + b = ((a+b)')'$, and so we have $a + b \in S(X)$. □

3.13. Theorem. *Let $(X_1; -_1)$ and $(X_2; -_2)$ be two subtraction algebras and $X_1 \cap X_2 = \{0\}$. We define the operation $-$ on $X = X_1 \cup X_2$ as follows*

$$x - y = \begin{cases} x -_1 y, & \text{if } x, y \in X_1, \\ x -_2 y, & \text{if } x, y \in X_2, \\ x, & \text{if } x \text{ and } y \text{ belong to different algebras.} \end{cases}$$

Then, X is a subtraction algebra.

Proof. It is easy to verify the axioms (S1)-(S3), and the proof is omitted. □

3.14. Definition. Let X_1 and X_2 be two subtraction algebras and $X_1 \cap X_2 = \{0\}$. If the set $X = X_1 \cup X_2$ is the subtraction algebra with the operation defined in Theorem 3.12, then X is called the *union of* X_1 *and* X_2, and is denoted by $X = X_1 \oplus X_2$.

Note that in $X = X_1 \oplus X_2$, if x and y do not belong to same algebra, then x and y are not comparable. Furthermore X_1 and X_2 are subalgebras of X.

Similarly, if X_i are subtraction algebras for all $i \in I$ and $X_i \cap X_j = \{0\}$ for $i, j \in I$, $i \neq j$, where I is an index set, the union algebra $X = \bigoplus_{i \in I} X_i$ can be defined in a similar way.

3.15. Example. Let $X_1 = \{0, a, b, c, d\}$ and $X_2 = \{0, e, f, g\}$ be two subtraction algebras with Cayley tables as follows:

$-_1$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

,

$-_2$	0	e	f	g
0	0	0	0	0
e	e	0	e	0
f	f	f	0	0
g	g	f	e	0

.

Then $X = \{0, a, b, c, d, e, f, g\}$ is the union subtraction algebra with the following Cayley table:

$-_1$	0	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0
a	a	0	a	a	a	a	a	a
b	b	b	0	b	b	b	b	b
c	c	c	c	0	c	c	c	c
d	d	d	d	d	0	d	d	d
e	e	e	e	e	e	0	e	0
f	f	f	f	f	f	f	0	0
g	g	g	g	g	g	f	e	0

3.16. Theorem. *Let $(X_1; -_1)$ and $(X_2; -_2)$ be two subtraction algebras which have at least two elements. Then the union $X = X_1 \oplus X_2$ is not a complicated subtraction algebra.*

Proof. Suppose x, y are non-zero elements in X and $x \in X_1, y \in X_2$. Then $x, y \in G(x, y)$, but there is no non-zero c such that $x \leq c$ and $y \leq c$. This means that $G(x, y)$ does not have a greatest element. So X is not a complicated subtraction algebra. □

3.17. Lemma. *Let x, y be any elements of a subtraction algebra X. If $A(x \wedge y) = \{0\}$ then $x - y = x$ and $y - x = y$.*

Proof. By (a4), since $x \wedge y \leq y$ and $x \wedge y \leq x$, we have $x \wedge y \in A(x)$ and $x \wedge y \in A(y)$. Since $A(x \wedge y) = \{0\}$ we have $x \wedge y = x - (x - y) = 0$ or $x \leq x \wedge y$. We also know that by (a3), $(x - y) - x = 0$ or $x - y \leq x$. Then $x - y = x$ is valid. Similarly it can be proved that $y - x = y$. □

3.18. Theorem. *Let X be a subtraction algebra, $\{X_i : i \in I\}$ a family of subsets of X. If the conditions*

a) $X = \bigcup\limits_{i \in I} X_i$,

b) $X_i \cap X_j = \{0\}$, $i \neq j$,

c) $x \in X_i$ *implies* $A(x) \subseteq X_i$ *for any* $i \in I$,

are satisfied, then all the X_i are subalgebras of X and X is the union of the X_i.

Proof. For any $x, y \in X_i$, from (a3) since $(x - y) - x = 0$, we have $x - y \leq x$ or $x - y \in A(x)$. Hence by (c), $x - y \in X_i$ and so X_i is a subalgebra of X. Now let $x \in X_i$ and $y \in X_j$, $i \neq j$. By using the hypothesis and Lemma 3.17, we have $x - y = x$ and $y - x = y$. So we obtain X is the union of all the X_i. □

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