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FACTORIZATIONS OF THE PASCAL MATRIX VIA A GENERALIZED SECOND ORDER RECURRENT MATRIX

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Abstract

In this paper, we consider positively and negatively subscripted terms of a generalized binary sequence $\{U_n\}$ with indices in arithmetic progression. We give a factorization of the Pascal matrix by a matrix associated with the sequence $\{U_{\pm kn}\}$ for a fixed positive integer k, generalizing results of Kılıç and Tascı; Lee, Kim and Lee; Stanica; and Zhizheng and Wang. Some new factorizations and combinatorial identities are derived as applications. Therefore we generalize the earlier results on the factorizations of the Pascal matrix.

Keywords: Factorization, Binary recurrences, Pascal matrix.

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1. Introduction

For $n > 0$, the $n \times n$ Pascal matrix $P_n = [p_{ij}]$ is defined as follows [4]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

In [6], it is shown that the matrix P_n satisfies

$$P_n = \mathcal{F}_n L_n,$$

where the $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{ij}]$ and the matrix $L_n = [l_{ij}]$ are defined by

$$[f_{ij}] = \begin{cases} F_{i-j+1} & \text{if } i - j + 1 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

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and

$$l_{ij} = \left(\binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}\right),$$

respectively, where F_n stands for the nth Fibonacci number.

In [7], the authors define an $n \times n$ matrix $R_n = [r_{i,j}]$ as follows:

$$r_{ij} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1},$$

and show that $P_n = R_n \mathcal{F}_n$. As an example, they give the following result:

$$\binom{n-1}{r-1} = F_{n-r+1} + (n-2)\, F_{n-r} + \tfrac{1}{2}\left(n^2 - 5n + 2\right) F_{n-r-1} + \sum_{k=r}^{n-3} \binom{n-1}{k-1} \left[2 - \frac{n}{k} - \frac{(n-k)(n-k-1)}{k(k+1)}\right] F_{k-r+1}.$$

Especially, for $r = 1$ they have

$$\sum_{k=1}^{n} \left(\binom{n-1}{k-1} - \binom{n-1}{k} - \binom{n-1}{k+1}\right) F_k = 1.$$

Furthermore they define an $n \times n$ matrix U_n of the form:

$$U_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -F_3 & 1 & 1 & 0 & \dots & 0 & 0 \\ -F_4 & 0 & 1 & 1 & \dots & 0 & 0 \\ -F_5 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -F_n & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix},$$

and the matrices $\overline{U}_k$ and $\overline{R}_n$ by $\overline{U}_k = I_{n-k} \oplus U_k$ and $\overline{R}_n = [1] \oplus R_{n-1}$. Then the authors give the following factorization:

$$R_n = \overline{R}_n U_n,$$
$$R_n = \overline{U}_1 \overline{U}_2 \cdots \overline{U}_{n-1} \overline{U}_n.$$

Let

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$S_k = S_0 \oplus I_k$ for $k \in \mathbb{N}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and $G_k = I_{n-k} \oplus S_{k-3}$ for $k \geq 3$.

In [5], the authors give the following factorization:

(1.1) $$\mathcal{F}_n = G_1 G_2 \cdots G_n,$$

where $\mathcal{F}_n$ is defined as before.

In [1], the authors show that the Stirling matrix $S_n = (S(i,j))_{ij}$ of the second kind can be written in terms of the Pascal matrix P_n:

$$S_n = P_n\left([1] \oplus S_{n-1}\right),$$

where $S(i,j)$ are the Stirling numbers of the second kind, defined by the following recurrence:

$$S(n,k) = S(n-1,k-1) + S(n-1,k).$$

In [3], the authors define the $n \times n$ matrix $W_n = [w_{ij}]$ and Pell matrix $E_n = [e_{ij}]$ as below

$$w_{ij} = \begin{cases} P_i & \text{if } j = 1, \\ 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

and $e_{ij} = P_{i-j+1}$ if $i - j + 1 \geq 0$ and 0 otherwise, where P_i is the i th Pell number. Then they show that

$$E_n = W_n \left(I_1 \oplus W_{n-1}\right) \left(I_2 \oplus W_{n-2}\right) \cdots \left(I_{n-2} \oplus W_2\right).$$

The Fibonacci and Lucas sequences have been discussed in very many studies. Various further generalizations and matrix representations of these sequences have also been introduced and investigated by many authors.

For $n > 0$ and nonnegative integers A and B such that $A^2 + 4B \neq 0$, the generalized Fibonacci and Lucas type sequences $\{U_n\}$ and $\{V_n\}$ are defined by

$$U_{n+1} = AU_n + BU_{n-1},$$
$$V_{n+1} = AV_n + BV_{n-1},$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = A$, respectively. When $A = B = 1$, $U_n = F_n$ (n th Fibonacci number) and $V_n = L_n$ (n th Lucas number).

The authors in [2] consider positively and negatively subscripted terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for a fixed positive integer k. They obtain relationships between these sequences and the determinants of certain tridiagonal matrices. Further, the authors give more general trigonometric factorizations and representations for the terms of $\{U_{\pm kn}\}$ and $\{V_{\pm kn}\}$. Generating functions and combinatorial representations for them are derived. Finally they obtain the following recurrence relations for $k > 0$ and $n > 1$,

$$U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)},$$
$$V_{kn} = V_k V_{k(n-1)} + (-1)^{k+1} B^k V_{k(n-2)}.$$

In this paper, we consider positively and negatively subscripted terms of the generalized binary sequence $\{U_n\}$. We give a factorization of the Pascal matrix by a matrix associated with the sequence $\{U_{\pm kn}\}$. Also, some new factorizations and combinatorial identities are derived as applications of our results. Therefore we generalize the results of some earlier studies on these factorizations.

2. Factorizations of the Pascal matrix via recurrent matrices associated with $\{U_{\pm kn}\}$

In this section, we define a matrix associated with the sequence $\{U_{\pm kn}\}$. Then we obtain some factorizations of the Pascal matrix by this new matrix, and derive new identities as an applications of these factorizations.

Let the $n \times n$ lower triangular matrix $H_n = [h_{ij}]$ be defined as follows:

$$h_{ij} = \begin{cases} U_{\pm(i-j+1)k} & \text{if } i - j + 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the matrix H_n is in the form

$$H_n = \begin{bmatrix} U_{\pm k} & & & & 0 \\ U_{\pm 2k} & U_{\pm k} & & & \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & & \\ \vdots & \vdots & \vdots & \ddots & \\ U_{\pm kn} & U_{\pm k(n-1)} & U_{\pm k(n-2)} & \dots & U_{\pm k} \end{bmatrix}.$$

Now, we define an $n \times n$ matrix $C_n = [c_{ij}]$ with

$$c_{ij} = \frac{1}{U_{\pm k}}\left(\binom{i-1}{j-1} - \frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-1}{j} + \binom{i-1}{j+1}(-B)^{\pm k}\right) \text{ if } i \geq j \text{ and } 0 \text{ otherwise.}$$

Then we can give the following theorem.

2.1. Theorem. *$P_n = C_n H_n$ for $n > 0$.*

Proof. To prove the theorem, it is sufficient to show $P_n H_n^{-1} = C_n$. The inverse of H_n is given by

$$H_n^{-1} = \left(h'_{ij}\right) = \begin{bmatrix} \frac{1}{U_{\pm k}} & & & & & \\ -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & & \\ \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \ddots & & & \\ 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & \ddots & \frac{1}{U_{\pm k}} & & \\ \vdots & \ddots & \ddots & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & \\ 0 & \dots & 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix}.$$

In order to prove that $P_n H_n^{-1} = C_n$ consider first the case where $i \geq 1$ and $j = 1$. By the definitions of P_n and H_n^{-1}, we write

$$\begin{aligned} \sum_{k=1}^{i} p_{i,k} h'_{k,1} &= p_{i,1} h'_{1,1} + p_{i,2} h'_{2,1} + p_{i,3} h'_{3,1} \\ &= \binom{i-1}{0}\frac{1}{U_{\pm k}} + \binom{i-1}{1}\left(-\frac{V_{\pm k}}{U_{\pm k}}\right) + \binom{i-1}{2}\left(\frac{(-B)^{\pm k}}{U_{\pm k}}\right) \\ &= \frac{1}{U_{\pm k}} - (i-1)\frac{V_{\pm k}}{U_{\pm k}} + \frac{(i-2)(i-1)}{2U_{\pm k}}(-B)^{\pm k}. \end{aligned}$$

By the definition of C_n,

$$c_{i,1} = \frac{1}{U_{\pm k}}\left(\binom{i-1}{0} - \binom{i-1}{1}\frac{U_{\pm 2k}}{U_{\pm k}} + \binom{i-1}{2}(-B)^{\pm k}\right),$$

and so we get the required conclusion

$$\sum_{k=1}^{i} p_{i,k} h'_{k,1} = c_{i,1}.$$

For $i \geq 1$ and $j \geq 2$, we obtain

$$\begin{aligned} \sum_{k=1}^{n} p_{i,k} h'_{k,j} &= p_{i,j} h'_{j,j} + p_{i,j+1} h'_{j+1,j} + p_{i,j+2} h'_{j+2,j} \\ &= \binom{i-1}{j-1}\frac{1}{U_{\pm k}} + \binom{i-1}{j}\left(-\frac{V_{\pm k}}{U_{\pm k}}\right) + \binom{i-1}{j+1}\frac{(-B)^{\pm k}}{U_{\pm k}} \\ &= \binom{i-1}{j-1}\frac{1}{U_{\pm k}} - \binom{i-1}{j}\frac{U_{\pm 2k}}{U_{\pm k}^2} + \binom{i-1}{j+1}\frac{(-B)^{\pm k}}{U_{\pm k}}. \end{aligned}$$

From the definition of C_n, we write

$$\sum_{k=1}^{n} p_{i,k} h'_{k,j} = c_{i,j}.$$

Then we obtain that $P_n H_n^{-1} = C_n$. Thus the proof is complete. □

As a result of Theorem 2.1, we may give the following identity without proof.

2.2. Corollary. *For $n \geq r > 0$,*

$$\tbinom{n-1}{r-1} = \sum_{j=r}^{n} \left(\tbinom{n-1}{j-1} - \tbinom{n-1}{j} V_{\pm k} + \tbinom{n-1}{j+1} (-B)^{\pm k} \right) \tfrac{U_{\pm(j-r+1)k}}{U_{\pm k}}.$$

In particular, if we take $r = 1$ in Corollary 2.2, we have

$$\sum_{j=1}^{n} \left(\tbinom{n-1}{j-1} - \tbinom{n-1}{j} V_{\pm k} + \tbinom{n-1}{j+1} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}} = 1.$$

2.3. Lemma. *For $3 \leq i, j \leq n$,*

$$\begin{aligned} &\sum_{j=3}^{i} \left(\tbinom{i-2}{j-2} - \tbinom{i-2}{j-1} V_{\pm k} + \tbinom{i-2}{j} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}^2} \\ &\qquad = (i-2) \tfrac{V_{\pm k}^2}{U_{\pm k}} - \left(\tbinom{i-2}{2} V_{\pm k} + \tbinom{i-2}{1} \right) \tfrac{(-B)^{\pm k}}{U_{\pm k}}. \end{aligned}$$

Proof. (By induction on i). Clearly the equation holds for $i = 3$. Assume that the equation holds for $i \geq 4$. Thus

$$\begin{aligned} &\sum_{j=3}^{i+1} \left(\tbinom{i-1}{j-2} - \tbinom{i-1}{j-1} V_{\pm k} + \tbinom{i-1}{j} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}^2} \\ &= \sum_{j=3}^{i+1} \left(\tbinom{i-2}{j-2} - \tbinom{i-2}{j-1} V_{\pm k} + \tbinom{i-2}{j} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}^2} \\ &\qquad + \sum_{j=3}^{i+1} \left(\tbinom{i-2}{j-3} - \tbinom{i-2}{j-2} V_{\pm k} + \tbinom{i-2}{j-1} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}^2} \\ &= \sum_{j=3}^{i} \left(\tbinom{i-2}{j-2} - \tbinom{i-2}{j-1} V_{\pm k} + \tbinom{i-2}{j} (-B)^{\pm k} \right) \tfrac{U_{\pm jk}}{U_{\pm k}^2} \\ &\qquad + \sum_{j=2}^{i+1} \left(\tbinom{i-2}{j-2} - \tbinom{i-2}{j-1} V_{\pm k} + \tbinom{i-2}{j} (-B)^{\pm k} \right) \tfrac{U_{\pm(j+1)k}}{U_{\pm k}^2} \\ &= (i-2) \tfrac{V_{\pm k}^2}{U_{\pm k}} - \left(\tbinom{i-2}{2} V_{\pm k} + \tbinom{i-2}{1} \right) \tfrac{(-B)^{\pm k}}{U_{\pm k}} \\ &\qquad + \sum_{j=2}^{i+1} \left(\tbinom{i-2}{j-2} - \tbinom{i-2}{j-1} V_{\pm k} + \tbinom{i-2}{j} (-B)^{\pm k} \right) \\ &\qquad\qquad \times \tfrac{\left(V_{\pm k} U_{\pm jk} - (-B)^{\pm k} U_{\pm(j-1)k} \right)}{U_{\pm k}^2}. \end{aligned}$$

After some calculations and using Corollary 2.2, we get

$$\sum_{j=3}^{i+1}\left(\binom{i-1}{j-2}-\binom{i-1}{j-1}V_{\pm k}+\binom{i-1}{j}(-B)^{\pm k}\right)\frac{U_{\pm jk}}{U_{\pm k}^2}$$
$$=(i-1)\frac{V_{\pm k}^2}{U_{\pm k}}-\left(\binom{i-1}{2}V_{\pm k}+\binom{i-1}{1}\right)\frac{(-B)^{\pm k}}{U_{\pm k}}.$$

Hence, the proof is complete. □

Now, we define the $n\times n$ matrices T_n, $\overline{C}_n$ and $\overline{T}_k$ by

$$T_n=\begin{bmatrix}\frac{1}{U_{\pm k}} & & & & & 0\\ 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 & & & & \\ -\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 & & & \\ -\frac{U_{\pm 4k}}{U_{\pm k}} & 0 & 1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ -\frac{U_{\pm nk}}{U_{\pm k}} & 0 & \dots & 0 & 1 & 1\end{bmatrix},$$

$\overline{C}_n=[1]\oplus C_{n-1}$ and $\overline{T}_k=I_{n-k}\oplus T_k$, where C_n is defined as before.

2.4. Lemma. *For $n>0$,*

$$C_n=\overline{C}_nT_n.$$

Proof. We denote the (i,j)th element of the matrix $\overline{C}_n$ by $\overline{c}_{i,j}$. Then,

$$\overline{c}_{i,j}=\begin{cases}1 & \text{if } i=1, j=1,\\ 0 & \text{if } i\neq 1, j=1 \text{ or } i=1, j\neq 1,\\ c_{i-1,j-1} & \text{otherwise.}\end{cases}$$

Let $\overline{C}_nT_n=[K_{i,j}]$ and $T_n=[t_{i,j}]$. Obviously $K_{1,1}=\frac{1}{U_{\pm k}}=c_{1,1}$, $K_{2,2}=\frac{1}{U_{\pm k}}=c_{2,2}$, $K_{2,1}=\frac{1-V_{\pm k}}{U_{\pm k}}=c_{2,1}$ and $K_{i,j}=0$ for $i<j$. Since $t_{i,1}=-\frac{U_{\pm ik}}{U_{\pm k}}$ for $i\geq 3$, $j=1$ and using Lemma 2.3, we have

$$\begin{aligned}K_{i,1}&=\sum_{j=2}^{i}\overline{c}_{i,j}t_{j,1}=\sum_{j=2}^{i}c_{i-1,j-1}t_{j,1}\\
&=\sum_{j=2}^{i}\left(\binom{i-2}{j-2}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{j-1}+\binom{i-2}{j}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}}t_{j,1}\\
&=\left(\binom{i-2}{0}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{1}+\binom{i-2}{2}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}}t_{2,1}\\
&\qquad-\sum_{j=3}^{i}\left(\binom{i-2}{j-2}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{j-1}+\binom{i-2}{j}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}^2}U_{\pm jk}\\
&=\left(1-(i-2)V_{\pm k}+\binom{i-2}{2}(-B)^{\pm k}\right)\left(\frac{1-V_{\pm k}}{U_{\pm k}}\right)\\
&\qquad-\left((i-2)\frac{V_{\pm k}^2}{U_{\pm k}}-\left(\binom{i-2}{2}V_{\pm k}+\binom{i-2}{1}\right)\left(\frac{(-B)^{\pm k}}{U_{\pm k}}\right)\right)\\
&=\left(\binom{i-1}{0}-\binom{i-1}{1}V_{\pm k}+\binom{i-1}{2}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}}\\
&=c_{i,1}.\end{aligned}$$

In general, for $i \geq 2$, $j \geq 2$, from the definition of C_n, we get

$$K_{i,j} = \sum_{m=1}^{i} \overline{c}_{i,m} t_{m,j} = c_{i-1,j-1} \cdot 1 + c_{i-1,j} \cdot 1 = c_{i,j}.$$

Thus the proof is complete. □

2.5. Lemma. *For $n > 0$,*

$$C_n = \overline{T}_1 \overline{T}_2 \cdots \overline{T}_{n-1} \overline{T}_n.$$

Proof. Follows directly from the definitions of C_n and $\overline{T}_n$. □

For example, when $n = 4$ in Lemma 2.5, we obtain

$$\begin{aligned} C_4 &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-2V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-3V_{\pm k}+3(-B)^{\pm k}}{U_{\pm k}} & \frac{3-3V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{U_{\pm k}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{U_{\pm k}} & 0 \\ 0 & 0 & 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 \end{bmatrix} \\ &\qquad \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 & 0 \\ 0 & 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 & 0 \\ 0 & -\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 & 0 & 0 \\ -\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 & 0 \\ -\frac{U_{\pm 4k}}{U_{\pm k}} & 0 & 1 & 1 \end{bmatrix} \\ &= \overline{T}_1 \overline{T}_2 \overline{T}_3 \overline{T}_4. \end{aligned}$$

Now, define

$$M_0 = \begin{bmatrix} U_{\pm k} & 0 & 0 \\ V_{\pm k} & 1 & 0 \\ -(-B)^{\pm k} & 0 & 1 \end{bmatrix}, \ M_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & U_{\pm k} & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} \end{bmatrix},$$

$M_k = M_0 \oplus I_k$, $k \in \mathbb{N}$, and $A_1 = I_n$, $A_2 = I_{n-3} \oplus M_{-1}$, $A_k = I_{n-k} \oplus M_{k-3}$, $k \geq 3$. Therefore, we easily obtain the following result which we give without proof.

2.6. Lemma. *For $n > 0$,*

$$H_n = A_1 A_2 \cdots A_n.$$

In particular, when $n = 4$,

$$H_4 = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & U_{\pm 2k} & U_{\pm k} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & V_{\pm k} & 1 & 0 \\ 0 & -(-B)^{\pm k} & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= A_1 A_2 A_3 A_4.$$

2.7. Corollary. *For $n > 0$,*

$$P_n = C_n H_n = \overline{T}_1 \overline{T}_2 ... \overline{T}_{n-1} \overline{T}_n A_1 A_2 \cdots A_n.$$

Proof. This follows from Theorem 2.1 and Lemma 2.6. □

Now, we define an $n \times n$ matrix $C'_n = \left[c'_{i,j}\right]$ with

$$c'_{i,j} = \frac{1}{U_{\pm k}} \left(\binom{i-1}{j-1} - \frac{U_{\pm 2k}}{U_{\pm k}} \binom{i-2}{j-1} + \binom{i-3}{j-1} (-B)^{\pm k} \right) \text{ if } i \geq j \text{ and } 0 \text{ otherwise.}$$

We can then give the following theorem.

2.8. Theorem. *Let P_n, H_n, C'_n be the $n \times n$ matrices defined above. Then we have:*

$$P_n = H_n C'_n.$$

Proof. It is sufficient to show $H_n^{-1} P_n = C'_n$. Let $H_n^{-1} P_n = [z_{i,j}]$. Here we note that the matrix H_n^{-1} is in the form

$$H_n^{-1} = \begin{bmatrix} \frac{1}{U_{\pm k}} & & & & & 0 \\ -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & & \\ \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & \\ 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & \\ 0 & \dots & 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix}.$$

Clearly, $z_{1,1} = c'_{1,1}$, $z_{2,1} = c'_{2,1}$, $z_{2,2} = c'_{2,2}$, and for $i < j$, $z_{i,j} = c_{ij} = 0$. Since all the elements of the first column of P_n are 1, we have $z_{i,j} = \frac{1 - V_{\pm k} + (-B)^{\pm k}}{U_{\pm k}}$ for $i \geq 3$ and $j = 1$.

For $i, j \geq 2$, from the definition of C'_n, we obtain

$$\begin{aligned} z_{i,j} &= \sum_{j=1}^{n} h'_{i,k} p_{k,j} = h'_{i,i} p_{i,j} + h'_{i,i-1} p_{i-1,j} + h'_{i,i-2} p_{i-2,j} \\ &= \frac{1}{U_{\pm k}} \binom{i-1}{j-1} + \left(-\frac{V_{\pm k}}{U_{\pm k}} \right) \binom{i-2}{j-1} + \frac{(-B)^{\pm k}}{U_{\pm k}} \binom{i-3}{j-1} \\ &= c'_{i,j}. \end{aligned}$$

Thus the proof is complete. □

From Theorem 2.8, we get the following result:

2.9. Corollary. *For $n \geq r > 0$,*

$$\binom{n-1}{r-1} = \sum_{j=r}^{n} \left(\frac{U_{\pm(n-j+1)k}}{U_{\pm k}}\right) \left(\binom{j-1}{r-1} - \binom{j-2}{r-1} V_{\pm k} + \binom{j-3}{r-1} (-B)^{\pm k}\right).$$

In particular, when $r = 1$, we obtain

$$\sum_{j=1}^{n} \left(\frac{U_{\pm(n-j+1)k}}{U_{\pm k}}\right) \left(1 - \binom{j-2}{0} V_{\pm k} + \binom{j-3}{0} (-B)^{\pm k}\right) = 1.$$

We define an $n \times n$ matrix Q_n by

$$Q_n = \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

If we take $\overline{C}'_n = [1] \oplus C'_{n-1}$, the following result is easily seen.

2.10. Lemma. *For $n > 0$,*

$$C'_n = Q_n \overline{C}'_n.$$

For example, when $n = 4$, we get

$$\begin{aligned} C'_4 &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-2V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 & 0 \\ 0 & \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ 0 & \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\ &= Q_4 \overline{C}'_4. \end{aligned}$$

2.11. Lemma. *Let the matrix Q_k be defined as before and $\overline{Q}_k = I_{n-k} \oplus Q_k$. Then*

$$C'_n = \overline{Q}_n \overline{Q}_{n-1} \ldots \overline{Q}_2 \overline{Q}_1.$$

We can give the following example:

$$
\begin{aligned}
C_3' &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 \\ 0 & \frac{1-V_{\pm k}}{U_{\pm k}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \overline{Q}_3 \overline{Q}_2 \overline{Q}_1 .
\end{aligned}
$$

We consider an $n \times n$ matrix $T_n' = \left[t_{i,j}' \right]$ with

$$
t_{i,j}' = \begin{cases} U_{\pm ik} & \text{if } i \geq 1, \ j = 1, \\ 1 & \text{if } i = j, \ i, j \geq 2, \\ 0 & \text{otherwise.} \end{cases}
$$

Now, we can give the following results:

2.12. Lemma. *For $n > 1$,*

$$
H_n = T_n' \left([1] \oplus H_{n-1} \right).
$$

Proof. Let $T_n' \left([1] \oplus H_{n-1} \right) = (y_{i,j})$. Since the $(1, 1)$ th element of the matrix $[1] \oplus H_{n-1}$ is 1, and the other elements in the first column of this matrix are zero, we get $y_{i,1} = U_{\pm ik}$. For $i \geq 1$, $j \geq 2$ and $i \geq j$, by using the definitions of T_n' and $[1] \oplus H_{n-1}$, we obtain

$$
y_{i,j} = U_{\pm (i-j+1)k}.
$$

For $i < j$, we obtain $y_{i,j} = 0$. Finally we get $y_{i,j} = h_{ij}$ for $1 \leq i, j \leq n$, which completes the proof. □

When $n = 6$ in Lemma 2.12,

$$
\begin{aligned}
H_6 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 6k} & U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & 1 & 0 & 0 & 0 & 0 \\ U_{\pm 3k} & 0 & 1 & 0 & 0 & 0 \\ U_{\pm 4k} & 0 & 0 & 1 & 0 & 0 \\ U_{\pm 5k} & 0 & 0 & 0 & 1 & 0 \\ U_{\pm 6k} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 & 0 & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ 0 & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ 0 & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ 0 & U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= T_6' \left([1] \oplus H_5 \right).
\end{aligned}
$$

2.13. Lemma. *If we define $\overline{T}_k' = I_{n-k} \oplus T_k'$, then*

$$
H_n = \overline{T}_n' \overline{T}_{n-1}' \cdots \overline{T}_2' \overline{T}_1'.
$$

For example, when $n = 5$ in Lemma 2.13, we have

$$H_5 = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix}$$

$$= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & 1 & 0 & 0 & 0 \\ U_{\pm 3k} & 0 & 1 & 0 & 0 \\ U_{\pm 4k} & 0 & 0 & 1 & 0 \\ U_{\pm 5k} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 & 0 \\ 0 & U_{\pm 2k} & 1 & 0 & 0 \\ 0 & U_{\pm 3k} & 0 & 1 & 0 \\ 0 & U_{\pm 4k} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 & 0 \\ 0 & 0 & U_{\pm 2k} & 1 & 0 \\ 0 & 0 & U_{\pm 3k} & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & 0 & U_{\pm 2k} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & U_{\pm k} \end{bmatrix}$$

$$= \overline{T}'_5 \overline{T}'_4 \overline{T}'_3 \overline{T}'_2 \overline{T}'_1.$$

Now, we define an $n \times n$ matrix D_n of the form:

$$D_n = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & \cdots & 0 \\ V_{\pm k} & 1 & 0 & 0 & \cdots & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then we have the following factorization.

2.14. Lemma. *For $n > 1$,*

$$H_n = ([1] \oplus H_{n-1})\, D_n.$$

Proof. Since the (i, j) th element of $[1] \oplus H_{n-1}$ is h_{ij}, and in view of the definition of D_n, the result is readily seen. □

For $n = 4$ in Lemma 2.14, we obtain

$$H_4 = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} & 0 \\ 0 & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= ([1] \oplus H_3)\, D_4.$$

If we define an $n \times n$ matrix $\overline{D}_k$ with $\overline{D}_k = I_{n-k} \oplus D_k$, then we can obtain the following result.

2.15. Lemma. *For $n > 1$,*

$$H_n = \overline{D}_1 \overline{D}_2 \cdots \overline{D}_{n-1} \overline{D}_n.$$

When $n = 4$ in Lemma 2.15, we get

$$H_4 = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & U_{\pm k} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & V_{\pm k} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & V_{\pm k} & 1 & 0 \\ 0 & -(-B)^{\pm k} & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \overline{D}_1 \overline{D}_2 \overline{D}_3 \overline{D}_4.$$

3. Conclusion

In the present paper we introduce the $n \times n$ matrix H_n whose entries are U_{kn} satisfying the general second order recurrence formula $U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)}$, with initial conditions 0, U_k for $k > 0$ and $n > 1$. We use the matrix H_n instead of the $n \times n$ Fibonacci matrix $\mathcal{F}_n$ in the factorizations $P_n = R_n \mathcal{F}_n$ and $P_n = \mathcal{F}_n L_n$ given in [7] and [6], respectively. Here we obtain new matrices corresponding to the matrices R_n and L_n. Therefore, we give more generalized factorizations of the $n \times n$ Pascal matrix P_n. Further, using these factorizations, the sequence $\{U_{\pm kn}\}$ and the matrix H_n associated with the sequence $\{U_{\pm kn}\}$, we generalize various results in [1, 3, 5, 6, 7].

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