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FUZZY DIFFERENTIAL SUBORDINATIONS FOR ANALYTIC FUNCTIONS INVOLVING WANAS OPERATOR

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Abstract

The purpose of the present paper is to establish some properties of fuzzy subordination of analytic functions associated with Wanas differential operator which defined in the open unit disk. Further, we obtain results related to fractional derivative (Riemann-Liouville derivative).

Keywords: Fuzzy set; Fuzzy differential subordination; Wanas differential operator; Fractional derivative.

1. Introduction

Denote by $\mathcal{M}_\lambda$ the class of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{n-\lambda} \quad (0 \leq \lambda < 1), \tag{1.1}$$

For functions $f_j \in \mathcal{M}_\lambda$ $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^{n-\lambda} \ (j = 1,2),$$

we define the Hadamard product (convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^{n-\lambda} = (f_2 * f_1)(z)\,.$$

A function $f \in \mathcal{M}_\lambda$ is said to be univalent starlike of order ρ $(0 \leq \rho < 1)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (z \in U).$$

Denote this class by $S(\rho)$.

Wanas [20] introduced the differential operator $W_{\alpha,\beta}^{k,\eta} : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ as follows

$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^n ,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $m, \eta \in \mathbb{N}_0 = \{0,1,2,3, \dots\}$.

It is easily verified that if $f \in \mathcal{M}_\lambda$, then we have

$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^{n-\lambda} , \qquad (1.2)$$

It follows from (1.2) that

$$z\left(W_{\alpha,\beta}^{k,\eta} f(z)\right)' = \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\left(\frac{\alpha}{\beta}\right)^m + 1 \right) \right] W_{\alpha,\beta}^{k,\eta+1} f(z)$$

$$- \left[\lambda + \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^m \right] W_{\alpha,\beta}^{k,\eta} f(z). \qquad (1.3)$$

Some of the special cases of the operator defined by (1.2) can be found in [1,3,4,16,19]. For more details see [22].

Definition 1.1 [23]. Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X: 0 < F_A(x) \leq 1\} = supp(A, F_A)$

is called fuzzy subset. The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.2 [13]. Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x), x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x), x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Let $D \subseteq \mathbb{C}$ and f, g analytic functions. We denote by

$$f(D) = supp\big(f(D), F_{f(D)}\big) = \left\{f(z) : 0 < F_{f(D)}\big(f(z)\big) \leq 1, z \in D\right\}$$

and

$$g(D) = supp\big(g(D), F_{g(D)}\big) = \left\{g(z) : 0 < F_{g(D)}\big(g(z)\big) \leq 1, z \in D\right\}.$$

Definition 1.3 [13]. Let $D \subseteq \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}\big(f(z)\big) \leq F_{g(D)}\big(g(z)\big), z \in D$.

Definition 1.4 [14]. Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination

$$F_{\psi(\mathbb{C}^3\times U)}\big(\psi(p(z), zp'(z), z^2p''(z); z)\big) \leq F_{h(U)}\big(h(z)\big), \tag{1.4}$$

i.e. $\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z)\ , z \in U,$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(z) \prec_F q(z), z \in U$ for all p satisfying (1.4). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in U$ for all fuzzy dominant q of (1.4) is said to be the fuzzy best dominant of (1.4).

In order to prove our main results, we need the following lemma.

Lemma 1.1 [6]. Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Suppose that

1) $Q(z)$ is starlike in U,

2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and $\psi\colon \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi\big(p(z), zp'(z)\big) = \theta\big(p(z)\big) + zp'(z).\phi\big(p(z)\big)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\big[\theta\big(p(z)\big) + zp'(z).\phi\big(p(z)\big)\big] \leq F_{h(U)}h(z),$$

implies $F_{p(U)}p(z) \leq F_{q(U)}q(z)$,

i.e. $p(z) \prec_F q(z)$ and q is the fuzzy best dominant, where

$$\psi(\mathbb{C}^2 \times U) = supp\left(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi\big(p(z), zp'(z)\big)\right)$$

$$= \left\{z \in \mathbb{C} : 0 < F_{\psi(\mathbb{C}^2\times U)}\psi\big(p(z), zp'(z)\big) \leq 1\right\},$$

and $$h(U) = supp\left(U, F_{h(U)}h(z)\right) = \left\{z \in \mathbb{C} : 0 < F_{h(U)}h(z) \leq 1\right\}.$$

Recently, Oros and Oros [14,15], Lupaş [7-11], Lupaş and Oros [12], Wanas and Majeed [21] and Altınkaya and [2] have obtained fuzzy differential subordination results for certain classes of analytic functions.

2. Fuzzy Subordination Results

Theorem 2.1. Let $\gamma, \delta, \mu \in \mathbb{C}$, $t \in \mathbb{C} \setminus \{0\}$, $\tau > 0$ and q be univalent function in U with $q(0) = 1, q(z) \neq 0$ and assume that

$$Re\left\{\frac{\gamma\mu}{t}q(z) + (\mu - 2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} + 1 + \frac{\delta}{t}(\mu - 1)\right\} > 0. \tag{2.1}$$

Suppose that $z(q(z))^{\mu-2}q'(z)$ is starlike in U. If $f \in \mathcal{M}_\lambda$ and $\chi(\gamma, \delta, \mu, \tau, k, \eta, \alpha, \beta; z)$ is analytic in U, where

$$\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)=\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\mu\tau}\left[\gamma+\delta\left(\frac{W_{\alpha,\beta}^{k,\eta}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}\right)^{\tau}\right.$$

$$\left.+t\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^{m}+1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}\right)^{\tau}\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)}-\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)\right].\quad(2.2)$$

then

$$F_{\psi(\mathbb{C}^2\times U)}[\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)]\le F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\mu}\left(\gamma+\frac{\delta}{q(z)}+t\frac{zq'(z)}{(q(z))^{2}}\right)\right]$$

$$=F_{h(U)}h(z),\quad(2.3)$$

implies

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\eta+1}}{W_{\alpha,\beta}^{k,\eta}}\right)^{\tau}(U)}\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}\le F_{q(U)}q(z),$$

i.e.

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}\prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. Define p by

$$p(z)=\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau}=\left(\frac{1+\sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m+n\beta^m}{\alpha^m+\beta^m}\right)\right]^{\eta+1}a_n z^{n-\lambda-1}}{1+\sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha^m+n\beta^m}{\alpha^m+\beta^m}\right)\right]^{\eta}a_n z^{n-\lambda-1}}\right)^{\tau}.\quad(2.4)$$

Then the function p is analytic in U and $p(0)=1$. After simple computation we have

$$(p(z))^{\mu}\left(\gamma+\frac{\delta}{p(z)}+t\frac{zp'(z)}{(p(z))^{2}}\right)=\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z),\quad(2.5)$$

where $\chi(\gamma,\delta,\mu,\tau,k,\eta,\alpha,\beta;z)$ is given by (2.2).

From (2.3) and (2.5), we obtain

$$F_{\psi(\mathbb{C}^2\times U)}\left[(p(z))^{\mu}\left(\gamma+\frac{\delta}{p(z)}+t\frac{zp'(z)}{(p(z))^{2}}\right)\right]\le F_{\psi(\mathbb{C}^2\times U)}\left[(q(z))^{\mu}\left(\gamma+\frac{\delta}{q(z)}+t\frac{zq'(z)}{(q(z))^{2}}\right)\right].$$

Define the functions θ and ϕ by

$$\theta(w)=(\gamma w+\delta)w^{\mu-1}\quad and\quad \phi(w)=t\,w^{\mu-2}.$$

Obviously, the functions θ and ϕ are analytic in $D=\mathbb{C}\setminus\{0\}$ and $\phi(w)\neq 0, w\in D$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = tz(q(z))^{\mu-2}q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = (q(z))^{\mu}\left(\gamma + \frac{\delta}{q(z)} + t\frac{zq'(z)}{(q(z))^2}\right).$$

Since $z(q(z))^{\mu-2}q'(z)$ is starlike univalent in U, we find that Q is starlike univalent in U.

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\gamma\mu}{t}q(z) + (\mu-2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} + 1 + \frac{\delta}{t}(\mu-1)\right\}. \quad (2.6)$$

Using (2.1), (2.6) becomes

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0.$$

Therefore, by Lemma 1.1, we get $F_{p(U)}p(z) \leq F_{q(U)}q(z)$. By using (2.4), we obtain

$$F_{\left(\frac{W_{\alpha,\beta}^{k,\eta+1}}{W_{\alpha,\beta}^{k,\eta}}\right)^{\tau}(U)}\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \leq F_{q(U)}q(z),$$

i.e. $\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \prec_F q(z)$ and q is the fuzzy best dominant.

By taking the fuzzy dominant $q(z) = \frac{1+z}{1-z}, \mu = t = 1$ and $\gamma = \delta = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. Let $Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0$. If $f \in \mathcal{M}_{\lambda}$ and

$$\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[\tau\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\left(\frac{\alpha}{\beta}\right)^m + 1\right)\left(\frac{W_{\alpha,\beta}^{k,\eta+2}f(z)}{W_{\alpha,\beta}^{k,\eta+1}f(z)} - \frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)\right] \leq F_{\psi(\mathbb{C}^2\times U)}\left[\frac{2z}{1-z^2}\right],$$

implies

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}f(z)}{W_{\alpha,\beta}^{k,\eta}f(z)}\right)^{\tau} \prec_F \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the fuzzy best dominant.

By fixing $\eta = 0$ in Corollary 2.1, we obtain the following corollary:

Corollary 2.2. Let $Re\left\{\frac{1+z^2}{1-z^2}\right\} > 0$. If $f \in \mathcal{M}_\lambda$ and $\tau\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$ is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[\tau\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] \le F_{\psi(\mathbb{C}^2\times U)}\left[\frac{2z}{1-z^2}\right],$$

implies

$$\left(\frac{zf'(z)}{f(z)}\right)^\tau \prec_F \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the fuzzy best dominant.

3. Fractional Derivative Operator Results

In this section, we introduce some applications of section 2 containing fractional derivative operators (Riemann-Liouville derivative).

Definition 3.1 [16]. The fractional derivative of order $\lambda, (0 \le \lambda < 1)$ of a function f is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\epsilon)}{(z-\epsilon)^\lambda} d\epsilon, \tag{3.1}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\epsilon)^{-\lambda}$ is removed by requiring $\log(z-\epsilon)$ to be real, when $(z-\epsilon) > 0$.

Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$. The Gaussian hypergeometric function ${}_2F_1$ (see [17]) is defined by

$${}_2F_1(a,b,c;z) = {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(x)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0) \\ x(x+1)\dots(x+n-1) & (n \in \mathbb{N}). \end{cases}$$

Definition 3.2 [4]. Let $0 \le \lambda < 1$ and $u, v \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,u,v}$ of a function f is defined by:

$$J_{0,z}^{\lambda,u,v} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz}\left\{z^{\lambda-u} \int_o^z (z-\epsilon)^{-\lambda} f(\epsilon).\ {}_2F_1\left(u-\lambda, -v; 1-\lambda; 1-\frac{\epsilon}{z}\right) d\epsilon\right\}, \\ \qquad\qquad\qquad (0 \le \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,u,v} f(z), \qquad (n \le \lambda < n+1, n \in \mathbb{N}), \end{cases} \tag{3.2}$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}),(z \to 0),$$

for $\varepsilon > max\ \{0, u - v\} - 1$, and the multiplicity of $(z - \epsilon)^{-\lambda}$ is removed by requiring $\log(z - \epsilon)$ to be real, when $(z - \epsilon) > 0$.

By comparing (3.1) with (3.2), we find

$$J_{0,z}^{\lambda,\lambda,v} f(z) = D_z^{\lambda} f(z), \ \ (0 \le \lambda < 1).$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,u,v} z^n = \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} z^{n-u}, \tag{3.3}$$

$$(\ 0 \le \lambda < 1, u, v \in \mathbb{R}\ and\ n > max\{0, u - v\} - 1\ ).$$

Now, we define

$$\Omega(z) = \sum_{n=2}^{\infty} \sigma_n z^n .$$

By Definition 3.1, we have

$$D_z^{\lambda}\Omega(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} \sigma_n z^{n-\lambda} = \sum_{n=2}^{\infty} a_n\, z^{n-\lambda},$$

where

$$a_n = \frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)} \sigma_n, \qquad n = 2,3, \dots\ .$$

Thus $G_1(z) = z + D_z^{\lambda}\Omega(z) \in \mathcal{M}_{\lambda}$, then we obtain the following result:

Theorem 3.1. Let the assumptions of Theorem 2.1 hold. Then

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1} G_1(z)}{W_{\alpha,\beta}^{k,\eta} G_1(z)}\right)^{\tau} \prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. It can easily observed that $G_1(z) = z + D_z^{\lambda}\Omega(z) \in \mathcal{M}_{\lambda}$. Thus by using Theorem 2.1, we obtain the result.

Also, by using (3.3), we have

$$J_{0,z}^{\lambda,u,v}\Omega(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} \sigma_n z^{n-u} = \sum_{n=2}^{\infty} a_n z^{n-u},$$

where

$$a_n = \frac{\Gamma(n+1)\Gamma(n-u+v+1)}{\Gamma(n-u+1)\Gamma(n-\lambda+v+1)} \sigma_n, \qquad n = 2,3, \dots\ .$$

Let $u = \lambda$. Then $G_2(z) = z + J_{0,z}^{\lambda,u,v}\Omega(z) \in \mathcal{M}_{\lambda}$, then we obtain the following result:

Theorem 3.2. Let the assumptions of Theorem 2.1 hold. Then

$$\left(\frac{W_{\alpha,\beta}^{k,\eta+1}G_2(z)}{W_{\alpha,\beta}^{k,\eta}G_2(z)}\right)^{\tau} \prec_F q(z)$$

and q is the fuzzy best dominant.

Proof. It can easily observed that $G_2(z) = z + J_{0,z}^{\lambda,u,v}\Omega(z) \in \mathcal{M}_\lambda$. Thus by using Theorem 2.1, we obtain the result.

4. Conclusions

In the present work, we have introduced some properties of fuzzy differential subordination of analytic functions by using Wanas differential operator. Further, fractional derivative (Riemann-Liouville derivative) is investigated in this study and therefore it may be considered as a useful tool for those who are interested in the above-mentioned topics for further research.

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References

[1] Al-Oboudi, F.M. (2004) On univalent functions defined by a generalized Salagean operator. Int. J. Math. Math. Sci., 27: 1429-1436.

[2] Altınkaya, Ş., Wanas, A.K. (2020) Some properties for fuzzy differential subordination defined by Wanas operator. Earthline Journal of Mathematical Sciences, 4(1): 51-62.

[3] Cho, N.E., Kim, T.H. (2003) Multiplier transformations and strongly close-to-convex functions. Bull. Korean Math. Soc., 40(3): 399-410.

[4] Cho, N.E., Srivastava, H.M. (2003) Argument estimates of certain analytic functions defined by a class of multiplier transformations. Math. Comput. Modelling, 37(1-2): 39-49.

[5] Goyal, S.P., Goyal, R. (2005) On a class of multivalent functions defined by a generalized Ruscheweyh derivatives involving a general fractional derivative operator. J. Indian Acad. Math., 27(2): 439-456.

[6] Haydar, A. (2015) On fuzzy differential subordination. Mthematica Moravica, 19(1): 123-129.

[7] Lupaş, A.A. (2013) A note on special fuzzy differential subordinations using generalized Sălăgean operator and Ruscheweyh derivative. J. Comp. Anal. Appl., 15(8): 1476-1483.

[8] Lupaş, A.A. (2016) A note on special fuzzy differential subordinations using multiplier transformation. Analele Universitatii Oradea, Fasc. Matematica, XXIII(2): 183-191.

[9] Lupaş, A.A. (2017) On special fuzzy differential subordinations using generalized Salagean operator and Ruscheweyh derivative. J. Adv. Appl. Comp. Math., 4: 26-34.

[10] Lupaş, A.A. (2017) On special fuzzy differential subordinations using multiplier transformation. J. Comp. Anal. Appl., 23(6): 1029-1035.

[11] Lupaş, A.A. (2018) A note on special fuzzy differential subordinations using multiplier transformation and Ruschewehy derivative. J. Comp. Anal. Appl., 25(6): 1125-1131.

[12] Lupaş, A.A., Oros, Gh. (2015) On special fuzzy differential subordinations using Sălăgean and Ruscheweyh operators. Appl. Math. Comp., 261: 119-127.
[13] Oros, G.I., Oros, Gh. (2011) The notion of subordination in fuzzy set theory. General Mathematics, 19(4): 97-103.
[14] Oros, G.I., Oros, Gh. (2012) Fuzzy differential subordination. Acta Universitatis Apulensis, 30: 55-64.
[15] Oros, G.I., Oros, Gh. (2012) Dominants and best dominants in fuzzy differential subordinations. Stud. Univ. Babeş-Bolyai Math., 57(2): 239-248.
[16] Salagean, G.S. (1983) Subclasses of univalent functions. Lecture Notes in Math., Springer Verlag, Berlin, 1013: 362-372.
[17] Srivastava H.M., Owa, S. (1989) Univalent Functions, Fractional Calculus and Their Applications, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
[18] Srivastava, H.M., Owa S. (Eds.) (1992) Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong.
[19] S. R. Swamy, Inclusion properties of certain subclasses of analytic functions, Int. Math. Forum, 7(36)(2012), 1751–1760.
[20] Wanas, A.K. (2019) New differential operator for holomorphic functions. Earthline Journal of Mathematical Sciences, 2(2): 527-537.
[21] Wanas, A.K., Majeed, A.H. (2018) Fuzzy differential subordination properties of analytic functions involving generalized differential operator. Sci. Int. (Lahore), 30(2): 297-302.
[22] Wanas, A.K., Murugusundaramoorthy, G. (2020) Differential sandwich results for Wanas operator of analytic functions. Mathematica Moravica, 24(1): 17-28.
[23] Zadeh, L.A. (1965) Fuzzy sets. Information and control, 8: 338-353.