Hacettepe Journal of Mathematics and Statistics
Volume 38 (2) (2009), 103 – 118

A CLASS OF FUNCTIONS AND SEPARATION AXIOMS WITH RESPECT TO AN OPERATION

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Received 05 : 05 : 2008 : Accepted 07 : 04 : 2009

Abstract

In this paper, a new kind of set called a γ-β-open set is introduced and investigated using the γ-operator due to Ogata (H. Ogata, *Operations on topological spaces and associated topology*, Math. Japonica **36** (1), 175–184, 1991). Such sets are used for studying new types of mappings, viz. γ-continuous, γ-β-continuous, γ-β-open, γ-β-closed, γ-β-generalized mappings, etc. A decomposition theorem for γ-continuous mappings, as well as a characterization of continuous mapping are obtained in terms of γ-β-continuous mappings. Finally, new separation axioms: γ-β-T_i ($i = 0, \frac{1}{2}, 1, 2$), γ-β-regularity and γ-β-normality are investigated along with the result that every topological space is γ-$\beta T_{\frac{1}{2}}$.

Keywords: β-open, γ-open, γ-β-open, γ-regular, γ-βg-closed, γ-continuous, γ-β-continuous, γ-βT_i ($i = 0, \frac{1}{2}, 1, 2$), γ-β-regular, γ-β-normal.

2000 AMS Classification: 54 A 05, 54 A 10, 54 D 10, 54 D 30, 54 C 10.

1. Introduction

In [9] Monsef *et al.* introduced β-open sets (semi-preopen sets [1]), Kasahara [5] defined an operation α on a topological space to introduce α-closed graphs, which were further investigated by Janković [4]. Following the same technique, Ogata [10] defined an operation γ on a topological space and introduced γ-open sets. Recently Krishnan *et al.* [6] used the operation γ for the introduction of γ-semi open sets.

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In this paper, we introduce γ-β-open sets which are shown to be independent of β-open sets but are a generalization of γ-open sets.

Section 3 deals with the introduction of γ-β-open sets, γ-β-generalized open sets and their various characterizations in terms of γ-β-interior, γ-β-closure, γ-β-frontier and γ-β-derived sets.

γ-continuity, γ-β-continuity, γ'-closedness, γ'-β-closedness, γ'-β-openness, γ'-βg-openness, γ'-βg-closedness, γ-γ'-irresoluteness and their interrelations are studied in section 4.

Section 5 is concerned with new separation axioms, viz. γ-β-T_i $(i = 0, \frac{1}{2}, 1, 2)$, γ-β-regular and γ-β-normal spaces. Such spaces, specially γ-β-regular spaces and γ-β-normal spaces, are characterized in various ways. Although γ-β-regularity and γ-β-normality are independent of regularity and normality, respectively, we have been able to achieve them in terms of regularity and normality respectively. Finally, the result that every topological space is γ-$\beta T_{\frac{1}{2}}$ is obtained.

Throughout this paper, unless otherwise stated, X or Y denotes a topological space without any separation axioms. We use $\mathrm{cl}\,(A)$ and $\mathrm{int}\,(A)$ to denote respectively the closure and interior of a subset A of X.

2. Preliminaries

A subset A of X is called β-open [9] if $A \subset \mathrm{cl}\,(\mathrm{int}\,(cl(A)))$. The complement of a β-open set is called a β-closed set. The family of all β-open sets of X is denoted by $\beta O(X)$. For a subset A of X, the union of all β-open sets of X contained in A is called the β-interior [9] (in short $\beta\mathrm{int}\,(A)$) of A, and the intersection of all β-open sets of X containing A is called the β-closure [9] (in short $\beta\mathrm{cl}\,(A)$) of A. A subset A of X is called generalized closed [7] (in short g-closed) if $\mathrm{cl}\,(A) \subset U$ whenever $A \subset U$ and U is open in X.

An operation γ [10] on a topology τ on X is a mapping $\gamma : \tau \to P(X)$, such that $V \subset V^{\gamma}$ for each $V \in \tau$, where $P(X)$ is the power set of X and V^{γ} denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [10] if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^{\gamma} \subset A$. Then, τ_{γ} denotes the set of all γ-open sets in X. Clearly $\tau_{\gamma} \subset \tau$. Complements of γ-open sets are called γ-closed. The γ-closure [10] of a subset A of X with an operation γ on τ is denoted by $\tau_{\gamma}\text{-}\mathrm{cl}\,(A)$ and is defined to be the intersection of all γ-closed sets containing A, and the τ_{γ}-interior [6] of A is denoted by $\tau_{\gamma}\text{-}int(A)$ and defined to be the union of all γ-open sets of X contained in A. A subset A of X with an operation γ on τ is called a γ-semiopen set [6] if and only if there exists a γ-open set U such that $U \subset A \subset \tau_{\gamma}\text{-}\mathrm{cl}\,(U)$. The family of all γ-semiopen sets in X is denoted by $\tau_{\gamma}\text{-}SO(X)$. A topological X with an operation γ on τ is said to be γ-regular [5] if for each $x \in X$ and open neighborhood V of x, there exists an open neighborhood U of x such that U^{γ} is contained in V. It is also to be noted that $\tau_{\gamma} = \tau$ if and only if X is a γ-regular space [10].

3. γ-β-open sets

3.1. Definition. Let (X, τ) be a topological space, γ an operation on τ and $A \subset X$. Then A is called a γ-β-open set if $A \subset \tau_{\gamma}\text{-}\mathrm{cl}\,(\tau_{\gamma}\text{-}\mathrm{int}\,(\tau_{\gamma}\text{-}\mathrm{cl}\,(A)))$.

γ-$\beta O(X)$ (resp. γ-$O(X)$ or τ_{γ}) denotes the collection of all γ-β-open (resp. γ-open) sets of (X, τ), and γ-$\beta O(X, x)$ (resp. γ-$O(X, x)$) is the collection of all γ-β-open (resp. γ-open) sets containing the point x of X.

A subset A of X is called γ-β-closed if and only if its complement is γ-β-open. Moreover, γ-$\beta C(X)$ (resp. $\gamma C(X)$) denotes the collection of all γ-β-closed (resp. γ-closed) sets of (X, τ).

It can be shown that a subset A of X is γ-β-closed if and only if $\tau_\gamma\text{-int}\,(\tau_\gamma\text{-cl}\,(\tau_\gamma\text{-int}\,(A))) \subset A$.

3.2. Remark.

- **(a)** The concepts of β-open and γ-β-open sets are independent, while in a γ-regular space [5] these concepts are equivalent.
- **(b)** A γ-open set is γ-β-open but the converse may not be true.
- **(c)** A γ-semiopen set [6] is γ-β-open and it is quite clear that the converse is true when A is γ-closed.

3.3. Example. **(a)** Let X={1,2,3}, $\tau = \{\emptyset, X, \{1,2\}, \{1\}, \{2\}\}$. Define an operation γ on τ by

$$A^\gamma = \begin{cases} \{1\} & \text{if } A = \{1\} \\ A \cup \{3\} & \text{if } A \neq \{1\} \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, X, \{1\}\}$. Then $\{2\}$ is β-open but not γ-β-open.

Again, if we define γ on τ by $A^\gamma = \text{cl}\,(A)$, then $\{3\}$ is γ-β-open but not β-open.

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X. Define an operation γ on τ by

$$A^\gamma = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{b\} & \text{if } A \neq \{a\} \end{cases}$$

Then, $\tau_\gamma = \tau$. But γ-$\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{b, d, a\}\}$.

3.4. Theorem. *An arbitrary union of γ-β-open sets is γ-β-open.*

Proof. Let $\{A_\alpha : \alpha \in \bigwedge\}$ be a family of γ-β-open sets. Then for each α, $A_\alpha \subset \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-int}\,(\tau_\gamma\text{-cl}\,(A_\alpha)))$ and so

$$\begin{aligned} \cup_\alpha A_\alpha &\subset \cup_\alpha \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-int}\,(\tau_\gamma\text{-cl}\,(A_\alpha))) \\ &\subset \tau_\gamma\text{-cl}\,(\cup_\alpha \tau_\gamma\text{-int}\,(\tau_\gamma\text{-cl}\,(A_\alpha))) \\ &\subset \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-int}\,(\cup_\alpha \tau_\gamma\text{-cl}\,(A_\alpha))) \\ &\subset \tau_\gamma\text{-cl}\,(\tau_\gamma\text{-int}\,(\tau_\gamma\text{-cl}\,(\cup_\alpha A_\alpha))). \end{aligned}$$

Thus, $\cup_\alpha A_\alpha$ is γ-β-open. □

3.5. Remark.

- **(a)** An arbitrary intersection of γ-β-closed sets is γ-β-closed.
- **(b)** The intersection of even two γ-β-open sets may not be γ-β-open.

3.6. Example. In Example 3.3(b), take $M = \{b, c\}$ and $N = \{a, c\}$. Then $M \cap N = \{c\}$, which is not a γ-β-open set.

3.7. Definition. Let A be a subset of a topological space (X, τ) and γ an operation on τ. The union of all γ-β-open sets contained in A is called the γ-β-interior of A and denoted by γ-$\beta\text{int}\,(A)$.

3.8. Definition. Let A be a subset of a topological space (X, τ) and γ an operation on τ. The intersection of all γ-β-closed sets containing A is called the γ-β-closure of A and denoted by γ-$\beta\text{cl}\,(A)$.

3.9. Definition. Let (X, τ) be a topological space with an operation γ on τ.

(a) The set denoted by γ-$\beta D(A)$ (resp. $\gamma D(A)$) and defined by

$$\{x : \text{for every } \gamma\text{-}\beta\text{-open (resp. } \gamma\text{-open) set } U \text{ containing } x, U \cap (A - \{x\}) \neq \emptyset\}$$

is called the γ-β-derived (resp. γ-derived set) set of A.

(b) The γ-β (resp. γ-frontier) frontier of A, denoted by γ-$\beta Fr(A)$ (resp. $\gamma Fr(A)$) is defined as γ-βcl $(A) \cap \gamma$-βcl $(X - A)$ (resp. τ_γ-cl $(A) \cap \tau_\gamma$-cl $(X - A)$).

We now state the following theorem without proof.

3.10. Theorem. *Let (X, τ) be a topological space and γ an operation on τ. For any subsets A, B of X we have the following:*

(i) *A is γ-β-open if and only if $A = \gamma$-βint (A).*
(ii) *A is γ-β-closed if and only if $A = \gamma$-βcl (A).*
(iii) *If $A \subset B$ then γ-βint $(A) \subset \gamma$-βint (B) and γ-βcl $(A) \subset \gamma$-βcl (B).*
(iv) *γ-βcl $(A) \cup \gamma$-βcl $(B) \subset \gamma$-βcl $(A \cup B)$.*
(v) *γ-βcl $(A \cap B) \subset \gamma$-$\beta$cl $(A) \cap \gamma$-βcl (B).*
(vi) *γ-βint $(A) \cup \gamma$-βint $(B) \subset \gamma$-βint $(A \cup B)$.*
(vii) *γ-βint $(A \cap B) \subset \gamma$-$\beta$int $(A) \cap \gamma$-βint (B).*
(viii) *γ-βint $(X - A) = X - \gamma$-βcl (A).*
(ix) *γ-βcl $(X - A) = X - \gamma$-βint (A).*
(x) *γ-βint $(A) = A - \gamma$-$\beta D(X - A)$.*
(xi) *γ-βcl $(A) = A \cup \gamma$-$\beta D(A)$,*
(xii) *τ_γ-int $(A) \subset \gamma$-βint (A).*
(xiii) *γ-βcl $(A) \subset \tau_\gamma$-cl (A).*

3.11. Remark. The reverse inclusions of (iii) to (vii) in Theorem 3.10 are not true, in general.

3.12. Example. In Example 3.3(b), let $A = \{a, b\}$, $B = \{c\}$, $C = \{a, c\}$, and $D = \{b, c\}$. Then γ-βcl $(A) = X$, γ-βcl $(B) = \{c\}$ but γ-βcl $(A \cap B) = \emptyset$. Also, γ-βcl $(C) = \{a, c\}$, γ-βcl $(D) = \{b, c\}$ but γ-βcl $(C \cup D) = X$.

Again, γ-βint $(A) = \{a, b\}$, γ-βint $(B) = \emptyset$ but γ-βint $(A \cup B) = \{a, b, c\}$ and γ-βint $(C) = \{a, c\}$, γ-βint $(D) = \{b, c\}$ but γ-βint $(C \cap D) = \emptyset$.

Also we note that γ-βint $(B) \subset \gamma$-βint (A) but $B \not\subset A$ and γ-βcl $(B) \subset \gamma$-βcl (A) but $B \not\subset A$.

3.13. Theorem. *Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then $x \in \gamma$-βcl (A) if and only if for every γ-β-open set U of X containing x, $A \cap U \neq \emptyset$.*

Proof. First suppose that $x \in \gamma$-βcl (A) and U is any γ-β-open set containing x such that $A \cap U = \emptyset$. Then $(X - U)$ is a γ-β-closed set containing A. Thus γ-βcl $(A) \subset (X - U)$. Then $x \notin \gamma$-βcl (A), which is a contradiction.

Conversely, suppose $x \notin \gamma$-βcl (A). Then there exists a γ-β-closed set V such that $A \subset V$ and $x \notin V$. Hence $(X - V)$ is a γ-β-open set containing x such that $A \cap (X - V) = \emptyset$. □

3.14. Theorem. *Let U, V be subsets of a topological space (X, τ) and γ an operation on τ. If $V \in \gamma$-$\beta O(X)$ is such that $U \subset V \subset \tau_\gamma$-cl (U), then U is a γ-β-open set.*

Proof. Since V is γ-β-open, $V \subset \tau_\gamma$-cl $(\tau_\gamma$-int $(\tau_\gamma$-cl $(V)))$. Again, τ_γ-cl $(V) \subset \tau_\gamma$-cl (U) for $V \subset \tau_\gamma$-cl (U). Then $U \subset V \subset \tau_\gamma$-cl $(\tau_\gamma$-int $(\tau_\gamma$-cl $(V))) \subset \tau_\gamma$-cl $(\tau_\gamma$- int $(\tau_\gamma$-cl $(U)))$. □

3.15. Definition. A subset A of a topological space (X,τ) with an operation γ on τ is called τ-γ-β-open (resp. γ-γ-β-open) if $\mathrm{int}\,(A) = \gamma$-$\beta\mathrm{int}\,(A)$ (resp. τ_γ-$\mathrm{int}\,(A) = \gamma$-$\beta\mathrm{int}\,(A)$).

3.16. Definition. A subset A of a topological space (X,τ) with an operation γ on τ is called a γ-β generalized closed set (γ-βg closed, for short) if γ-$\beta\mathrm{cl}\,(A) \subset U$ whenever $A \subset U$ and U is a γ-β-open set in X.

The complement of a γ-βg closed set is called a γ-βg open set. Clearly, A is γ-βg open if and only if $F \subset \gamma$-$\beta\mathrm{int}\,(A)$ whenever $F \subset A$ and F is γ-β closed in X.

3.17. Theorem. *A subset A of a topological space (X,τ) with an operation γ on τ, is γ-βg closed if and only if γ-$\beta\mathrm{cl}\,\{x\} \cap A \neq \emptyset$ for every $x \in \gamma$-$\beta\mathrm{cl}\,(A)$.*

Proof. Let A be a γ-βg closed set in X and suppose if possible there exists an $x \in \gamma$-$\beta\mathrm{cl}\,(A)$ such that γ-$\beta\mathrm{cl}\,\{x\} \cap A = \emptyset$. Therefore $A \subset X - \gamma$-$\beta\mathrm{cl}\,\{x\}$, and so γ-$\beta\mathrm{cl}\,(A) \subset X - \gamma$-$\beta\mathrm{cl}\,\{x\}$. Hence $x \notin \gamma$-$\beta\mathrm{cl}\,(A)$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any γ-β open set containing A. Let $x \in \gamma$-$\beta\mathrm{cl}\,(A)$. Then γ-$\beta\mathrm{cl}\,\{x\} \cap A \neq \emptyset$, so there exists $z \in \gamma$-$\beta\mathrm{cl}\,\{x\} \cap A$ and so $z \in A \subset U$. Thus by the Theorem 3.13, $\{x\} \cap U \neq \emptyset$. Hence $x \in U$, which implies γ-$\beta\mathrm{cl}\,(A) \subset U$. □

3.18. Theorem. *Let A be a γ-βg closed set in a topological space (X,τ) with operation γ on τ. Then γ-$\beta\mathrm{cl}\,(A) - A$ does not contain any nonempty γ-β closed set.*

Proof. If possible, let F be a nonempty γ-β closed set such that $F \subset \gamma$-$\beta\mathrm{cl}\,(A) - A$. Let $x \in F$ and so $x \in \gamma$-$\beta\mathrm{cl}\,(A)$. Again we observe that

$$F \cap A = \gamma\text{-}\beta\mathrm{cl}\,(F) \cap A \supset \gamma\text{-}\beta\mathrm{cl}\,\{x\} \cap A \neq \emptyset,$$

which gives $F \cap A \neq \emptyset$, a contradiction. □

3.19. Theorem. *In a topological space (X,τ) with an operation γ on τ, either $\{x\}$ is γ-β closed or $X - \{x\}$ is γ-βg closed.*

Proof. If $\{x\}$ is not γ-β closed, then $X - \{x\}$ is not γ-β-open. Then X is the only γ-β-open set such that $X - \{x\} \subset X$. Hence $X - \{x\}$ is a γ-βg closed set. □

4. γ-β-functions

4.1. Definition. Let (X,τ) and (Y,τ') be two topological spaces and γ an operation on τ. Then a function $f : (X,\tau) \to (Y,\tau')$ is said to be γ-β-continuous (resp. γ-continuous) at x if for each open set V containing $f(x)$, there exists an $U \in \gamma$-$\beta O(X,x)$ (resp. γ-$O(X,x)$) such that $f(U) \subset V$.

If f is γ-β-continuous (resp. γ-continuous) at each point x of X, then f is called γ-β-continuous (resp. γ-continuous) on X.

4.2. Theorem. *Let (X,τ) be a topological space with an operation γ on τ. For a function $f : (X,\tau) \to (Y,\tau')$, the following are equivalent:*

(a) *f is γ-β-continuous (resp. γ-continuous).*
(b) *For each open subset V of Y, $f^{-1}(V) \in \gamma$-$\beta O(X)$ (resp. γ-$O(X)$).*
(c) *For each closed subset V of Y, $f^{-1}(V) \in \gamma$-$\beta C(X)$ (resp. $\gamma C(X)$).*
(d) *For any subset V of Y, γ-$\beta\mathrm{cl}\,(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}\,(V))$ (resp. τ_γ-$\mathrm{cl}\,(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}\,(V))$).*
(e) *For any subset U of X, $f(\gamma$-$\beta\mathrm{cl}\,(U)) \subset \mathrm{cl}\,(f(U))$ (resp. $f(\tau_\gamma$-$\mathrm{cl}\,U) \subset \mathrm{cl}\,(f(U))$).*
(f) *For any subset U of X, $f(\gamma$-$\beta D(U)) \subset \mathrm{cl}\,(f(U))$ (resp. $f(\gamma D(U)) \subset \mathrm{cl}\,(f(U))$).*

(g) *For any subset V of Y, $f^{-1}(\text{int}\,(V)) \subset \gamma$-$\beta\text{int}\,(f^{-1}(V))$ (resp. $f^{-1}(\text{int}\,(V)) \subset \tau_\gamma$-$\text{int}\,(f^{-1}(V))$).*
(h) *For each subset V of Y, γ-$\beta Fr(f^{-1}(V)) \subset f^{-1}(Fr(V))$ (resp. γ-$Fr(f^{-1}(V)) \subset f^{-1}(Fr(V))$).*

Proof. We prove the theorem for γ-β-continuous functions only. The proof for γ-continuity is quite similar.

(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) Obvious.

(c) $\Longrightarrow$ (d) Let V be any subset of Y. By (c), we have $f^{-1}(cl(V))$ is a γ-β-closed set containing $f^{-1}(V)$ and hence γ-$\beta cl(f^{-1}(V)) \subset f^{-1}(cl(V))$.

(d) $\Longrightarrow$ (e) Obvious.

(e) $\Longrightarrow$ (c) Let V be a closed set in Y. Then by (e), we obtain $f(\gamma\text{-}\beta\text{cl}\,(f^{-1}(V))) \subset \text{cl}\,(f(f^{-1}(V))) \subset \text{cl}\,(V) = V$, which implies γ-$\beta\text{cl}\,(f^{-1}(V)) \subset f^{-1}(V)$. Thus $f^{-1}(V) \in \gamma$-$\beta C(X)$.

(c) $\Longrightarrow$ (f) Let U be any subset of X. Since (c) implies (e), then by the fact that γ-$\beta\text{cl}\,(U) = U \cup \gamma$-$\beta D(U)$, we get $f(\gamma\text{-}\beta D(U)) \subset f(\gamma\text{-}\beta\text{cl}\,(U)) \subset \text{cl}\,(f(U))$.

(f) $\Longrightarrow$ (c) Let V be any closed set in Y. By (f), we obtain $f(\gamma\text{-}\beta Df^{-1}(V)) \subset \text{cl}\,(f(f^{-1}(V))) \subset \text{cl}\,(V) = V$. This implies γ-$\beta Df^{-1}(V) \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is γ-β-closed in X.

(b) $\Longleftrightarrow$ (g) Let V be any open in Y. Then by (g), we get $f^{-1}(V) = f^{-1}(\text{int}\,(V)) \subset \gamma$ -$\beta\text{int}\,(f^{-1}(V))$. Thus $f^{-1}(V) \in \gamma$-$\beta O(X)$.

On the other hand by (b), we get $f^{-1}(\text{int}\,(V)) \in \gamma$-$\beta O(X)$, for any subset V of Y. Therefore, we obtain $f^{-1}(\text{int}\,(V)) = \gamma$-$\beta\text{int}\, f^{-1}(int(V)) \subset \gamma$-$\beta\text{int}\,(f^{-1}(V))$.

(b) $\Longrightarrow$ (h) Let V be any subset of Y. Since (a) implies (d), we have

$$\begin{aligned} f^{-1}(Fr(V)) &= f^{-1}(\text{cl}\,(V) - \text{int}\,(V)) \\ &= f^{-1}(\text{cl}\,(V)) - f^{-1}(\text{int}\,(V)) \supset \gamma\text{-}\beta\text{cl}\,(f^{-1}(V)) - f^{-1}(\text{int}\,(V)) \\ &= \gamma\text{-}\beta\text{cl}\,(f^{-1}(V)) - \gamma\text{-}\beta\text{int}\,(f^{-1}(intV)) \\ &\supset \gamma\text{-}\beta\text{cl}\,(f^{-1}(V)) - \gamma\text{-}\beta\text{int}\,(f^{-1}(V)) \\ &= \gamma\text{-}\beta Frf^{-1}(V), \end{aligned}$$

and hence $f^{-1}(Fr(V)) \supset \gamma$-$\beta Fr(f^{-1}(V))$.

(h) $\Longrightarrow$ (b) Let U be open in Y and $V = Y - U$. Then by (h), we obtain γ-$\beta Fr(f^{-1}(V)) \subset f^{-1}(Fr(V)) \subset f^{-1}(cl(V)) = f^{-1}(V)$ and hence

$$\gamma\text{-}\beta\text{cl}\,(f^{-1}(V)) = \gamma\text{-}\beta\text{int}\,(f^{-1}(V)) \cup \gamma\text{-}\beta Fr(f^{-1}(V)) \subset f^{-1}(V).$$

Thus $f^{-1}(V)$ is γ-β-closed and hence $f^{-1}(U)$ is γ-β-open in X. □

4.3. Remark. Every γ-continuous function is γ-β-continuous, but the converse is not true.

4.4. Example. Let X be a topological space and γ an operation as in Example 3.3(b). Suppose that $Y = \{1, 2, 3\}$, $\tau' = \{\emptyset, Y, \{1, 2\}, \{1\}, \{2\}\}$. Define a map $f : (X, \tau) \to (Y, \tau')$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{d, b\} \\ 2 & \text{if } x \in \{a\} \\ 3 & \text{if } x \in \{c\} \end{cases}$$

Then the mapping $f : (X,\tau) \to (Y,\tau')$ is γ-β-continuous but not γ-continuous, since $f^{-1}(1) = \{b,d\} \notin \tau_\gamma$.

4.5. Remark. Let γ and γ' be operations on the topological spaces (X,τ) and (Y,τ') respectively. If the functions $f : (X,\tau) \to (Y,\tau')$ and $g : (Y,\tau') \to (Z,\tau'')$ are γ-β-continuous and continuous, respectively, then $g \circ f$ is γ-β-continuous. But the composition of a γ-β-continuous and a γ'-β-continuous function may not be γ-β-continuous.

4.6. Example. Let us consider the topological spaces (X,τ) and (Y,τ') as in Example 4.4. Also let $Z = \{p,q,r\}$, $\tau'' = \{\emptyset, Z, \{p\}, \{q,r\}\}$.

We take γ on τ as in Example 4.4, and define γ' on τ' by

$$B^{\gamma'} = \begin{cases} \{1\} & \text{if } B = \{1\}, \\ \{2\} \cup B & \text{if } B \neq \{1\}. \end{cases}$$

Now we define $g : X \to Y$ and $h : Y \to Z$ as follows:

$$g(x) = \begin{cases} 1 & \text{if } x \in \{a,b\}, \\ 2 & \text{if } x = c, \\ 3 & \text{if } x = d \end{cases}$$

and

$$h(y) = \begin{cases} p & \text{if } y = 1, \\ q & \text{if } y = 3, \\ r & \text{if } y = 2. \end{cases}$$

Then g and h are γ-β-continuous and γ'-β-continuous, respectively, but $h \circ g$ is not γ-β-continuous.

4.7. Theorem. *Let (X,τ) be a topological space with an operation γ on τ and let $f : (X,\tau) \to (Y,\tau')$ be a function. Then*

$$X - \gamma\text{-}\beta C(f) = \bigcup\{\gamma\text{-}\beta Fr(f^{-1}(V)) : V \in \tau',\ f(x) \in V,\ x \in X\},$$

where γ-$\beta C(f)$ denotes the set of points at which f is γ-β-continuous.

Proof. Let $x \in X - \gamma\text{-}\beta C(f)$. Then there exists $V \in \tau'$ containing $f(x)$ such that $f(U) \not\subset V$, for every γ-β-open set U containing x. Hence $U \cap [X - f^{-1}(V)] \neq \emptyset$ for every γ-β-open set U containing x. Therefore, by Theorem 3.13, $x \in \gamma\text{-}\beta\mathrm{cl}\,(X - f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \gamma\text{-}\beta\mathrm{cl}\,(X - f^{-1}(V)) \subset \gamma\text{-}\beta Fr(f^{-1}(V))$. So,

$$X - \gamma\text{-}\beta C(f) \subset \bigcup\{\gamma\text{-}\beta Fr(f^{-1}(V)) : V \in \tau',\ f(x) \in V,\ x \in X\}.$$

Conversely, let $x \notin X - \gamma\text{-}\beta C(f)$. Then for each $V \in \tau'$ containing $f(x)$, $f^{-1}(V)$ is a γ-β-open set containing x. Thus $x \in \gamma\text{-}\beta\mathrm{int}\,(f^{-1}(V))$ and hence $x \notin \gamma\text{-}\beta Fr(f^{-1}(V))$, for every $V \in \tau'$ containing $f(x)$. Therefore,

$$X - \gamma\text{-}\beta C(f) \supset \bigcup\{\gamma\text{-}\beta Fr(f^{-1}(V) : V \in \tau',\ f(x) \in V,\ x \in X\}.$$

□

4.8. Theorem. *Let (X,τ) be a topological space with an operation γ on τ and let $f : (X,\tau) \to (Y,\tau')$ be a function. Then $X - \gamma\text{-}C(f) = \bigcup\{\gamma\text{-}Fr(f^{-1}(V)) : V \in \tau',\ f(x) \in V,\ x \in X\}$, where γ-$C(f)$ denotes the set of points at which f is γ-continuous.*

4.9. Definition. Let (X,τ) be a topological space with an operation γ on τ. A function $f : (X,\tau) \to (Y,\tau')$ is called τ-γ-β continuous (resp. γ-γ-β continuous) if for each open set V in Y, $f^{-1}(V)$ is τ-γ-β open (resp. γ-γ-β open) in X.

4.10. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be a mapping and γ an operation on τ. Then the following are equivalent:*

(i) *f is γ-continuous.*
(ii) *f is γ-β continuous and γ-γ-β continuous.*

Proof. (i) $\Longrightarrow$ (ii) Let f be γ-continuous. Then f is γ-β continuous. Now, let G be any open set in Y, then $f^{-1}(G)$ is γ-open in X. Then

$$\tau_\gamma\text{-int}\,(f^{-1}(G)) = f^{-1}(G) = \gamma\text{-}\beta\text{int}\, f^{-1}(G).$$

Thus, $f^{-1}(G)$ is γ-γ-β open in X. Therefore f is γ-γ-β continuous.

(ii) $\Longrightarrow$ (i) Let f be γ-β continuous and γ-γ-β continuous. Then for any open set G in Y, $f^{-1}(G)$ is both γ-β open and γ-γ-β open in X. So

$$f^{-1}(G) = \gamma\text{-}\beta\text{int}\, f^{-1}(G) = \tau_\gamma\text{-int}\,(f^{-1}(G)).$$

Thus $f^{-1}(G) \in \tau_\gamma$ and hence f is γ-continuous. □

4.11. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be τ-γ-β continuous, where γ is an operation on τ. Then f is continuous if and only if f is γ-β continuous.*

Proof. Let $V \in \tau'$. Since f is continuous as well as τ-γ-β continuous, $f^{-1}(V)$ is open as well as τ-γ-β open in X and hence $f^{-1}(V) = \text{int}\,(f^{-1}(V)) = \gamma\text{-}\beta\text{int}\,(f^{-1}(V)) \in \gamma\text{-}\beta O(X)$. Therefore, f is γ-β continuous.

Conversely, let $V \in \tau'$. Then $f^{-1}(V)$ is γ-β open and τ-γ-β open. So $f^{-1}(V) = \gamma\text{-}\beta\text{int}\,(f^{-1}(V)) = \text{int}\,(f^{-1}(V))$. Hence $f^{-1}(V)$ is open in X. Therefore f is continuous. □

4.12. Definition. A function $f : (X, \tau) \to (Y, \tau')$, where γ and γ' are operations on τ and τ', respectively, is called γ'-closed (resp. γ'-β-closed, γ'-βg-closed, γ'-βgγ-β-closed) if for every closed (resp. closed, closed, γ-β-closed) set F in X, $f(F)$ is γ'-closed (resp.γ'-β-closed, γ'-βg-closed, γ'-βg-closed) in Y.

4.13. Theorem. *A surjective function $f : (X, \tau) \to (Y, \tau')$, where γ' is an operation on τ', is γ'-βg-closed (resp. γ'-β-closed) if and only if for each subset A of Y and each open set V of X containing $f^{-1}(A)$, there exists a γ'-βg-open (resp. γ'-β-open) set W of Y such that $A \subset W$ and $f^{-1}(W) \subset V$.*

Proof. First we suppose that $f : (X, \tau) \to (Y, \tau')$ is γ'-βg-closed (resp. γ'-β-closed), $A \subset Y$ and V is open in X such that $f^{-1}(A) \subset V$. Now we put $X - V = G$, then $f(G)$ is a γ'- βg-closed (resp. γ'-β-closed) set in Y. If $W = Y - f(G)$ then W is a γ'-βg-open (resp. γ'-β-open) set in Y, $A \subset W$ and $f^{-1}(W) \subset V$.

Conversely, let B be any closed set in X. Then $A = Y - f(B)$ is a subset of Y and $f^{-1}(A) \subset X - B$, where $(X - B)$ is open in X. Therefore, by hypothesis there exists a γ'-βg-open (resp. γ'-β-open) set W of Y such that $A = Y - f(B) \subset W$ and $f^{-1}(W) \subset X - B$. Now, $f^{-1}(W) \subset X - B$ gives $W \subset f(X - B) \subset Y - f(B)$. Therefore, $Y - f(B) = W$ and hence $f(B)$ is a γ'-βg-closed (resp. γ'-β-closed) set. □

4.14. Theorem. *A surjective function $f : (X, \tau) \to (Y, \tau')$, where γ and γ' are operations on τ and τ', respectively, is γ'-$\beta g\gamma$-β-closed if and only if for each subset B of Y and each γ-β-open set U of X containing $f^{-1}(B)$, there exists a γ'-βg-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

4.15. Definition. A function $f : (X, \tau) \to (Y, \tau')$, where γ is an operation on τ, is called γ-β-anti-closed (resp. γ-β-anti-open) if the image of each γ-β-closed (resp. γ-β-open) set in X is closed (resp. open) in Y.

4.16. Theorem. *A surjective function $f : (X, \tau) \to (Y, \tau')$, where γ is an operation on τ, is γ-β-anti-closed if for each subset A of Y and each γ-β-open set V containing $f^{-1}(A)$, there exists an open set W such that $A \subset W$ and $f^{-1}(W) \subset V$.*

4.17. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be a function and γ' an operation on τ'. Then the following conditions are equivalent:*

(i) *f is γ'-β-closed.*
(ii) *γ'-$\beta\mathrm{cl}\,(f(U)) \subset f(\mathrm{cl}\,(U))$, for each subset U of X.*
(iii) *γ'-$\beta D(f(U)) \subset f(\mathrm{cl}\,(U))$, for each subset U of X.*

Proof. (i) $\Longrightarrow$ (ii) Here, for any subset U of X, $f(\mathrm{cl}\,(U))$ is a γ'-β-closed set in Y and $f(U) \subset f(\mathrm{cl}\,(U))$, hence γ'-$\beta\mathrm{cl}\,(f(U)) \subset f(\mathrm{cl}\,(U))$.

(ii) $\Longrightarrow$ (iii) For each $U \subset X$, we have γ'-$\beta D(f(U)) \subset \gamma'$-$\beta\mathrm{cl}\,(f(U)) \subset f(\mathrm{cl}\,(U))$.

(iii) $\Longrightarrow$ (i) Let V be any closed set in X. Then γ'-$\beta D(f(V)) \subset f(\mathrm{cl}\,(V)) = f(V)$. Hence $f(V)$ is γ'-β-closed. □

4.18. Remark. If $f : (X, \tau) \to (Y, \tau')$ is a bijection and γ' an operation on τ', then f is γ'-β-closed if and only if f^{-1} is γ'-β-continuous.

4.19. Definition. A function $f : (X, \tau) \to (Y, \tau')$, where γ' is an operation on τ', is said to be γ'-β-open if for each open set U in X, $f(U)$ is γ'-β-open in Y.

4.20. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be a mapping and γ' an operation on τ'. Then the following conditions are equivalent:*

(i) *f if γ'-β-open.*
(ii) *$f(\mathrm{int}\,(V)) \subset \gamma'$-$\beta\mathrm{int}\,(f(V))$.*

4.21. Remark. If f is a bijection, then f is γ'-β-open if and only if f^{-1} is γ'-β-continuous.

4.22. Remark. The concepts of γ'-β-closedness and γ'-β-openness are independent.

4.23. Example. Let (X, τ) be the topological space with operation γ on τ as in Example 3.3(b), and let (Y, τ') be a topological space where $Y = \{1, 2, 3\}$, $\tau' = \{\emptyset, Y, \{1, 2\}, \{1\}, \{2\}\}$. Let $f : (Y, \tau') \to (X, \tau)$ be defined as follows:

$$f(y) = \begin{cases} d & \text{if } y = 1, \\ c & \text{if } y = 2, \\ a & \text{if } y = 3. \end{cases}$$

Then f is γ-β-closed but not γ-β-open.

Let (Y, τ') be the topological space with γ' an operation on τ' as in Example 4.6. We define a function $g : (X, \tau) \to (Y, \tau')$ as follows:

$$g(x) = \begin{cases} 1 & \text{if } x \in \{b, c\} \\ 2 & \text{if } x \in \{a, d\} \end{cases}$$

Then g is γ'-β-open but not γ'-β-closed.

4.24. Definition. Let (X, τ), (Y, τ') be two topological spaces and γ, γ' operations on τ, τ', respectively. A mapping $f : (X, \tau) \to (Y, \tau')$ is called (γ, γ')-β irresolute at x if and only if for each γ'-β-open set V in Y containing $f(x)$, there exists a γ-β-open set U in X containing x such that $f(U) \subset V$.

If f is (γ, γ')-β irresolute at each $x \in X$ then f is called (γ, γ')-β irresolute on X.

4.25. Theorem. *Let (X,τ), (Y,τ') be topological spaces and γ, γ' operations on τ, τ', respectively. If $f:(X,\tau)\to(Y,\tau')$ is (γ,γ')-β irresolute and γ'-$\beta g\gamma$-β-closed, and A is γ-βg-closed in X, then $f(A)$ is γ'-βg-closed in Y.*

Proof. Suppose A is a γ-βg-closed set in X and that U is a γ'-β-open set in Y such that $f(A)\subset U$. Then $A\subset f^{-1}(U)$. Since f is (γ,γ')-β irresolute, $f^{-1}(U)$ is γ-β-open set in X.

Again A is a γ-βg-closed set, therefore γ-$\beta\mathrm{cl}\,(A)\subset f^{-1}(U)$ and hence $f(\gamma$-$\beta\mathrm{cl}\,(A))\subset U$. Since f is a γ'-βgγ-β-closed map, $f(\gamma$-$\beta cl(A))$ is a γ'-βg-closed set in Y. Therefore γ'-$\beta\mathrm{cl}\,(f(\gamma$-$\beta\mathrm{cl}\,(A))\subset U$, which implies γ'-$\beta\mathrm{cl}\,(f(A))\subset U$. □

4.26. Theorem. *Let $f:(X,\tau)\to(Y,\tau')$ be a mapping and γ, γ' operations on τ, τ', respectively. Then the following are equivalent:*

(i) *f is (γ,γ')-β irresolute.*
(ii) *The inverse image of each γ'-β-open set in Y is a γ-β-open set in X.*
(iii) *The inverse image of each γ'-β-closed set in Y is a γ-β-closed set in X.*
(iv) *γ-$\beta\mathrm{cl}\,(f^{-1}(V))\subset f^{-1}(\gamma'$-$\beta$-$\mathrm{cl}\,(V))$, for all $V\subset Y$.*
(v) *$f(\gamma$-$\beta\mathrm{cl}\,(U))\subset\gamma'$-$\beta\mathrm{cl}\,(f(U))$, for all $U\subset X$.*
(vi) *γ-$\beta Fr(f^{-1}(V))\subset f^{-1}(\gamma'$-$\beta Fr(V))$, for all $V\subset Y$.*
(vii) *$f(\gamma$-$\beta D(U))\subset\gamma'$-$\beta$-$\mathrm{cl}\,(f(U))$, for all $U\subset X$.*
(viii) *$f^{-1}(\gamma'$-$\beta\mathrm{int}\,(V))\subset\gamma$-$\beta\mathrm{int}\,(f^{-1}(V))$, for all $V\subset Y$.*

4.27. Theorem. *Let (X,τ), (Y,τ') be topological spaces and γ, γ' operations on τ, τ', respectively. Also let $f:(X,\tau)\to(Y,\tau')$ be a mapping. Then the set of points at which f is not (γ,γ')-β irresolute is*

$$\bigcup\{\gamma'\text{-}\beta Fr(V) : V \text{ is } \gamma'\text{-}\beta \text{ open set in } Y \text{ containing } f(x)\}.$$

5. γ-β-separation axioms

5.1. Definition. A topological space (X,τ) with an operation γ on τ is called γ-βT_0 if and only if for each pair of distinct points x, y in X, there exists an γ-β-open set U such that either $x\in U$ and $y\notin U$ or $x\notin U$ and $y\in U$.

5.2. Definition. A topological space (X,τ) with an operation γ on τ is called γ-βT_1 if and only if for each pair of distinct points x, y in X, there exists two γ-β-open sets U and V such that $x\in U$ but $y\notin U$ and $y\in V$ and $x\notin V$.

5.3. Definition. A topological space (X,τ) with an operation γ on τ is called γ-βT_2 if and only if for each pair of distinct points x, y in X, there exist two disjoint γ-β-open sets U and V containing x and y respectively.

5.4. Definition. A topological space (X,τ) with an operation γ on τ is called γ-$\beta T_{\frac{1}{2}}$ if every γ-βg-closed set is γ-β-closed.

5.5. Theorem. *A topological space (X,τ) with an operation γ on τ is γ-βT_0 if and only if for every pair of distinct points x, y of X, γ-$\beta\mathrm{cl}\,\{x\}\neq\gamma$-$\beta\mathrm{cl}\,\{y\}$.*

5.6. Theorem. *A topological space (X,τ) with an operation γ on τ is γ-βT_1 if and only if every singleton $\{x\}$ is γ-β-closed.*

5.7. Theorem. *The following are equivalent for a topological space (X,τ) with an operation γ on τ:*

(i) *X is γ-βT_2.*
(ii) *Let $x\in X$. For each $y\neq x$, there exists a γ-β-open set U containing x such that $y\notin\gamma$-$\beta\mathrm{cl}\,(U)$.*

(iii) *For each* $x \in X$, $\bigcap\{\gamma\text{-}\beta\mathrm{cl}\,(U) : U \in \gamma\text{-}\beta O(X, x)\} = \{x\}$.

Proof. (i) $\Longrightarrow$ (ii) Since X is γ-βT_2, there exist disjoint γ-β-open sets U and W containing x and y respectively. So $U \subset X - W$. Therefore, γ-$\beta\mathrm{cl}\,(U) \subset X - W$. So $y \notin \gamma$-$\beta\mathrm{cl}\,(U)$.

(ii) $\Longrightarrow$ (iii) If possible for $y \neq x$, let $y \in \gamma$-$\beta\mathrm{cl}\,(U)$ for every γ-β-open set U containing x, which then contradicts (ii).

(iii) $\Longrightarrow$ (i) Let $x, y \in X$ and $x \neq y$. Then there exists a γ-β-open set U containing x such that $y \notin \gamma$-$\beta\mathrm{cl}\,(U)$. Let $V = X - \gamma$-$\beta\mathrm{cl}\,(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \emptyset$. □

5.8. Theorem. *The following are equivalent for a topological space* (X, τ) *with an operation* γ *on* τ,

(i) (X, τ) *is* γ-$\beta T_{\frac{1}{2}}$.
(ii) *Each singleton* $\{x\}$ *of* X *is either* γ-β-*closed or* γ-β-*open.*

Proof. (i) $\Longrightarrow$ (ii) Suppose $\{x\}$ is not γ-β-closed. Then by Theorem 3.19, $X - \{x\}$ is γ-βg closed. Now since (X, τ) is γ-$\beta T_{\frac{1}{2}}$, $X - \{x\}$ is γ-β-closed i.e. $\{x\}$ is γ-β-open.

(ii) $\Longrightarrow$ (i) Let A be any γ-βg closed set in (X, τ) and $x \in \gamma$-$\beta\mathrm{cl}\,(A)$. By (ii) we have $\{x\}$ is γ-β-closed or γ-β-open. If $\{x\}$ is γ-β-closed then $x \notin A$ will imply $x \in \gamma$-$\beta\mathrm{cl}\,(A) - A$, which is not possible by Theorem 3.18. Hence $x \in A$. Therefore, γ-$\beta\mathrm{cl}\,(A) = A$ i.e. A is γ-β-closed. So, (X, τ) is γ-$\beta T_{\frac{1}{2}}$. On the other hand if $\{x\}$ is γ-β-open then as $x \in \gamma$-$\beta\mathrm{cl}\,(A)$, $\{x\} \cap A \neq \emptyset$. Hence $x \in A$. So A is γ-β-closed. □

5.9. Theorem. *Every topological space* (X, τ) *with an operation* γ *on* τ *is* γ-$\beta T_{\frac{1}{2}}$.

Proof. Let $x \in X$. To prove (X, τ) is γ-$\beta T_{\frac{1}{2}}$, it is sufficient to show that $\{x\}$ is γ-β-closed or γ-β-open (by Theorem 5.8). Now if $\{x\}$ is γ-open then it is obviously γ-β-open. If $\{x\}$ is not γ-open then τ_γ-$\mathrm{int}\,(\{x\}) = \emptyset$ and hence τ_γ-$\mathrm{int}\,(\tau_\gamma$-$\mathrm{cl}\,(\tau_\gamma$-$\mathrm{int}\,(\{x\}))) = \emptyset \subset \{x\}$. Therefore, $\{x\}$ is γ-β-closed. □

5.10. Remark.

(a) Because of Theorem 5.9 above, the concepts γ'-β-closedness and γ'-βg-closedness are identical, and so also are γ-$\beta T_{\frac{1}{2}}$ and γ-βT_0.
(b) γ-$\beta T_2 \implies \gamma$-$\beta T_1 \implies \gamma$-$\beta T_{\frac{1}{2}}$. But the reverse implications are not true in general.

5.11. Example. Let $X = \{a, b, c\}$ and let τ be the discrete topology on X. Then $\beta O(X) = \tau$. Define an operation γ on τ by

$$A^\gamma = \begin{cases} \{a\} & \text{if } A = \{a\}, \\ A \cup \{c\} & \text{if } A \neq \{a\}. \end{cases}$$

Then X is γ-$\beta T_{\frac{1}{2}}$ but not γ-βT_1.

5.12. Example. Let $X = \{a, b, c\}$ and let τ be the discrete topology on X. Define an operation γ on τ by

$$A^\gamma = \begin{cases} A \cup \{b\} & \text{if } A = \{a\}, \\ A \cup \{c\} & \text{if } A = \{b\}, \\ A \cup \{a\} & \text{if } A = \{c\}, \\ A & \text{if } A \neq \{a\}, \{b\}, \{c\}. \end{cases}$$

Then X is γ-βT_1 but not γ-βT_2.

5.13. Definition. A topological space (X,τ) with an operation γ on τ is called γ-β-regular if for each γ-β-closed set F of X not containing x, there exist disjoint γ-β-open sets U and V such that $x \in U$ and $F \subset V$.

Tahiliani [11] characterized β-regular spaces. In a similar fashion give several characterizations of γ-β-regular spaces.

5.14. Theorem. *The following are equivalent for a topological space (X,τ) with an operation γ on τ:*

(a) *X is γ-β-regular.*
(b) *For each $x \in X$ and each $U \in \gamma$-$\beta O(X,x)$, there exists a $V \in \gamma$-$\beta O(X,x)$ such that $x \in V \subset \gamma$-$\beta\mathrm{cl}\,(V) \subset U$.*
(c) *For each γ-β-closed set F of X, $\bigcap\{\gamma$-$\beta\mathrm{cl}\,(V) : F \subset V,\ V \in \gamma$-$\beta O(X)\} = F$.*
(d) *For each A subset of X and each $U \in \gamma$-$\beta O(X)$ with $A \cap U \neq \emptyset$, there exists a $V \in \gamma$-$\beta O(X)$ such that $A \cap V \neq \emptyset$ and γ-$\beta\mathrm{cl}\,(V) \subset U$.*
(e) *For each nonempty subset A of X and each γ-β-closed subset F of X with $A \cap F = \emptyset$, there exists $V, W \in \gamma$-$\beta O(X)$ such that $A \cap V \neq \emptyset$, $F \subset W$ and $W \cap V = \emptyset$.*
(f) *For each γ-β-closed set F and $x \notin F$, there exists $U \in \gamma$-$\beta O(X)$ and a γ-βg-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.*
(g) *For each $A \subset X$ and each γ-β-closed set F with $A \cap F = \emptyset$, there exists $U \in \gamma$-$\beta O(X)$ and a γ-βg-open set V such that $A \cap U \neq \emptyset$, $F \subset V$ and $U \cap V = \emptyset$.*
(h) *For each γ-β-closed set F of X, $F = \bigcap\{\gamma$-$\beta\mathrm{cl}\,(V) : F \subset V,\ V$ is γ-βg-open$\}$.*

Proof. (a) $\Longrightarrow$ (b) Let $x \notin X - U$, where $U \in \gamma$-$\beta O(X,x)$. Then there exists $G, V \in \gamma$-$\beta O(X)$ such that $(X - U) \subset G$, $x \in V$ and $G \cap V = \emptyset$. Therefore $V \subset (X - G)$ and so $x \in V \subset \gamma$-$\beta\mathrm{cl}\,(V) \subset (X - G) \subset U$.

(b) $\Longrightarrow$ (c) Let $X - F \in \gamma$-$\beta O(X,x)$. Then by (b) there exists an $U \in \gamma$-$\beta O(X,x)$ such that $x \in U \subset \gamma$-$\beta\mathrm{cl}\,(U) \subset (X - F)$. So, $F \subset X - \gamma$-$\beta\mathrm{cl}\,(U) = V$, $V \in \gamma$-$\beta O(X)$ and $V \cap U = \emptyset$. Then by Theorem 3.13, $x \notin \gamma$-$\beta\mathrm{cl}\,(V)$. Thus

$$F \supset \bigcap\{\gamma\text{-}\beta\mathrm{cl}\,(V) : F \subset V,\ V \in \gamma\text{-}\beta O(X)\}.$$

(c) $\Longrightarrow$ (d) Let $U \in \gamma$-$\beta O(X)$ with $x \in U \cap A$. Then $x \notin (X - U)$ and hence by (c) there exists a γ-β-open set W such that $X - U \subset W$ and $x \notin \gamma$-$\beta\mathrm{cl}\,(W)$. We put $V = X - \gamma$-$\beta\mathrm{cl}\,(W)$, which is a γ-β-open set containing x and hence $V \cap A \neq \emptyset$. Now $V \subset (X - W)$ and so γ-$\beta\mathrm{cl}\,(V) \subset (X - W) \subset U$.

(d) $\Longrightarrow$ (e) Let F be a set as in the hypothesis of (e). Then $(X - F)$ is γ-β-open and $(X - F) \cap A \neq \emptyset$. Then there exists $V \in \gamma$-$\beta O(X)$ such that $A \cap V \neq \emptyset$ and γ-$\beta\mathrm{cl}\,(V) \subset (X - F)$. If we put $W = X - \gamma$-$\beta\mathrm{cl}\,(V)$, then $F \subset W$ and $W \cap V = \emptyset$.

(e) $\Longrightarrow$ (a) Let F be a γ-β-closed set not containing x. Then by (e), there exist $W, V \in \gamma$-$\beta O(X)$ such that $F \subset W$ and $x \in V$ and $W \cap V = \emptyset$.

(a) $\Longrightarrow$ (f) Obvious.

(f) $\Longrightarrow$ (g) For $a \in A$, $a \notin F$ and hence by (f) there exists $U \in \gamma$-$\beta O(X)$ and a γ-βg-open set V such that $a \in U$, $F \subset V$ and $U \cap V = \emptyset$. So, $A \cap U \neq \emptyset$.

(g) $\Longrightarrow$ (a) Let $x \notin F$, where F is γ-β-closed. Since $\{x\} \cap F = \emptyset$, by (g) there exists $U \in \gamma$-$\beta O(X)$ and a γ-βg-open set W such that $x \in U$, $F \subset W$ and $U \cap W = \emptyset$. Now put $V = \gamma$-$\beta\mathrm{int}\,(W)$. Using Definition 3.16 of γ-βg-open sets we get $F \subset V$ and $V \cap U = \emptyset$.

(c) $\Longrightarrow$ (h) We have

$$\begin{aligned} F &\subset \bigcap\{\gamma\text{-}\beta\mathrm{cl}\,(V) : F \subset V \text{ and } V \text{ is } \gamma\text{-}\beta\text{g-open}\} \\ &\subset \bigcap\{\gamma\text{-}\beta\mathrm{cl}\,(V) : F \subset V \text{ and } V \text{ is } \gamma\text{-}\beta\text{-open}\} \\ &= F. \end{aligned}$$

(h) $\Longrightarrow$ (a) Let F be a γ-β-closed set in X not containing x. Then by (h) there exists a γ-βg-open set W such that $F \subset W$ and $x \in X - \gamma\text{-}\beta\mathrm{cl}\,(W)$. Since F is γ-β-closed and W is γ-βg-open, $F \subset \gamma\text{-}\beta\mathrm{int}\,(W)$. Take $V = \gamma\text{-}\beta\mathrm{int}\,(W)$. Then $F \subset V$, $x \in U = X - \gamma\text{-}\beta\mathrm{cl}\,(V)$ and $U \cap V = \emptyset$. □

5.15. Definition. A mapping $f : (X, \tau) \to (Y, \tau')$, where γ' is an operation on τ', is called γ'-β-anti-continuous if the inverse image of each γ'-β-open set in Y is open in X.

We now give examples which shows that regularity and γ-β-regularity are independent concepts:

5.16. Example. The topological space (X, τ) with the operation γ on τ as defined in Example 5.11 is regular but not γ-β-regular.

5.17. Example. The topological space (X, τ) defined as in Example 3.3(b) is not regular but is γ-β-regular.

Although regularity and γ-β-regularity are independent concepts, we have been able to obtain γ-β-regularity from regularity and vice versa. The following theorems show these facts.

5.18. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be a γ-β-continuous, γ-β-anti-closed and γ-β-anti-open surjective function on (X, τ) with an operation γ on τ. If X is γ-β-regular then Y is regular.*

Proof. Let K be closed in Y and $y \notin K$. Since f is γ-β-continuous and X is γ-β-regular, then for each point $x \in f^{-1}(y)$, there exist disjoint $V, W \in \gamma\text{-}\beta O(X)$ such that, $x \in V$ and $f^{-1}(K) \subset W$. Now since f is γ-β-anti-closed, there exists an open set U containing K such that $f^{-1}(U) \subset W$. As f is a γ-β-anti-open map, we have $y = f(x) \in f(V)$ and $f(V)$ is open in Y. Now, $f^{-1}(U) \cap V = \emptyset$ and hence $U \cap f(V) = \emptyset$. Therefore Y is regular. □

5.19. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$ be a γ'-β-anti-continuous, γ'-βg-closed and γ'-β-open surjection, where γ' is an operation on τ'. If X is regular, then Y is γ'-β-regular.*

Proof. Let $y \in Y$ and F be any γ'-β-open set in Y containing y. Since f is γ'-β-anti-continuous, $f^{-1}(F)$ is open in X and contains x, where $y = f(x)$. Again since X is regular, there exists an open set V in X containing x such that $x \in V \subset cl(V) \subset f^{-1}(F)$, which is equivalent to $y \in f(V) \subset f(\mathrm{cl}\,(V)) \subset F$. Since f is γ'-β-open and γ'-βg-closed, $f(V) \in \gamma'\text{-}\beta O(Y)$ and $f(\mathrm{cl}\,(V))$ is a γ'-βg-closed set in Y. Then $\gamma'\text{-}\beta\mathrm{cl}\,(f(\mathrm{cl}\,(V)) \subset F$ and so,

$$y \in f(V) \subset \gamma'\text{-}\beta\mathrm{cl}\,(f(V)) \subset \gamma'\text{-}\beta\mathrm{cl}\,(f(\mathrm{cl}\,(V)) \subset F.$$

Hence Y is γ'-β-regular by Theorem 5.14. □

5.20. Definition. A topological space (X, τ) with an operation γ on τ, is said to be γ-β-normal if for any pair of disjoint γ-β-closed sets A, B of X, there exist disjoint γ-β-open sets U and V such that $A \subset U$ and $B \subset V$.

Tahiliani [11] characterized β-normal spaces. In a similar fashion we give several characterizations of γ-β-normal spaces.

5.21. Theorem. *For a topological space (X, τ) with an operation γ on τ, the following are equivalent:*

(a) *X is γ-β-normal.*
(b) *For each pair of disjoint γ-β-closed sets A, B of X, there exist disjoint γ-βg-open sets U and V such that $A \subset U$ and $B \subset V$.*
(c) *For each γ-β-closed A and any γ-β-open set V containing A, there exists a γ-βg-open set U such that $A \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset V$.*
(d) *For each γ-β-closed set A and any γ-βg-open set B containing A, there exists a γ-βg-open set U such that $A \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset \gamma\text{-}\beta\mathrm{int}\,(B)$.*
(e) *For each γ-β-closed set A and any γ-βg-open set B containing A, there exists a γ-β-open set G such that $A \subset G \subset \gamma\text{-}\beta\mathrm{cl}\,(G) \subset \gamma\text{-}\beta\mathrm{int}\,(B)$.*
(f) *For each γ-βg-closed set A and any γ-β-open set B containing A, there exists a γ-β-open set U such that $\gamma\text{-}\beta\mathrm{cl}\,(A) \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset B$.*
(g) *For each γ-βg-closed set A and any γ-β-open set B containing A, there exists a γ-βg-open set G such that $\gamma\text{-}\beta\mathrm{cl}\,(A) \subset G \subset \gamma\text{-}\beta\mathrm{cl}\,(G) \subset B$.*

Proof. (a) $\Longrightarrow$ (b) Follows from the fact that every γ-β-open set is γ-βg-open.

(b) $\Longrightarrow$ (c) Let A be a γ-β-closed set and V any γ-β-open set containing A. Since A and $(X - V)$ are disjoint γ-β-closed sets, there exist γ-βg-open sets U and W such that $A \subset U$, $(X - V) \subset W$ and $U \cap W = \emptyset$. By Definition 3.16, we get

$$(X - V) \subset \gamma\text{-}\beta\mathrm{int}\,(W).$$

Since $U \cap \gamma\text{-}\beta\mathrm{int}\,(W) = \emptyset$, we have $\gamma\text{-}\beta\mathrm{cl}\,(U) \cap \gamma\text{-}\beta\mathrm{int}\,(W) = \emptyset$, and hence

$$\gamma\text{-}\beta\mathrm{cl}\,(U) \subset X - \gamma\text{-}\beta\mathrm{int}\,(W) \subset V.$$

Therefore $A \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset V$.

(c) $\Longrightarrow$ (a) Let A and B be any two disjoint γ-β-closed sets of X. Since $(X - B)$ is an γ-β-open set containing A, there exists a γ-βg-open set G such that

$$A \subset G \subset \gamma\text{-}\beta\mathrm{cl}\,(G) \subset (X - B).$$

Since G is a γ-βg-open set, using Definition 3.16, we have $A \subset \gamma\text{-}\beta\mathrm{int}\,(G)$. Taking $U = \gamma\text{-}\beta\mathrm{int}\,(G)$ and $V = X - \gamma\text{-}\beta\mathrm{cl}\,(G)$, we have two disjoint γ-β-open sets U and V such that $A \subset U$ and $B \subset V$. Hence X is γ-β-normal.

(e) $\Longrightarrow$ (d) Obvious.

(d) $\Longrightarrow$ (c) Obvious.

(e) $\Longrightarrow$ (c) Let A be any γ-β-closed set and V any γ-β-open set containing A. Since every γ-β-open set is γ-βg-open, there exists a γ-β-open set G such that

$$A \subset G \subset \gamma\text{-}\beta\mathrm{cl}\,(G) \subset \gamma\text{-}\beta\mathrm{int}\,(V).$$

Also we have a γ-βg-open set G such that $A \subset G \subset \gamma\text{-}\beta\mathrm{cl}\,(G) \subset \gamma\text{-}\beta\mathrm{int}\,(V) \subset V$.

(f) $\Longrightarrow$ (g) Obvious.

(g) $\Longrightarrow$ (c) Obvious.

(c) $\Longrightarrow$ (e) Let A be a γ-β-closed set and B any γ-βg-open set containing A. Using Definition 3.16 of a γ-βg-open set we get $A \subset \gamma\text{-}\beta\mathrm{int}\,(B) = V$, say. Then applying (c), we get a γ-βg-open set U such that $A = \gamma\text{-}\beta\mathrm{cl}\,(A) \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset V$. Again, using the same Definition 3.16 we get $A \subset \gamma\text{-}\beta\mathrm{int}\,(U)$, and hence

$$A \subset \gamma\text{-}\beta\mathrm{int}\,(U) \subset U \subset \gamma\text{-}\beta\mathrm{cl}\,(U) \subset V;$$

which implies $A \subset \gamma\text{-}\beta\text{int}\,(U) \subset \gamma\text{-}\beta\text{cl}\,(\gamma\text{-}\beta\text{int}\,(U)) \subset \gamma\text{-}\beta\text{cl}\,(U) \subset V$, i.e.

$$A \subset G \subset \gamma\text{-}\beta\text{cl}\,(G) \subset \gamma\text{-}\beta\text{int}\,(B),$$

where $G = \gamma\text{-}\beta\text{int}\,(U)$.

(c) $\Longrightarrow$ (g) Let A be a γ-βg-closed set and B any γ-β-open set containing A. Since A is a γ-βg-closed set, we have $\gamma\text{-}\beta\text{cl}\,(A) \subset B$, therefore by (c) we can find a γ-βg-open set U such that $\gamma\text{-}\beta\text{cl}\,(A) \subset U \subset \gamma\text{-}\beta\text{cl}\,(U) \subset B$.

(g) $\Longrightarrow$ (f) Let A be a γ-βg-closed set and B any γ-β-open set containing A, then by (g) there exists a γ-βg-open set G such that $\gamma\text{-}\beta\text{cl}\,(A) \subset G \subset \gamma\text{-}\beta\text{cl}\,(G) \subset B$. Since G is a γ-βg-open set, then by Definition 3.16, we get $\gamma\text{-}\beta\text{cl}\,(A) \subset \gamma\text{-}\beta\text{int}\,(G)$. If we take $U = \gamma\text{-}\beta\text{int}\,(G)$, the proof follows. □

The following examples show that normality and γ-β-normality are independent concepts:

5.22. Example. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Define an operation γ on τ by $A^\gamma = cl(A)$. This space is γ-β-normal but not normal.

5.23. Example. Let $X = \{a, b, c\}$ and let τ be the discrete topology on X. Define an operation γ on τ by

$$A^\gamma = \begin{cases} \{b\} & \text{if } A = \{b\}, \\ A \cup \{b\} & \text{if } A \neq \{b\}. \end{cases}$$

Then X is normal but not γ-β-normal.

Although from the above two examples we have seen that normality and γ-β-normality are independent to each other, one can be obtained from the other. The following theorems show these facts.

5.24. Theorem. *Let (X, τ) be a topological space and γ be an operation on τ. If $f : (X, \tau) \to (Y, \tau')$ is a γ-β-continuous, γ-β-anti-closed surjective function and X is γ-β-normal, then Y is normal.*

Proof. Let A and B be two disjoint closed sets in Y. Since f is γ-β-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint γ-β-closed sets in X. Now as X is γ-β-normal, there exist disjoint γ-β-open sets V and W such that $f^{-1}(A) \subset V$ and $f^{-1}(B) \subset W$. Since f is γ-β-anti-closed, there exist open sets M and N such that $A \subset M$, $B \subset N$, $f^{-1}(M) \subset V$ and $f^{-1}(N) \subset W$. Since $V \cap W = \emptyset$, we have $M \cap N = \emptyset$. So Y is normal. □

5.25. Theorem. *Let $f : (X, \tau) \to (Y, \tau')$, where γ, γ' are operations on τ, τ', respectively, be a γ'-β-anti-continuous and γ'-βg-closed surjection. If X is normal, then Y is γ'-β-normal.*

Proof. For any pair of disjoint γ'-β-closed sets F_1 and F_2 in Y, since f is γ'-β-anti continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets in X. Since X is normal we have two disjoint open sets V and W such that $f^{-1}(F_1) \subset V$ and $f^{-1}(F_2) \subset W$. Since f is γ'-βg-closed, by Theorem 4.13, we get γ'-β-open sets M and N in Y such that $F_1 \subset M$ and $F_2 \subset N$ with $f^{-1}(M) \subset V$ and $f^{-1}(N) \subset W$. Again $f^{-1}(M) \cap f^{-1}(N) \subset V \cap W = \emptyset$. So $M \cap N = \emptyset$. □

References

[1] Andrijević, D. *Semi-preopen sets*, Math. Vesnik **38**, 24–32, 1986.

[2] Dontchev, J. *On generalising semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. (Math) **16**, 35–48, 1995.

[3] Dugundji, J. *Topology* (Allyn and Bacon, Boston, Mass, 1966).

[4] Janković, D. S. *On functions with α-closed graphs*, Glasnik Mathematicki **18**, 141–148, 1983.

[5] Kasahara, S. *Operation compact spaces*, Math. Japonica **24** (1), 97–105, 1979.

[6] Krishnan, G. S. S. and Balachandran, K. *On γ-semiopen sets in topological spaces*, Bull. Cal. Math. Soc. **98** (6), 517–530, 2006.

[7] Levine, N. *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70**, 36–41, 1963.

[8] Maki, H., Umehara, J. and Noiri, T. *Every topological space is pre-$T_{\frac{1}{2}}$*, Mem. Fac. Sci. Kochi Univ. (Math) **17**, 33–42, 1996.

[9] Abd. El-Monsef, M. E., El-Deeb, S. N. and Mahmood, R. A. *β-open sets and β-contnuous mappings*, Bull. Fac. Sci. Assiut Univ. **12**, 77–90, 1983.

[10] Ogata, H. *Operations on topological spaces and associated topology*, Math. Japonica **36** (1), 175–184,1991.

[11] Tahiliani, S. *Generalised β-closed functions*, Bull. Cal. Math. Soc. **98** (4), 367–376, 2006.