Hacettepe Journal of Mathematics and Statistics
Volume 38 (2) (2009), 119 – 128

THE STOLARSKY TYPE FUNCTIONS AND THEIR MONOTONICITIES

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Received 04 : 07 : 2008 : Accepted 23 : 03 : 2009

Abstract

In this paper, we give the definition of a Stolarsky type function, and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in n variables.

Keywords: Two-parameter, Monotonicity, Mean, Vandermonde determinant.

2000 AMS Classification: 26 D 15.

1. Introduction

The so-called Stolarsky means $S(a, b; \alpha)$ were defined first by Stolarsky in [9] as follows:

$$S(a,b;\alpha) = \left[\frac{a^{\alpha+1} - b^{\alpha+1}}{(\alpha+1)(a-b)}\right]^{1/\alpha}, \qquad \alpha(\alpha+1)(a-b) \neq 0; \tag{1.1}$$

$$S(a,b;-1) = \frac{a-b}{\ln a - \ln b}, \qquad \alpha(a-b) \neq 0,\ \alpha = -1; \tag{1.2}$$

$$S(a,b;0) = \exp\left(-1 + \frac{a\ln a - b\ln b}{a-b}\right), \qquad (\alpha+1)(a-b) \neq 0,\ \alpha = 0; \tag{1.3}$$

$$S(a,a;\alpha) = a, \qquad a = b. \tag{1.4}$$

The monotonicity of $S(a, b; \alpha)$ has been discussed by Leach and Sholander [3, 4], and by Qi [7, 8] also using different ideas and simpler methods.

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In [7], Qi studied the following generalized weighted Stolarsky type mean values $E_{f,p}(a,b;\alpha)$ with parameter α, and proved that $E_{f,p}(x,y;\alpha)$ is an increasing function in α:

$$(1.5)\qquad E_{f,p}(a,b;\alpha)=\left(\frac{\int_a^b p(u)f^{\alpha}(u)du}{\int_a^b p(u)du}\right)^{\frac{1}{\alpha}},\qquad (\alpha-\beta)(a-b)\neq 0;$$

$$(1.6)\qquad E_{f,p}(a,b;0)=\exp\left(\frac{\int_a^b p(u)\ln f(u)du}{\int_a^b p(u)du}\right),\qquad \alpha=0, a-b\neq 0;$$

$$(1.7)\qquad E_{f,p}(a,a;\alpha)=f(a),\qquad \alpha=\beta, a=b;$$

where $a,b,\alpha,\beta\in\mathbb{R}$, $p\geq 0, and\ f>0$ is an integrable function on the interval $[a,b]\subset\mathbb{R}$.

We know by the definition of the power mean that

$$(1.8)\qquad M(x;\alpha)=\left(\frac{\sum_{k=1}^n x_k^{\alpha}}{n}\right)^{\frac{1}{\alpha}},\qquad \alpha\neq 0;$$

$$(1.9)\qquad M(x;0)=\exp\left(\frac{\sum_{k=1}^n \ln x_k}{n}\right),\qquad \alpha=0;$$

where $a_k\in\mathbb{R}_+$, and $\alpha\in\mathbb{R}$.

We note that for each of these two means the one-parameter means are of the type $(F(\alpha)/F(0))^{1/\alpha}$ if $\alpha\neq 0$, and $\exp(F'(\alpha)/F(\alpha))$ if $\alpha=0$, where $F(\alpha)$ is a certain univariate function involving an α-order power.

In this paper, we define a Stolarsky type function and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in n variables.

2. Main results

Throughout the paper we assume $\mathbb{R}$ to be the set of real numbers, $\mathbb{R}_+$ the set of strictly positive real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean Space,

$$\mathbb{R}^n_+=\{(x_1,x_2,\ldots,x_n): x_i>0,\ i=1,2,\ldots,n\},$$

and

$$\alpha x=(\alpha x_1,\alpha x_2,\ldots,\alpha x_n),\qquad e^x=(e^{x_1},e^{x_2},\ldots,e^{x_n}),$$
$$x^{\alpha}=(x_1^{\alpha},x_2^{\alpha},\ldots,x_n^{\alpha}),\qquad \ln x=(\ln x_1,\ln x_2,\ldots,\ln x_n),$$

where $\alpha\in\mathbb{R}$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n_+$, and $y=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n_+$.

2.1. Definition. Let $\alpha,\beta\in\mathbb{R}$, and f be continuous involving an $(\alpha\beta)$-order power function on $I\subseteq\mathbb{R}^n_+$. If $F(\alpha)=f(x;\alpha\beta)$, $\beta\neq 0$, and f is a differentiable function with respect to $\alpha\in\mathbb{R}$, then the Stolarsky type function $S_f(x;\alpha,\beta)$ is defined as follows,

$$(2.1)\qquad S_f(x;\alpha)=\left(\frac{f(x;\alpha\beta)}{f(x;0)}\right)^{\frac{1}{\alpha}},\qquad (\alpha\neq 0),$$

$$(2.2)\qquad S_f(x;0)=\lim_{\alpha\to 0}\exp\left(\frac{f'_{\alpha}(x;\alpha\beta)}{f(x;\alpha\beta)}\right),\qquad (\alpha=0),$$

where f'_{α} is the partial derivative with respect to α of $f(x;\alpha\beta)$.

2.2. Remark. For convenience, we write

$$(2.3)\qquad S_f(x;\alpha)=S_f(x)=S_f(\alpha)=S_f,$$

shifting notation to suit the context.

2.3. Theorem. *Let $\alpha, \beta \in \mathbb{R}, \beta \neq 0$, and f be continuous involving an $(\alpha\beta)$-order power function on $I \subseteq \mathbb{R}_+^n$. If*

$$f(x;\alpha\beta) f''_{\alpha\alpha}(x;\alpha\beta) > [f'_\alpha(x;\alpha\beta)]^2, \tag{2.4}$$

then $S_f(x;\alpha)$ is a monotonic increasing function in α, and monotonic decreasing if the inequality (2.4) *is reversed.*

Proof. Suppose the inequality (2.4) holds. Setting $T(\alpha) = \ln \mid f(x;\alpha\beta) \mid$, then $T'(\alpha) = f'_\alpha(x;\alpha\beta)/f(x;\alpha\beta)$, and

$$T''(\alpha) = \frac{f(x;\alpha\beta) f''_{\alpha\alpha}(x;\alpha\beta) - [f'_\alpha(x;\alpha\beta)]^2}{[f(x;\alpha\beta)]^2} > 0.$$

When $\alpha = 0$, $\ln S_f = f'_\alpha(x;\alpha\beta)/f(x;\alpha\beta) = T'(\alpha)$, and $\partial \ln S_f/\partial\alpha = T''(\alpha) > 0$, which implies that $S_f(x;\alpha)$ is a monotonic increasing function in α.

When $\alpha \neq 0$, using the mean value theorem, we find

$$\frac{\partial \ln S_f}{\partial \alpha} = \frac{T'(\alpha)}{\alpha} - \frac{T(\alpha)}{\alpha^2} = \frac{T'(\alpha) - T(\alpha)/\alpha}{\alpha} = \frac{T'(\alpha) - T'(\zeta)}{\alpha} = \frac{\alpha - \eta}{\alpha} T''(\eta) > 0,$$

where ζ is between 0 and α, and η is between α and ζ. That is to say, $S_f(x;\alpha)$ is a monotonic increasing function in α. Theorem 2.3 is thus proved. □

3. The generalized weighted Stolarsky type functional mean

3.1. Theorem. *The generalized weighted Stolarsky type functional mean values $S_{f,p}(x;\alpha)$ are monotonic increasing functions with α in R, where*

$$S_{f,p}(x;\alpha) = \left(\frac{\int_E p(t) f^\alpha(A(x;t))dt}{\int_E p(t)dt} \right)^{1/\alpha}, \qquad \alpha \neq 0, \tag{3.1}$$

$$S_{f,p}(x;0) = \exp\left(\frac{\int_E p(t) \ln f(A(x;t))dt}{\int_E p(t)dt} \right), \qquad \alpha = 0, \tag{3.2}$$

and $E = \left\{(t_1, t_2, \ldots, t_n) \mid \sum_{i=1}^n t_i \leqq 1, t_i \geqq 0,\ i = 1, 2, \cdots, n\right\}$, $t_0 = 1 - \sum_{i=1}^n t_i$, $A(x;t) = x_0 + \sum_{i=1}^n (x_i - x_0)t_i = \sum_{i=0}^n x_i t_i$, $x_i \in I \subseteq \mathbb{R}_+$, and $p \geqq 0$, $f > 0$ integrable functions respectively on E and I.

Proof. By taking $T(x;\alpha) = \int_E p(t) f^\alpha(A(x;t))dt$, and using Cauchy's integral inequality, we have

$$\begin{aligned} &T(x;\alpha\beta) T''_{\alpha\alpha}(x;\alpha\beta) - [T'_\alpha(x;\alpha\beta)]^2 \\ &\quad = \int_E p(t) f^\alpha(A(x;t))dt \cdot \int_E p(t) f^\alpha(A(x;t)) \ln^2 f(A(x;t))dt \\ &\qquad - \left(\int_E p(t) f^\alpha(A(x;t)) \ln f(A(x;t))dt \right)^2 > 0, \end{aligned}$$

which implies Theorem 3.1 from Theorem 2.3. □

3.2. Corollary. *The generalized weighted Stolarsky type functional mean values $E_{f,p}(a,b;\alpha)$ are monotonic increasing functions with α in $\mathbb{R}$, where $E_{f,p}(a,b;\alpha)$ is given by* (1.5)–(1.7).

Proof. Setting $u = x_0 + (x_1 - x_0)t_1$, then $du = (x_1 - x_0)dt_1$. Setting $a = x_0$ and $b = x_1$, from Theorem 3.1, we immediately obtain Corollary 3.2. The proof is completed. □

4. The generalized weighted Stolarsky type functional mean with two parameters

4.1. Definition. Let $\alpha, \beta \in R$, E, t_0 and p, f be defined as in Theorem 3.1. If

$$M_\beta(x;t) = \left(x_0^\beta + \sum_{i=1}^{n}(x_i^\beta - x_0^\beta)t_i\right)^{1/\beta} = \left(\sum_{i=0}^{n} x_i^\beta t_i\right)^{1/\beta},$$

and $M_0(x;t) = G(x;t) = \prod_{i=0}^{n} x_i^{t_i}$, then the first generalized weighted Stolarsky type functional mean values, $S_{f,p}^{[1]}(x;\alpha,\beta)$, with two parameters α and β are as follows

$$S_{f,p}^{[1]}(x;\alpha,\beta) = \left(\frac{\int_E p(t) f^\alpha(M_\beta(x;t))dt}{\int_E p(t)dt}\right)^{1/\alpha}, \qquad \alpha\beta \neq 0; \tag{4.1}$$

$$S_{f,p}^{[1]}(x;0,\beta) = \exp\left(\frac{\int_E p(t) \ln f(M_\beta(x;t))dt}{\int_E p(t)dt}\right), \qquad \alpha = 0, \beta \neq 0; \tag{4.2}$$

$$S_{f,p}^{[1]}(x;\alpha,0) = \left(\frac{\int_E p(t) f^\alpha(G(x;t))dt}{\int_E p(t)dt}\right)^{1/\alpha}, \qquad \alpha \neq 0, \beta = 0; \tag{4.3}$$

$$S_{f,p}^{[1]}(x;0,0) = \exp\left(\frac{\int_E p(t) \ln f(G(x;t))dt}{\int_E p(t)dt}\right), \qquad \alpha = \beta = 0. \tag{4.4}$$

In a manner similar to Section 3, from Definition 4.1 we obtain the following theorem.

4.2. Theorem. *The first generalized weighted Stolarsky type functional mean values* $S_{f,p}^{[1]}(x;\alpha,\beta)$ *are monotonic increasing functions in* $\alpha \in R$.

4.3. Theorem. *The first generalized weighted Stolarsky type functional mean values* $S_{f,p}^{[1]}(x;\alpha,\beta)$ *are monotonic increasing functions with* $\beta \in R$ *if* f *is a monotonic increasing function.*

Proof. This follows from the weighted power mean inequality, Definition 4.1 and the fact that f is a monotonic increasing function. □

4.4. Remark. We have $S_{f,p}^{[1]}(x;\alpha,1) = S_{f,p}(x;\alpha)$.

4.5. Definition. Let $\alpha, \beta \in R$, E, t_0 and p, f be defined as in Theorem 3.1. If

$$M_\beta(x^\alpha;t) = \left[x_0^{\alpha\beta} + \sum_{i=1}^{n}(x_i^{\alpha\beta} - x_0^{\alpha\beta})t_i\right]^{1/\beta} = \left(\sum_{i=0}^{n} x_i^{\alpha\beta} t_i\right)^{1/\beta},$$

$M_0(x^\alpha;t) = G(x^\alpha;t) = \prod_{i=0}^{n} x_i^{\alpha t_i}$, and $f'(1)$ exists, then the second generalized weighted Stolarsky type functional mean values $S_{f,p}^{[2]}(x;\alpha,\beta)$ with two parameters α and β are as follows

$$S_{f,p}^{[2]}(x;\alpha,\beta) = \left(\frac{\int_E p(t) f(M_\beta(x^\alpha;t))dt}{\int_E p(t)dt}\right)^{1/\alpha}, \qquad \alpha\beta \neq 0; \tag{4.5}$$

$$S_{f,p}^{[2]}(x;0,\beta) = \exp\left(\frac{\int_E p(t) f'(1)\left(\sum_{k=1}^{n} t_k \ln x_k\right)dt}{\int_E p(t)dt}\right), \qquad \alpha = 0, \beta \in \mathbb{R}; \tag{4.6}$$

$$S_{f,p}^{[2]}(x;\alpha,0) = \left(\frac{\int_E p(t) f(G(x^\alpha;t))dt}{\int_E p(t)dt}\right)^{1/\alpha}, \qquad \alpha \neq 0, \beta = 0. \tag{4.7}$$

4.6. Theorem. *The second generalized weighted Stolarsky type functional mean values* $S_{f,p}^{[2]}(x;\alpha,\beta)$ *are monotonic increasing functions with* α *in* R *if* $f' > 0$, $ff'' > (f')^2$ *and* $\beta > 0$.

Proof. By taking $T(x;\alpha\beta)=\int_E p(t)f(M_\beta(x^\alpha;t))dt$, if $\beta\neq 0$, then

$$T'_\alpha(x;\alpha\beta)=\int_E p(t)f'(M_\beta(x^\alpha;t))\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-1}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)dt, \tag{4.8}$$

$$\begin{aligned}T''_\alpha(x;\alpha\beta)=&\int_E p(t)f''(M_\beta(x^\alpha;t))\left[\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-1}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)\right]^2dt\\&+\int_E p(t)f'(M_\beta(x^\alpha;t))\left[(1-\beta)\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-2}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)^2\right.\\&\left.+\beta\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-1}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln^2 x_k\right)\right]dt.\end{aligned} \tag{4.9}$$

Using Cauchy's integral inequality, from (4.8)-(4.9), and $f'>0, ff''>(f')^2, \beta>0$, yields

$$\begin{aligned}&T(x;\alpha\beta)T''_{\alpha\alpha}(x;\alpha\beta)-[T'_\alpha(x;\alpha\beta)]^2\\&=\int_E p(t)f(M_\beta(x^\alpha;t))dt\cdot\int_E p(t)f''(M_\beta(x^\alpha;t))\\&\qquad\cdot\left[\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-1}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)\right]^2dt\\&-\left[\int_E p(t)f'(M_\beta(x^\alpha;t))\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-1}\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)dt\right]^2\\&+\int_E p(t)f(M_\beta(x^\alpha;t))dt\cdot\int_E p(t)f''(M_\beta(x^\alpha;t))\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)^{1/\beta-2}\\&\quad\cdot\left\{\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)^2+\beta\left[\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\right)\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln^2 x_k\right)\right.\right.\\&\qquad\left.\left.-\left(\sum_{k=0}^n x_k^{\alpha\beta}t_k\ln x_k\right)^2\right]\right\}dt>0\end{aligned}$$

which implies Theorem 4.6 from Theorem 2.3. If $\beta=0$ we can obtain Theorem 4.6 similarly. □

5. Some mean values in n variables

5.1. Notation and lemmas. Throughout this section we assume $x=(x_0,x_1,\dots,x_n)\in\mathbb{R}_+^{n+1}$, and that φ is a function in $\mathbb{R}$. Put

$$V(x;\varphi)=\begin{vmatrix}1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & \varphi(x_0)\\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & \varphi(x_1)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & \varphi(x_n)\end{vmatrix}. \tag{5.1}$$

Assuming $\varphi(t)=t^{n+r}\ln^k t$, then

$$V(x;r,k)=\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^{n+r}\ln^k x_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^{n+r}\ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^{n+r}\ln^k x_n \end{vmatrix}. \tag{5.2}$$

Note the case $r=0$ and $k=0$ is just the determinant of Van der Monde's matrix of the n-th order:

$$V(x;0,0)=\prod_{0\leqq i<j\leqq n}(x_j-x_i). \tag{5.3}$$

Write $\ln x=(\ln x_0,\ln x_1,\ldots,\ln x_n)$, then

$$V(\ln x;r,k)=\begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r\ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r\ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r\ln^k x_n \end{vmatrix}. \tag{5.4}$$

Also, let $0\leqq i\leqq n$, and set

$$V_{[i]}(x;\varphi)=\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & \varphi(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & \varphi(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_i & x_i^2 & \cdots & x_i^{n-1} & \varphi(x_i) \\ 0 & 1 & 2x_i & \cdots & (n-1)x_i^{n-2} & \varphi'(x_i) \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-1} & \varphi(x_{i+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & \varphi(x_n) \end{vmatrix} \tag{5.5}$$

and for $\varphi(t)=t^{n+r+1}$ in (5.5), we have

$$V_{[i]}(x;r)=\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n} & x_0^{n+r+1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n} & x_1^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_i & x_i^2 & \cdots & x_i^{n} & x_i^{n+r+1} \\ 0 & 1 & 2x_i & \cdots & nx_i^{n-1} & (n+r+1)x_i^{n+r} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n} & x_{i+1}^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n} & x_n^{n+r+1} \end{vmatrix},\quad (i\leqq i\leqq n), \tag{5.6}$$

and

$$V_{[i]}(x;0)=(-1)^{i+1}V(x;0,0)\prod_{j=0,j\neq i}^{n}(x_j-x_i)=(-1)^{i+1}V^2(x;0,0)/V_i(x), \tag{5.7}$$

where

$$V_i(x)=\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{i-1} & x_{i-1}^2 & \cdots & x_{i-1}^{n-1} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix},\quad (0\leqq i\leqq n). \tag{5.8}$$

5.1. Lemma. (see [12, 13, 14]) *If $n \in \mathbb{N}$, and φ is a n-order differentiable function on an interval $I \subset \mathbb{R}_+$, then,*

$$V(x;\varphi) = V(x;0,0)\int_E \varphi^{(n)}(A(x;t))dt, \tag{5.9}$$

$$\sum_{i=0}^{n}(-1)^{i+1}\lambda_i V_{[i]}(x;\varphi)V_i(x) = V^2(x;0,0)\int_E A(\lambda;t)\varphi^{(n)}(A(x;t))dx, \tag{5.10}$$

where $dt = dt_1 dt_2 \cdots dt_n$, and $E, A(x;t)$ are as in Theorem 3.1.

5.2. Lemma. (see [10]) *Let r be an integer, then*

$$V(a;r,0) = V(a;0,0)\cdot \sum_{i_0+i_1+\cdots+i_n=r}^{i_0,i_2,\cdots,i_n\geqq 0}\prod_{k=0}^{n} a_k^{i_k}, \qquad r>0; \tag{5.11}$$

$$V(a;r,0) = 0, \qquad r = 0,1,\ldots,-(n-1); \tag{5.12}$$

$$V(a;r,0) = (-1)^n V(a;0;0)\cdot \sum_{i_0+i_1+\cdots+i_n=-r}^{i_0,i_1,\cdots,i_n\geqq 1}\prod_{k=0}^{n} a_k^{-i_k}, \qquad r<-n. \tag{5.13}$$

5.2. The Stolarsky type mean with one parameter in n variables.

5.3. Definition. (see [11]) The Stolarsky type generalized mean values $S_\alpha(x)$ with parameter α in n variables are

$$S_\alpha(x) = \left[n!\int_E \varphi_1^{(n)}(\alpha, A(x;t))dt\right]^{1/\alpha}, \qquad \alpha \neq 0, \tag{5.14}$$

$$S_0(x) = \exp\left(n!\int_E \varphi_2^{(n)}(0, A(x;t))dt\right), \qquad \alpha = 0, \tag{5.15}$$

where $\varphi_1^{(n)}(\alpha,t) = t^\alpha$ and $\varphi_2^{(n)}(\alpha,t) = t^\alpha \ln t$.

5.4. Theorem. *If the generalized mean values $S_\alpha(x)$ with two parameters α and β, in n variables are as given by Definition 5.3, then*

$$S_\alpha(x) = \left[\frac{n!}{\prod_{k=1}^{n}(k+\alpha)}\cdot\frac{V(x;\alpha,0)}{V(x;0,0)}\right]^{\frac{1}{\alpha}} \qquad \alpha \neq 0,-1,-2,\cdots,-n; \tag{5.16}$$

$$S_0(x) = \exp\left(\frac{V(x;0,1)}{V(x;0,0)} - \sum_{k=1}^{n}\frac{1}{k}\right), \qquad \alpha = 0; \tag{5.17}$$

$$S_\alpha(x) = \left[\frac{n!\cdot V(x;\alpha,1)}{(-1)^{\alpha+1}(-\alpha-1)!\cdot(n+\alpha)!\cdot V(x;0,0)}\right]^{\frac{1}{\alpha}}, \qquad \alpha = -1,\cdots,-n; \tag{5.18}$$

where $S_\alpha(x)$ are monotonic increasing functions with α in R.

Proof. Consider the following two functions:

$$\varphi_1(\alpha,t) = \prod_{k=1}^{n}(k+\alpha)^{-1}t^{n+\alpha}, \tag{5.19}$$

if $\alpha \neq 0,-1,-2,\ldots,-n$; and

$$\varphi_2(0,t) = (n!)^{-1}t^n\left(\ln t - \sum_{k=1}^{n}\frac{1}{k}\right), \tag{5.20}$$

if $\alpha = 0$; and

$$\varphi_1(\alpha,t) = \left[(-1)^{\alpha+1}(-\alpha-1)!(n+\alpha)!\right]^{-1} t^{n+\alpha}\ln t, \tag{5.21}$$

if $\alpha = -1, -2, \ldots, -n$. Then $\varphi_1^{(n)}(\alpha, t) = t^\alpha$ and $\varphi_2^{(n)}(0, t) = \ln t$.

According to Lemma 5.1 and (5.19)–(5.21), we know that the expressions (5.16)–(5.18) hold true.

Let $f^\alpha(A(x;t)) = \varphi_1^{(n)}(\alpha, A(x;t))$. Then $\ln f(A(x;t)) = \varphi_2^{(n)}(0; A(x;t))$. Taking $p(x) \equiv 1$, we find from Theorem 3.1 that the $S_\alpha(x)$ are monotonic increasing functions with α in R. The proof of Theorem 5.4 is completed. □

5.5. Remark. (see [10]–[17]) $S_0(x)$ is the so-called identric mean in n variables, and $S_{-1}(x)$ the first logarithmic mean $L(x)$. It is noted that $S_0(x) := I(x)$, and

$$L(x) := S_{-1}(x) = \frac{V(x;0,0)}{nV(x;-1,1)}. \tag{5.22}$$

5.6. Remark. (see [1]) If α is a nonnegative integer, from Lemma 5.2, $[S_\alpha(x)]^\alpha$ is the r-th generalized elementary symmetric mean in n variables, i.e.

$$\sum\nolimits_n^{[\alpha]}(x) := [S_\alpha(x)]^\alpha = \binom{n+\alpha}{\alpha}^{-1} \sum_{\substack{i_0+i_1+\cdots+i_n=\alpha,\\ i_0,i_1,\ldots,i_n\in\mathbb{N}_0}} \prod_{k=1}^{n} x_k^{i_k}. \tag{5.23}$$

5.3. The Stolarsky type mean with two parameters in n variables.

5.7. Definition. (see [12]) The Stolarsky type generalized mean values $S_{\alpha,\beta}(x)$ with two parameters α and β in n variables are

$$S_{\alpha,\beta}(x) = \left[n! \int_E \varphi_1^{(n)}(\alpha, M_\beta(x;t))dt\right]^{1/\alpha}, \qquad \alpha \neq 0,\ \beta \neq 0; \tag{5.24}$$

$$S_{0,\beta}(x) = \exp\left(n! \int_E \varphi_2^{(n)}(0, M_\beta(x;t))dt\right), \qquad \alpha = 0,\ \beta \neq 0; \tag{5.25}$$

$$S_{\alpha,0}(x) = \left[n! \int_E \varphi_1^{(n)}(\alpha, G(x;t))dt\right]^{1/\alpha}, \qquad \alpha \neq 0,\ \beta = 0; \tag{5.26}$$

$$S_{0,0}(x) = \left(\prod_{i=0}^{n} a_i\right)^{1/(n+1)}, \qquad \alpha = 0,\ \beta = 0; \tag{5.27}$$

where $\varphi_1^{(n)}(\alpha, t) = t^\alpha$ and $\varphi_2^{(n)}(\alpha, t) = t^\alpha \ln t$.

5.8. Theorem. *We have that $S_{\alpha,\beta}(x)$ are monotonic increasing functions with α in $\mathbb{R}$, and*

$$S_{\alpha,\beta}(x) = \left[\frac{n!\cdot\beta^n}{\prod_{k=1}^{n}(k\beta+\alpha)} \cdot \frac{V(x^\beta;\alpha/\beta,0)}{V(x^\beta;0,0)}\right]^{1/\alpha}, \qquad \beta \neq 0,\ \alpha \neq -k\beta,\ 0 \leqq k \leqq n; \tag{5.28}$$

$$S_{\alpha,\beta}(x) = \left[(-1)^{k+1}k\beta\binom{n}{k}\frac{V(x^\beta;-k,1)}{V(x^\beta;0,0)}\right]^{-1/(k\beta)}, \qquad \beta \neq 0,\ \alpha = -k\beta,\ 1 \leqq k \leqq n; \tag{5.29}$$

$$S_{\alpha,0}(x) = \left[\frac{n!}{\alpha^n}\cdot\frac{V(\ln x;\alpha,0)}{V(\ln x;0,n)}\right]^{1/\alpha}, \qquad \beta = 0,\ \alpha \neq 0; \tag{5.30}$$

$$S_{0,\beta}(x) = \exp\left(\frac{V(x^\beta;0,1)}{V(x^\beta;0,0)} - \frac{1}{\beta}\sum_{k=1}^{n}\frac{1}{k}\right), \qquad \alpha = 0,\ \beta \neq 0; \tag{5.31}$$

$$S_{0,0}(x) = \left(\prod_{i=0}^{n} x_i\right)^{1/(n+1)}, \qquad \alpha = \beta = 0. \tag{5.32}$$

5.9. Remark. Replacing α by $\alpha-\beta$, the generalized Stolarsky type mean $S_{\alpha-\beta,\beta}(x)$ is the Pečarić-Šimić mean in [6].

5.10. Remark. (see [15] and also [5, 16]) If $\alpha = 1$, then $S_{1,0}(x)$ is the second logarithmic mean in n variables:

$$l(x) := S_{1,0}(x) = \frac{n!V(\ln x; 1, 0)}{V(\ln x; 0, n)}. \tag{5.33}$$

5.11. Remark. (see [15] and also [5]) Change β to $1/\beta$, and set $\alpha = 1$. If β is a nonnegative integer, from Lemma 5.2 we see that $S_{1,1/\beta}(x)$ is the generalized Heron's mean in n variables:

$$H_\beta(x) := S_{1,1/\beta}(x) = \binom{n+\beta}{\beta}^{-1} \sum_{\substack{i_0+i_1+\cdots+i_n=\beta,\\ i_0,i_1,\ldots,i_n\in\mathbb{N}_0}} \prod_{k=1}^{n} x_k^{i_k/\beta}, \tag{5.34}$$

5.4. The r-th weighted elementary symmetric mean in n variables.

5.12. Definition. (see [17]) Let x be a tuple of n non-negative real numbers and the weight λ a tuple of n positive real numbers, then

$$E_\alpha(x,\lambda) = \sum_{\substack{i_0+i_1+\cdots+i_n=\alpha,\\ i_0,i_1,\ldots,i_n\in\mathbb{N}_0}} \sum_{k=0}^{n} \lambda_k(1+i_k)x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n} \tag{5.35}$$

is called the α-th weighted elementary symmetric function of x for the positive weight λ, where the sum is over all $\binom{n+\alpha+1}{\alpha}$-tuples of non-negative integers such that $i_1 + i_2 + \cdots + i_n = \alpha$; In addition, $E_0(x,\lambda) = \sum_{i=1}^{n}\lambda_i$. The α-th weighted elementary symmetric mean of x for λ is defined by

$$\sum{}_\alpha(x,\lambda) = \frac{E_\alpha(\mathrm{x},\lambda)}{\binom{n+\alpha+1}{\alpha}\sum_{i=1}^{n}\lambda_i}. \tag{5.36}$$

5.13. Theorem. (see [17]) *If $r \in \mathbb{N}$, then*

$$\sum{}_\alpha(x,\lambda) = \frac{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right)\left(\sum_{i=0}^{n} a_i x_i\right)^\alpha dx}{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right) dx} \tag{5.37}$$

is a monotonic increasing function with α in $\mathbb{N}$.

Acknowledgement. We are thankful to the referee for some valuable suggestions.

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