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# ON A SUBCLASS OF HARMONIC FUNCTIONS ASSOCIATED WITH WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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**Abstract**

Wright's generalized hypergeometric function is used here to introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. Among the results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for the generalized class of functions defined here.

**Keywords:** Harmonic univalent functions, Wright's generalized hypergeometric function, Distortion bounds, Extreme points, Convolution, Inclusion property.

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## 1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain $\Omega$ if both $u$ and $v$ are real and harmonic in $\Omega$. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and orientation preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (see [2]).

Denote by $\mathcal{H}$ the family of functions

(1.1) $\quad f = h + \overline{g}$

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which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that $f$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions $h$ and $g$ analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n, \;\; (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n}, \;\; (|b_1| < 1). \tag{1.2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $S$ of normalized univalent functions if the co-analytic part of $f$ is identically zero, i.e. $g \equiv 0$.

Also, we denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n}, \;\; (|b_1| < 1). \tag{1.3}$$

The Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \tag{1.4}$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \tag{1.5}$$

is defined (as usual) by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \lambda_n \mu_n z^n.$$

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_p, A_p$ and $\beta_1, B_1, \ldots, \beta_q, B_q$, $(p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ satisfying the condition that

$$1 + \sum_{n=1}^{q} B_n - \sum_{n=1}^{p} A_n \geq 0, (z \in \mathbb{U}),$$

Wright's generalized hypergeometric function [9]

$${}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] = {}_p\Psi_q[(\alpha_n, A_n)_{1,p} (\beta_n, B_n)_{1,q}; z]$$

is defined by

$${}_p\Psi_q[(\alpha_m, A_m)_{1,p} (\beta_m, B_m)_{1,q}; z] = \sum_{n=0}^{\infty} \left\{ \prod_{m=1}^{p} \Gamma(\alpha_m + nA_m) \right\} \left\{ \prod_{m=1}^{q} \Gamma(\beta_m + nB_m) \right\}^{-1} \frac{z^n}{n!}$$

for $z \in \mathbb{U}$. If $A_m = 1$, $(m = 1, \ldots, p)$ and $B_m = 1$, $(m = 1, ..., q)$, then we have the following obvious relationship:

$$\begin{aligned} \Omega_p \Psi_q[(\alpha_m, 1)_{1,p} (\beta_m, 1)_{1,q}; z] &\equiv {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \end{aligned} \tag{1.6}$$

where ${}_pF_q(\alpha_1,\dots\alpha_p;\beta_1,\dots,\beta_q;z)$ is the generalized hypergeometric function (see for details [4]) where $(\alpha)_n$ is the familiar Pochhammer symbol and

$$\Omega=\left(\prod_{m=0}^{p}\Gamma(\alpha_m)\right)^{-1}\left(\prod_{m=0}^{q}\Gamma(\beta_m)\right). \tag{1.7}$$

By using the generalized hypergeometric function, Dziok and Srivastava [4] introduced a linear operator which was subsequently extended by Dziok and Raina [3] using the Wright's generalized hypergeometric function .

Let $W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]:S\to S$ be a linear operator defined by

$$W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]\phi(z):=\{\Omega z\ {}_p\Psi_q[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q};z]\}*\phi(z),$$

then on using (1.4) and (1.7), we get

$$W[(\alpha_m,A_m)_{1,p};(\beta_m,B_m)_{1,q}]\phi(z)=z+\sum_{n=2}^{\infty}\sigma_n(\alpha_1)\ \lambda_n z^n, \tag{1.8}$$

where $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1)=\frac{\Omega\Gamma(\alpha_1+A_1(n-1))\dots\Gamma(\alpha_p+A_p(n-1))}{(n-1)!\Gamma(\beta_1+B_1(n-1))\dots\Gamma(\beta_q+B_q(n-1))} \tag{1.9}$$

and $\Omega$ is given by (1.7).

For the sake of convenience, we use the contracted notation $W_q^p[\alpha_1]$ to represent the following:

$$W_q^p[\alpha_1]\phi(z)=W[(\alpha_1,A_1),\dots,(\alpha_l,A_p);(\beta_1,B_1),\dots,(\beta_q,B_q)]\phi(z), \tag{1.10}$$

which is used throughout the sequel. The linear operator $W_q^p[\alpha_1]$ contains the Dziok-Srivastava operator (see [4]), and as special cases contains such further linear operators as the Hohlov operator, Carlson-Shaffer operator, Ruscheweyh derivative operator, generalized Bernardi-Libera-Livingston operator and the fractional derivative operator. Details and references about these operators can be found in [3] and [4].

In view of the relationship (1.6), and the linear operator (1.8) for the harmonic function $f=h+\overline{g}$ given by (1.1), we define the operator

$$W_q^p[\alpha_1]f(z)=W_q^p[\alpha_1]h(z)+\overline{W_q^p[\alpha_1]g(z)}, \tag{1.11}$$

and introduce below a new subclass $W_H([\alpha_1],\lambda,\gamma)$ of $\mathcal{H}$ in terms of the operator defined by (1.11) .

Let $W_H([\alpha_1],\gamma)$ denote a subclass of $\mathcal{H}$ consisting of functions of the form $f=h+\overline{g}$ given by (1.2) satisfying the condition that

$$\frac{\partial}{\partial\theta}\left(argW_q^p[\alpha_1]f(z)\right)>\gamma=\text{Re}\left\{\frac{z(W_q^p[\alpha_1]h'(z))-\overline{z(W_q^p[\alpha_1]g'(z))}}{W_q^p[\alpha_1]h(z)+\overline{(W_q^p[\alpha_1]g(z))}}\right\}\geq\gamma, \tag{1.12}$$

$$\left(z=re^{i\theta};\ 0\leq\theta<2\pi;\ 0\leq r<1;\ 0\leq\gamma<1;\ z\in\mathbb{U}\right),$$

where $W_q^p[\alpha_1]f(z)$ is given by (1.11). We also let $W_{\overline{H}}([\alpha_1],\gamma)=W_H([\alpha_1],\gamma)\cap\overline{\mathcal{H}}$.

In this paper, we obtain coefficient conditions for the classes $W_H([\alpha_1],\gamma)$ and $W_{\overline{H}}([\alpha_1],\gamma)$. We also establish a representation theorem, inclusion properties and distortion bounds for the class $W_{\overline{H}}([\alpha_1],\gamma)$.

## 2. Coefficient bounds

The following result gives a sufficient coefficient condition for a harmonic function $f$ to belong to the class $W_H([\alpha_1], \gamma)$.

**2.1. Theorem.** *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty} \left[ \frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n| \right] \sigma_n(\alpha_1) \leq 2 \tag{2.1}$$

*where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in W_H([\alpha_1], \gamma)$.*

*Proof.* Suppose the condition (2.1) holds true. To show that $f \in W_H([\alpha_1], \gamma)$, we show (in view of (1.12)) that

$$\Re \left\{ \frac{z\left(W_q^p[\alpha_1]\,h(z)\right)' - \overline{z\left(W_q^p[\alpha_1]\,g(z)\right)'}}{z\left(W_q^p[\alpha_1]\,h(z)\right)' + \overline{z\left(W_q^p[\alpha_1]\,g(z)\right)'}} \right\} = \Re\left\{ \frac{A(z)}{B(z)} \right\} \geq \gamma, \ \ (z \in \mathbb{U}),$$

and upon using (8) and (11), the expressions for $A(z)$ and $B(z)$ become

$$\begin{aligned} A(z) &= z(W_q^p[\alpha_1]h(z))' - \overline{z(W_q^p[\alpha_1]g(z))'} \\ &= z + \sum_{n=2}^{\infty} n\sigma_n(\alpha_1)a_n z^n - \sum_{n=1}^{\infty} n\sigma_n(\alpha_1)\overline{b}_n\overline{z}^n \end{aligned}$$

and

$$B(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)a_n z^n + \sum_{n=1}^{\infty} \sigma_n(\alpha_1)\overline{b}_n\overline{z}^n.$$

Using the fact that Re $\{w\} \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2), and performing elementary calculations, we find that

$$\begin{aligned} &|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\ &\qquad \geq 2(1-\gamma)|z| \left\{ 2 - \sum_{n=1}^{\infty} \left[ \frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n| \right] \sigma_n(\alpha_1)|z|^{n-1} \right\} \\ &\qquad \geq 2(1-\gamma) \left\{ 2 - \sum_{n=1}^{\infty} \left[ \frac{n-\gamma}{1-\gamma}|a_n| + \frac{n+\gamma}{1-\gamma}|b_n| \right] \sigma_n(\alpha_1) \right\} \\ &\qquad \geq 0, \end{aligned}$$

which implies that $f(z) \in W_H([\alpha_1], \gamma)$. □

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\gamma}{(n-\gamma)\sigma_n(\alpha_1)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{(n+\gamma)\sigma_n(\alpha_1)} \overline{y}_n(\overline{z})^n, \tag{2.3}$$

where $\sum\limits_{n=2}^{\infty} |x_n| + \sum\limits_{n=1}^{\infty} |y_n| = 1$, shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $W_H([\alpha_1],\gamma)$ because

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{(n-\gamma)\sigma_n(\alpha_1)}{1-\gamma}|a_n|+\frac{(n+\gamma)\sigma_n(\alpha_1)}{1-\gamma}|b_n|\right)&\\ &=1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|\\ &=2.\end{aligned}$$

Our next theorem gives a necessary and sufficient condition for functions of the form (1.3) to be in the class $W_{\overline{H}}([\alpha_1],\gamma)$.

**2.2. Theorem.** *For $a_1=1$ and $0\le\gamma<1$, $f=h+\overline{g}\in W_{\overline{H}}([\alpha_1],\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{n-\gamma}{1-\gamma}|a_n|+\frac{n+\gamma}{1-\gamma}|b_n|\right]\sigma_n(\alpha_1)\le 2. \tag{2.4}$$

*Proof.* Since $W_{\overline{H}}([\alpha_1],\gamma)\subset W_H([\alpha_1],\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f$ of the form (1.3), we notice that the condition

$$\Re\left\{\frac{z(W_q^p[\alpha_1]h'(z))-\overline{z(W_q^p[\alpha_1]g'(z))}}{W_q^p[\alpha_1]h(z)+\overline{(W_q^p[\alpha_1]g(z))}}\right\}\ge\gamma$$

implies that

$$\Re\left\{\frac{(1-\gamma)z-\sum_{n=2}^{\infty}(n-\gamma)\sigma_n(\alpha_1)a_nz^n-\sum_{n=1}^{\infty}(n+\gamma)\sigma_n(\alpha_1)\overline{b}_n\overline{z}^n}{z-\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_nz^n+\sum_{n=1}^{\infty}\sigma_n(\alpha_1)\overline{b}_n\overline{z}^n}\right\}\ge 0.$$

The above required condition must hold for all values of $z$ in $\mathbb{U}$. Upon choosing the values of $z$ on the positive real axis where $0\le z=r<1$, we must have

$$\frac{(1-\gamma)-\sum_{n=2}^{\infty}(n-\gamma)\sigma_n(\alpha_1)a_nr^{n-1}-\sum_{n=1}^{\infty}(n+\gamma)\sigma_n(\alpha_1)b_nr^{n-1}}{1-\sum_{n=2}^{\infty}\sigma_n(\alpha_1)a_nr^{n-1}+\sum_{n=1}^{\infty}\sigma_n(\alpha_1)b_nr^{n-1}}\ge 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for $r$ sufficiently close to 1. Hence, there exist $z_0=r_0$ in $(0,1)$ for which the quotient of (2.5) is negative. This contradicts the required condition for $f(z)\in W_{\overline{H}}([\alpha_1],\gamma)$. This completes the proof of the theorem. □

## 3. Distortion bounds and extreme points

By applying the condition (2.1), the following results (Theorem 3.1 and Theorem 3.3; Corollary 3.2) giving the distortion bounds and covering result for functions belonging to the class $W_{\overline{H}}([\alpha_1],\gamma)$, and the extreme points of the closed convex hulls of $W_{\overline{H}}([\alpha_1],\gamma)$ denoted by $\operatorname{clco} W_{\overline{H}}([\alpha_1],\gamma)$ can be proved by using similar steps of derivation as given in [5, 6, 7].

**3.1. Theorem.** *Let $f\in W_{\overline{H}}([\alpha_1],\gamma)$, then (for $|z|=r<1$),*

$$\begin{aligned}(1-b_1)r-\frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}-\frac{1+\gamma}{2-\gamma}b_1\right)r^2&\\ &\le|f(z)|\\ &\le(1+b_1)r+\frac{1}{\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{2-\gamma}-\frac{1+\gamma}{2-\gamma}b_1\right)r^2.\end{aligned}$$

**3.2. Corollary.** *If $f(z) \in W_{\overline{H}}([\alpha_1], \gamma)$. Then*

$$\left\{ w : |w| < \frac{2\sigma_2(\alpha_1) - 1 - [\sigma_2(\alpha_1) - 1]\gamma}{(2-\gamma)\sigma_2(\alpha_1)} - \frac{2\sigma_2(\alpha_1) - 1 - [\sigma_2(\alpha_1) + 1]\gamma}{(2-\gamma)\sigma_2(\alpha_1)} b_1 \right\} \subset f(U).$$

**3.3. Theorem.** *A function $f(z) \in \operatorname{clco} W_{\overline{H}}([\alpha_1], \gamma)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} \left( X_n h_n(z) + Y_n g_n(z) \right),$$

*where*

$$h_1(z) = z, h_n(z) = z - \frac{1-\gamma}{(n-\gamma)\sigma_n(\alpha_1)} z^n \ (n \geq 2),$$

$$g_n(z) = z + \frac{1-\gamma}{(n+\gamma)\sigma_n(\alpha_1)} \overline{z}^n, \ (n \geq 2),$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1, \ X_n \geq 0 \text{ and } Y_n \geq 0.$$

*In particular, the extreme points of $W_{\overline{H}}([\alpha_1], \gamma)$ are $\{h_n\}$ and $\{g_n\}$.*

## 4. Inclusion results

The following result gives the convex combinations of the class $W_{\overline{H}}([\alpha_1], \gamma)$.

**4.1. Theorem.** *The family $W_{\overline{H}}([\alpha_1], \gamma)$ is closed under convex combinations.*

*Proof.* Let $f_i \in W_{\overline{H}}([\alpha_1], \gamma)$, $(i = 1, 2, \ldots)$, where

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n + \sum_{n=2}^{\infty} |b_{i,n}| \overline{z}^n.$$

The convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i,n}| \right) z^n + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{i,n}| \right) \overline{z}^n,$$

provided that $\sum_{i=1}^{\infty} t_i$, $(0 \leq t_i \leq 1)$. Applying the inequality (2.4) of Theorem 2.2, we obtain

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(n-\gamma)\sigma_n(\alpha_1)}{1-\gamma} \left( \sum_{i=1}^{\infty} t_i |a_{i,n}| \right) + \sum_{n=1}^{\infty} \frac{(n+\gamma)\sigma_n(\alpha_1)}{1-\gamma} \left( \sum_{i=1}^{\infty} t_i |b_{i,n}| \right) \\ &\qquad = \sum_{i=1}^{\infty} t_i \left( \sum_{n=2}^{\infty} \frac{(n-\gamma)\sigma_n(\alpha_1)}{1-\gamma} |a_{i,n}| + \sum_{n=1}^{\infty} \frac{(n+\gamma)\sigma_n(\alpha_1)}{1-\gamma} |b_{i,n}| \right) \\ &\qquad \leq \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

and therefore, $\sum_{i=1}^{\infty} t_i f_i \in W_{\overline{H}}([\alpha_1], \gamma)$. □

**4.2. Theorem.** *Let $f(z), F(z) \in W_{\overline{H}}([\alpha_1], \gamma)$, $(0 \leq \delta \leq \gamma < 1)$, then*

$$f(z) * F(z) \in W_{\overline{H}}([\alpha_1], \gamma) \subset W_{\overline{H}}([\alpha_1], \delta).$$

*Proof.* Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \in W_{\overline{H}}([\alpha_1], \gamma)$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n \in W_{\overline{H}}([\alpha_1], \delta)$. Then $f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \overline{z}^n$.

From the assertion that $f(z)*F(z) \in W_{\overline{H}}([\alpha_1], \delta)$, we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. and in view of Theorem 2.2 and the inequality $0 \leq \delta \leq \gamma < 1$, we have

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(n-\delta)\sigma_n(\alpha_1)}{1-\delta} |a_n|\,|A_n| + \sum_{n=1}^{\infty} \frac{(n+\delta)\sigma_n(\alpha_1)}{1-\delta} |b_n|\,|B_n| \\ \leq \sum_{n=2}^{\infty} \frac{(n-\delta)\sigma_n(\alpha_1)}{1-\delta} |a_n| + \sum_{n=1}^{\infty} \frac{(n+\delta)\sigma_n(\alpha_1)}{1-\delta} |b_n| \\ \leq \sum_{n=2}^{\infty} \frac{(n-\gamma)\sigma_n(\alpha_1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{(n+\gamma)\sigma_n(\alpha_1)}{1-\gamma} |b_n| \leq 1,\end{aligned}$$

which implies by Theorem 2.2 that $f(z) \in W_{\overline{H}}([\alpha_1], \gamma)$. Hence

$$f(z) * F(z) \in W_{\overline{H}}([\alpha_1], \gamma) \subset W_{\overline{H}}([\alpha_1], \delta).$$

□

Lastly, we consider the closure property of the class $W_{\overline{H}}([\alpha_1], \gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \ (c > -1).$$

We prove the following result.

**4.3. Theorem.** *Let* $f(z) \in W_{\overline{H}}([\alpha_1], \gamma)$. *Then* $\mathcal{L}_c(f(z)) \in W_{\overline{H}}([\alpha_1], \gamma)$.

*Proof.* Using (1.1) and (1.3), we get

$$\begin{aligned}\mathcal{L}_c(f) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left[ h(t) + \overline{g(t)} \right] dt \\ &= \frac{c+1}{z^c} \left( \int_0^z t^{c-1} \left( t - \sum_{n=2}^{\infty} |a_n| t^n \right) dt + \overline{\int_0^z t^{c-1} \left( \sum_{n=1}^{\infty} |b_n| t^n \right) dt} \right) \\ &= z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n\end{aligned}$$

where

$$A_n = \frac{c+1}{c+n} |a_n| ; \ B_n = \frac{c+1}{c+n} |b_n|..$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty} \left[ \frac{n-\gamma}{1-\gamma} \left( \frac{c+1}{c+n} |a_n| \right) + \frac{n+\gamma}{1-\gamma} \left( \frac{c+1}{c+n} |b_n| \right) \right] \sigma_n(\alpha_1) \\ \leq \sum_{n=1}^{\infty} \left[ \frac{n-\gamma}{1-\gamma} |a_n| + \frac{n+\gamma}{1-\gamma} |b_n| \right] \sigma_n(\alpha_1) \\ \leq 2(1-\gamma),\end{aligned}$$

and since $f(z) \in W_{\overline{H}}([\alpha_1], \gamma)$, therefore by Theorem 2.2, $\mathcal{L}_c(f(z)) \in W_{\overline{H}}([\alpha_1], \gamma)$. □

**Concluding Remarks:** If $A_m = 1,\ (m = 1, \ldots, p)$ and $B_m = 1,\ (m = 1, \ldots, q)$ in (1.10), then as also pointed out in Section 1, Wright's generalized hypergeometric function contains, as further special cases, such other linear operators as the Hohlov operator, the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator, and so on. The various results presented in this paper would, therefore, provide extensions and generalizations of those results which were considered earlier for simpler harmonic function classes(see [5, 6, 8]). The details involved in the derivations of such specializations of the results presented in this paper are fairly straightforward, and are left to the interested reader.

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