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NEW RESULTS RELATED TO THE CONVEXITY AND STARLIKENESS OF THE BERNARDI INTEGRAL OPERATOR

Georgia Irina Oros*

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Abstract

In (Lewandowski, Z., Miller, S. S. and Zlotkiewicz, E. *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56**, 111–117, 1976) and (Pascu, N. N. *Alpha-close-to-convex functions*, Romanian–Finish Seminar on Complex Analysis, Springer Berlin, 331–335, 1979) it has been proved that the integral operator defined by S. D. Bernardi (*Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135**, 429–446, 1969) and given by

$$(1) \qquad L_\gamma(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad z \in U$$

preserves certain classes of univalent functions, such as the class of starlike functions, the class of convex functions and the class of close-to-convex functions.

In this paper we determine conditions that a function $f \in A$ needs to satisfy in order that the function F given by (1) be convex. We also prove two duality theorems between the classes $K\Big(-\dfrac{1}{2\gamma}\Big)$ and S^*, and between $K\Big(-\dfrac{1}{2\gamma}\Big)$ and $S^*\Big(-\dfrac{1}{2\gamma}\Big)$, respectively.

Keywords: Analytic function, Univalent function, Integral operator, Convex function, Starlike function, Close-to-convex function.

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*Department of Mathematics, University of Oradea, Str. Universităţii, No.1, 410087 Oradea, Romania. E-mail: georgia_oros_ro@yahoo.co.uk

1. Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbb{C}: \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. Also, let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$ and

$$S = \{f \in A: \ f \text{ is univalent in } U\}.$$

Let

$$K = \left\{ f \in A, \ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\},$$

denote the class of normalized convex functions in U,

$$S^* = \left\{ f \in A, \ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0, \ z \in U \right\}$$

denote the class of starlike functions in U,

$$K(\alpha) = \left\{ f \in A: \ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > \alpha, \ z \in U \right\}$$

denote the class of normalized convex functions of order α, where $\alpha < 1$,

$$S^*(\alpha) = \left\{ f \in A: \ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \ z \in U \right\}$$

denote the class of starlike functions of order α, with $\alpha < 1$ and

$$C = \left\{ f \in A: \ \exists\, \varphi \in K, \ \mathrm{Re}\,\frac{f'(z)}{\varphi'(z)} > 0, \ z \in U \right\}$$

denote the class of close-to-convex functions.

In order to prove our original results, we use the following lemmas:

1.1. Lemma. [3], [4], [6, Theorem 2.3.i, p. 35] *Let* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ *satisfy the condition*

$$\mathrm{Re}\,\psi(is, t; z) \le 0, \quad z \in U,$$

for $s, t \in \mathbb{R}$, $t \le -\frac{1}{2}(1 + s^2)$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ *satisfies*

$$\mathrm{Re}\,[p(z), zp'(z); z] > 0$$

then

$$\mathrm{Re}\,p(z) > 0, \quad z \in U.$$

More general forms of this lemma can be found in [6].

1.2. Lemma. [7, Theorem 4.6.3, p. 84] *The function* $f \in A$, *with* $f'(z) \neq 0$, $z \in U$ *is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta > -\pi, \quad z = re^{i\theta},$$

for all θ_1, θ_2 *with* $0 \le \theta_1 < \theta_2 \le 2\pi$ *and all* $r \in (0, 1)$.

If $L_\gamma : A \to A$ is the integral operator defined by $L_\gamma[f] = F$, where F is given by

$$L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt$$

and $\operatorname{Re}\gamma \geq 0$, $z \in U$, then it is well known that

(i) $L_\gamma[S^*] \subset S^*$,
(ii) $L_\gamma[K] \subset K$, and
(iii) $L_\gamma[C] \subset C$.

These results are obtained in [2] and [8].

2. Main results

We determine conditions such that, for a function $f \in A$, the image under the Bernardi integral operator is convex or starlike.

2.1. Theorem. *Let $f \in A$, $\gamma \geq 1$ and*

$$L_\gamma(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad z \in U.$$

If

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > -\frac{1}{2\gamma}, \quad z \in U, \tag{2}$$

then the function F given by (1) *is convex.*

Proof. Let $f \in A$, $f(z) = z + a_2z^2 + \cdots$, $z \in U$. Then, from (1), we have:

$$\begin{aligned} F(z) &= \frac{\gamma+1}{z^\gamma}\int_0^z (t + a_2t^2 + \ldots)t^{\gamma-1}dt \\ &= \frac{\gamma+1}{z^\gamma}\left(\frac{z^{\gamma+1}}{\gamma+1} + \frac{a_2z^{\gamma+2}}{\gamma+2} + \ldots\right) \\ &= z + A_2z^2 + \cdots, \end{aligned}$$

hence $F \in A$.

Since $\gamma \geq 1$, $0 \leq \frac{1}{2\gamma} \leq \frac{1}{2}$, we have $-\frac{1}{2} \leq -\frac{1}{2\gamma} < 0$, and

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] \geq -\frac{1}{2\gamma} > -\frac{1}{2}.$$

Then, according to Lemma 1.2 we obtain $f \in C$, hence it is univalent. If $f \in C$ then from (iii), we have $L_\gamma[f] = F \in C$, hence F is univalent.

From (1), we have

$$z^\gamma F(z) = (1+\gamma)\int_0^z f(t)t^{\gamma-1}dt, \quad z \in U. \tag{3}$$

By differentiating (3), we obtain

$$\gamma F(z) + zF'(z) = (\gamma+1)f(z), \quad z \in U. \tag{4}$$

By differentiating (4) and by a simple calculation, we obtain

$$F'(z)\left[1 + \frac{zF''(z)}{F'(z)}\right] + \gamma F'(z) = (\gamma+1)f'(z), \quad z \in U. \tag{5}$$

Let

$$1 + \frac{zF''(z)}{F'(z)} = p(z), \quad z \in U. \tag{6}$$

Then (5) is equivalent to

$$F'(z)[p(z)+\gamma] = (\gamma+1)f'(z), \quad z \in U. \tag{7}$$

Since $F'(z) \neq 0$, $p(z)+\gamma \neq 0$, $f \in C$, we have $f'(z) \neq 0$, $z \in U$, and by differentiating (7), we obtain

$$1 + \frac{zF''(z)}{F'(z)} + \frac{zp'(z)}{p(z)+\gamma} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \tag{8}$$

Using (6), we have

$$p(z) + \frac{zp'(z)}{p(z)+\gamma} = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in U. \tag{9}$$

Using (2), we obtain

$$\operatorname{Re}\left[p(z) + \frac{zp'(z)}{p(z)+\gamma}\right] = \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > -\frac{1}{2\gamma}, \quad z \in U,\ \gamma \geq 1$$

which is equivalent to

$$\operatorname{Re}\left[p(z) + \frac{zp'(z)}{p(z)+\gamma} + \frac{1}{2\gamma}\right] > 0, \quad z \in U,\ \gamma \geq 1. \tag{10}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z), zp'(z); z) = p(z) + \frac{zp'(z)}{p(z)+\gamma} + \frac{1}{2\gamma}, \quad z \in U,\ \gamma \geq 1. \tag{11}$$

Then (10) is equivalent to

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0, \quad z \in U. \tag{12}$$

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate

$$\begin{aligned}
\operatorname{Re}\psi(is, t; z) &= \operatorname{Re}\left[is + \frac{1}{2\gamma} + \frac{t}{is+\gamma}\right] \\
&= \operatorname{Re}\left[is + \frac{1}{2\gamma} + \frac{t(\gamma - is)}{\gamma^2+s^2}\right] \\
&= \frac{1}{2\gamma} + \frac{t\gamma}{\gamma^2+s^2} \\
&\leq \frac{1}{2\gamma} - \frac{\gamma(1+s^2)}{2(\gamma^2+s^2)} \\
&= \frac{\gamma^2+s^2-\gamma^2-\gamma^2 s^2}{2(\gamma^2+s^2)} \\
&= \frac{s^2(1-\gamma^2)}{2(\gamma^2+s^2)} \leq 0,
\end{aligned}$$

since $\gamma \geq 1$. Now, using Lemma 1.1 we get that $\operatorname{Re} p(z) > 0$, $z \in U$, i.e.

$$\operatorname{Re}\frac{zF''(z)}{F'(z)} + 1 > 0, \quad z \in U, \text{ hence } F \in K.$$

□

2.2. Remark. Since $\gamma \geq 1$, $0 < \frac{1}{2\gamma} \leq \frac{1}{2}$, and $-\frac{1}{2} \leq \frac{1}{-2\gamma} < 0$, we let

$$K\Big(-\frac{1}{2\gamma}\Big) = \left\{f \in A;\ \operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma},\ z \in U,\ \gamma \geq 1\right\}.$$

For the classes $K\Big(-\dfrac{1}{2\gamma}\Big)$ and S^* the following duality theorem can be proved:

2.3. Theorem. *The function $f \in A$ belongs to the class $K\Big(-\dfrac{1}{2\gamma}\Big)$ if and only if $F \in S^*$, where F is given by*

(13) $$F(z) = z[f'(z)]^{\frac{2\gamma}{2\gamma+1}}, \quad z \in U$$

and $[f'(z)]^{\frac{2\gamma}{2\gamma+1}}$ is the holomorphic determination for $[f'(z)]^{\frac{2\gamma}{2\gamma+1}}\Big|_{z=0} = 1$.

Proof. If $f \in K\Big(-\dfrac{1}{2\gamma}\Big)$, according to Lemma 1.2 we obtain $f \in C$, hence $f'(z) \neq 0$, $z \in U$. Then function F given by (13) is

$$F(z) = z[f'(z)]^{\frac{2\gamma}{2\gamma+1}} = z(1 + 2a_2 z + 3a_3 z^2 + \cdots)^{\frac{2\gamma}{2\gamma+1}},$$

holomorphic in U, with $F(0) = 0$, $F'(0) = 1$, $\dfrac{F(z)}{z} \neq 0$, $z \in U$.

Relation (13) is equivalent to

(14) $$\Big(\frac{F(z)}{z}\Big)^{\frac{2\gamma+1}{2\gamma}} = f'(z), \quad z \in U.$$

By differentiating (14), we obtain

$$\frac{2\gamma+1}{2\gamma}\left[\frac{zF'(z)}{F(z)} - 1\right] = \frac{zf''(z)}{f'(z)}, \quad z \in U,$$

which is equivalent to

$$\frac{2\gamma+1}{2\gamma} \cdot \frac{zF'(z)}{F(z)} - \frac{1}{2\gamma} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U.$$

Since

$$\frac{2\gamma+1}{2\gamma}\mathrm{Re}\,\frac{zF'(z)}{F(z)} - \frac{1}{2\gamma} = \mathrm{Re}\,\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma}$$

we obtain

$$\mathrm{Re}\,\frac{zF'(z)}{F(z)} > 0, \quad z \in U, \text{ i.e. } F \in S^*.$$

For the converse, if $F \in S^*$, let f be the function:

(15) $$f'(z) = \left[\frac{F(z)}{z}\right]^{\frac{2\gamma+1}{2\gamma}}, \quad z \in U, \ \gamma \geq 1.$$

Differentiating (15), we have

(16) $$\frac{zf''(z)}{f'(z)} = \frac{2\gamma+1}{2\gamma}\left[\frac{zF'(z)}{F(z)} - 1\right], \quad z \in U, \ \gamma \geq 1,$$

which is equivalent to

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{2\gamma+1}{2\gamma} \cdot \frac{zF'(z)}{F(z)} - \frac{1}{2\gamma}, \quad z \in U, \ \gamma \geq 1.$$

Since

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] = \frac{2\gamma+1}{2\gamma}\mathrm{Re}\,\frac{zF'(z)}{F(z)} - \frac{1}{2\gamma} > -\frac{1}{2\gamma}$$

we deduce

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma}, \text{ i.e. } f \in K\left(-\frac{1}{2\gamma}\right).$$

□

2.4. Theorem. *The function $f \in A$, belongs to the class $K\Big(-\dfrac{1}{2\gamma}\Big)$ if and only if $F \in S^*\left(-\dfrac{1}{2\gamma}\right)$, where F is given by*

$$F(z) = zf'(z), \quad z \in U. \tag{17}$$

Proof. If $f \in K\Big(-\dfrac{1}{2\gamma}\Big)$, then according to Lemma 1.2 we obtain $f \in C$. Hence $f'(z) \neq 0$, $z \in U$. Then F given by (17) is

$$F(z) = z(1 + 2a_2 z + \dots) = z + A_2 z^2 + \dots \in A,$$

holomorphic in U, with $F(0) = 0$, $F'(0) = 1$, $\dfrac{F(z)}{z} \neq 0$, $z \in U$. Relation (17) is equivalent to

$$\frac{F(z)}{z} = f'(z), \quad z \in U. \tag{18}$$

By differentiating (18), we obtain

$$\frac{zF'(z)}{F(z)} - 1 = \frac{zf''(z)}{f'(z)}, \quad z \in U,$$

which is equivalent to

$$\frac{zF'(z)}{F(z)} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \tag{19}$$

Since

$$\operatorname{Re} \frac{zF'(z)}{F(z)} = \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > -\frac{1}{2\gamma}, \quad z \in U, \ \gamma \geq 1$$

we deduce $F \in S^*\left(-\dfrac{1}{2\gamma}\right)$.

Conversely, if $F \in S^*\left(-\dfrac{1}{2\gamma}\right)$, we have

$$f'(z) = \frac{F(z)}{z}, \quad z \in U. \tag{20}$$

Differentiating (20), we obtain

$$\frac{zf''(z)}{f'(z)} = \frac{zF'(z)}{F(z)} - 1, \quad z \in U,$$

which is equivalent to

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{zF'(z)}{F(z)}, \quad z \in U.$$

Since

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) = \operatorname{Re} \frac{zF'(z)}{F(z)} > -\frac{1}{2\gamma}, \quad z \in U, \ \gamma \geq 1$$

we have $f \in K\Big(-\dfrac{1}{2\gamma}\Big)$. □

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