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STRONG VERSIONS OF THE THEOREMS OF WEIERSTRASS, MONTEL AND HURWITZ

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Abstract

In this article, using the notion of statistical convergence, we relax the hypotheses of the well-known theorems from classical complex analysis, such as Weierstrass' Theorem, Montel's Theorem and Hurwitz's Theorem. So, we obtain more powerful results than the classical ones in complex analysis.

Keywords: Statistical convergence, Weierstrass' theorem, Montel's theorem, Hurwitz's theorem.

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1. Introduction

In classical complex analysis, the theorems of Weierstrass, Montel and Hurwitz are of great use in very many contexts. The main goal of the present paper is to relax their strong hypotheses via the concept of A-statistical convergence, where A is a non-negative regular summability matrix. The A-statistical convergence method is defined in the following way. Let

$$A := [a_{jn}] \ (j, n \in \mathbb{N} := \{1, 2, 3, ...\})$$

be an infinite summability matrix. For a given (complex) sequence $x := \{x_n\}$, the A-*transform of* x, denoted by $Ax := \{(Ax)_j\}$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided that the series converges for each $j \in \mathbb{N}$. We say that A is *regular* (see [8]) if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$.

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Assume that A is a non-negative regular summability matrix. Then the sequence $x = \{x_n\}$ is called *A-statistically convergent* to L provided that, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\ |x_n - L| \geq \varepsilon} a_{jn} = 0. \tag{1.1}$$

We denote this limit by $\mathrm{st}_A - \lim_n x_n = L$ (*cf.* [4]). Many useful properties and some other generalizations of A-statistical convergence may be found in the papers [2, 9, 10, 11, 13]. Now we recall some basic properties of A-statistical convergence as follows:

- Actually, this convergence method is based on the concept of A-density. Recall that the *A-density* of a subset $K \subset \mathbb{N}$, denoted by $\delta_A\{K\}$, is given by

  $$\delta_A\{K\} = \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K(n)$$

  provided that the limit exists, where χ_K is the characteristic function of K; or equivalently

  $$\delta_A\{K\} = \lim_j \sum_{n \in K} a_{jn}.$$

  So, by (1.1), we easily see that $\mathrm{st}_A - \lim x = L$ if and only if

  $$\delta_A\{n : |x_n - L| \geq \varepsilon\} = 0$$

  for every $\varepsilon > 0$.
- If we take $A = C_1 := [c_{jn}]$, the *Cesáro matrix,* then A-statistical convergence reduces to the concept of *statistical convergence* (*cf.* [3]; see also [1, 5]). In this case, we write $\mathrm{st} - \lim x = L$ instead of $\mathrm{st}_{C_1} - \lim x = L$.
- Taking $A = I$, the *identity matrix*, A-statistical convergence coincides with ordinary convergence, i.e., $\mathrm{st}_I - \lim x = \lim x = L$.
- Observe that every convergent sequence (in the usual sense) is A-statistically convergent to the same value for any non-negative regular matrix A, but the converse is not always true. Actually, in [9], Kolk proved that A-statistical convergence is stronger than convergence when $A = [a_{jn}]$ is a non-negative regular summability matrix such that

  $$\lim_j \max_n \{a_{jn}\} = 0.$$

  So, one can construct a sequence that is A-statistically convergent but non-convergent.
- Not all properties of convergent sequences are true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded for an A-statistically convergent sequence.
- A characterization for statistical convergence, i.e., the case of $A = C_1$, was proved by Connor [1]: $\mathrm{st} - \lim x = L$ if and only if there exists a subsequence $\{x_{n_k}\}$ of x such that $\delta\{n_1, n_2, \ldots\} = 1$ and $\lim_k x_{n_k} = L$, where $\delta\{K\} := \delta_{C_1}\{K\}$. It is easy to check that a similar characterization is also valid for A-statistical convergence when A is any non-negative regular summability matrix.
- We say that a sequence $\{x_n\}$ is *A-statistically bounded* if there exists a number M such that $\delta_A\{n : |x_n| \leq M\} = 1$. Then, it is easy to see that every A-statistically bounded sequence contains an ordinary convergent subsequence. Indeed, if $x = (x_n)$ is A-statistically bounded, then, by the definition, it has a

bounded subsequence, say $y = (y_k) = (x_{n_k})$ for which $\delta_A\{n_k : k \in \mathbb{N}\} = 1$. If we apply the classical Bolzano-Weierstrass theorem to this sequence $y = (y_k)$, then y has a convergent subsequence, say $z = (z_j) = (y_{k_j})$, where $\{k_j : j \in \mathbb{N}\}$ is an increasing index set. Since $z = (z_j) = (y_{k_j}) = (x_{n_{k_j}})$, we conclude that z is a convergent subsequence of the original sequence $x = (x_n)$ that is A-statistically bounded. Some related results may also be found in the papers [6, 7].

With these properties, using A-statistical convergence rather than ordinary convergence enables us to obtain more powerful results than the classical ones. In the present paper, we use it to obtain stronger results than some well-known theorems from classical complex analysis, such as Weierstrass' Theorem, Montel's Theorem and Hurwitz's Theorem.

We now recall some basic concepts from the complex functions theory. Let Ω be an open set in $\mathbb{C}$, the set of all complex numbers. As usual, a complex valued function f defined on Ω is called *holomorphic* on Ω if, for every $a \in \Omega$, there exist a neighborhood U of a, $U \subset \Omega$, and a sequence $\{c_k\}$, $k = 0, 1, \ldots$, of complex numbers such that, for any $z \in U$, the series

$$\sum_{k=0}^{\infty} c_k (z-a)^k$$

converges to $f(z)$. By $\mathcal{H}(\Omega)$ we denote the set of all holomorphic functions on Ω. Assume that E is a discrete subset of Ω. A holomorphic function $f \in \mathcal{H}(\Omega \setminus E)$ is said to be *meromorphic* on Ω if, for any $a \in E$, there is a disc U with center a, $U \subset \Omega$, and two functions $g, h \in \mathcal{H}(U)$ such that h is not identically zero on U and $(h \cdot f)|_{U\setminus E} = g|_{U\setminus E}$. In other words, f can be locally written as the quotient of two holomorphic functions even at points of E where it is not *a priori* defined. Now let Ω be a connected open set in $\mathbb{C}$ and let f be a meromorphic function not identically zero on Ω. Suppose that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

is the Laurent expansion of f at a point $a \in \Omega$. Then, the *order* $\operatorname{ord}_a(f)$ of f at a is defined by

$$\operatorname{ord}_a(f) = \inf\{k : c_k \neq 0\}.$$

Throughout the paper, we use the following sets:

$$\begin{aligned} U(a,r) &:= \{u \in \mathbb{C} : |u-a| < r\}, \\ \overline{U}(a,r) &:= \{u \in \mathbb{C} : |u-a| \leq r\}, \\ C_r &:= \{u \in \mathbb{C} : |u-a| = r\}. \end{aligned}$$

Then we obtain the following results.

1.1. Theorem (*Modified Weierstrass' Theorem*)**.** *Let Ω be an open set in $\mathbb{C}$ and $A = [a_{jn}]$ a non-negative regular summability matrix. Let $\{f_n\}$ be a sequence of complex valued functions defined on Ω. Assume that*

$$\delta_A\{K\} = 1 \text{ with } K := \{n : f_n \in \mathcal{H}(\Omega)\}. \tag{1.2}$$

If, for any compact subset D of Ω,

$$\mathrm{st}_A - \lim_n \|f_n - f\|_D = 0 \tag{1.3}$$

for a certain function f, where $\|\cdot\|_D$ denotes the usual sup-norm on D, then f belongs to $\mathcal{H}(\Omega)$. Moreover, the derivative f_n' exists for every $n \in K$, and

$$\mathrm{st}_A - \lim_n \left\|f_n' - f'\right\|_D = 0$$

holds for any compact subset D of Ω.

1.2. Theorem (*Modified Montel's Theorem*)**.** *Let Ω be an open set in $\mathbb{C}$ and $A=[a_{jn}]$ a non-negative regular summability matrix. Let $\{f_n\}$ be a sequence of complex valued functions defined on Ω. Assume that* (1.2) *holds. If, for any compact subset D of Ω, there exists a positive number M_D such that*

$$\delta_A\left\{n : \|f_n\|_D \leq M_D\right\} = 1, \tag{1.4}$$

then there exists a subsequence $\{f_{n_k}\}$ of holomorphic functions on Ω, which converges uniformly to a function $f \in \mathcal{H}(\Omega)$ on every compact subset of Ω.

1.3. Theorem (*Modified Hurwitz's Theorem*)**.** *Let Ω be a connected open set in $\mathbb{C}$ and $A=[a_{jn}]$ a non-negative regular summability matrix. Assume that* (1.2) *and* (1.3) *hold. If, for every $n \in K$, f_n is everywhere nonzero on Ω, then, either $f \equiv 0$ or f has no zeros on Ω.*

1.4. Theorem. *Let Ω be a connected open set in $\mathbb{C}$ and $A=[a_{jn}]$ a non-negative regular summability matrix. Assume that* (1.2) *and* (1.3) *hold. If, for every $n \in K$, f_n is injective and f is non-constant, then f is also injective.*

1.5. Remarks. If we take $A = I$, the identity matrix, then condition (1.3) reduces to $\lim_n \|f_n - f\|_D = 0$ for every compact subset D of Ω, i.e., $\{f_n\}$ converges uniformly to f on every compact subset of Ω. If we also replace (1.2) by the stronger condition "$f_n \in \mathcal{H}(\Omega)$ for each $n \in \mathbb{N}$", then our Theorem 1.1 coincides with the classical theorem of Weierstrass.

In order to get the theorems of Montel and Hurwitz one can make similar choices in Theorems 1.2 and 1.3. According to our results, observe that f_n does not need to be holomorphic on Ω for each $n \in \mathbb{N}$, additionally infinitely many terms of $\{f_n\}$ need not be holomorphic on Ω provided that the A-density of the set consisting of these terms is zero. For example, we may choose $A = C_1$, the Cesáro matrix, and define $\{f_n\}$ on $\Omega = \mathbb{C}$ by

$$f_n(z) = \begin{cases} \overline{z}, & \text{if } n = m^2,\ (m = 1, 2, \ldots), \\ e^z/n, & \text{otherwise.} \end{cases} \tag{1.5}$$

Now take $K := \{n : n \neq m^2,\ m = 1, 2, \ldots\}$. Then it is clear that $\delta_{C_1}\{K\} = 1$ and $\mathrm{st} - \lim_n \|f_n - f\|_D = 0$, with $f = 0$ for every compact subset of $\mathbb{C}$. Observe that the sequence $\{f_n\}$ given by (1.5) satisfies all the hypotheses of Theorems 1.1, 1.2 and 1.3 but not of their classical ones.

2. Proofs of the main results

Proof of Theorem 1.1. Let $a \in \Omega$ and $r > 0$ be so that $\overline{U}(a, r) \subset \Omega$. We may write from (1.2) that

$$\delta_A\{\mathbb{N} \setminus K\} = 0. \tag{2.1}$$

Let $0 < \rho < r$. Then it follows from the Cauchy integral formula for a disc that

$$f_n(w) = \frac{1}{2\pi i} \oint\limits_{C_r} \frac{f_n(z)}{z - w} dz \qquad \text{for any } n \in K \text{ and } w \in \overline{U}(a, \rho). \tag{2.2}$$

Observe that $\left|\dfrac{1}{z-w}\right| \leq \dfrac{1}{r-\rho}$ for $z \in C_r$, $w \in \overline{U}(a, \rho)$ and by (1.3),

$$\mathrm{st}_A - \lim_n \|f_n - f\|_{C_r} = 0$$

since C_r is a compact subset of Ω. So we obtain that

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{C_r}\frac{f_n(z)}{z-w}dz-\frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}dz\right| &\le \frac{1}{2\pi}\oint_{C_r}\frac{|f_n(z)-f(z)|}{|z-w|}dz \\ &\le \frac{r}{r-\rho}\,\|f_n-f\|_{C_r}\end{aligned} \tag{2.3}$$

holds for every $n\in K$ and $w\in\overline{U}(a,\rho)$. Now, given $\varepsilon>0$, define the following sets:

$$S:=\left\{n:\left|\frac{1}{2\pi i}\oint_{C_r}\frac{f_n(z)}{z-w}dz-\frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}dz\right|\ge\varepsilon\right\},$$

$$T:=\left\{n:\|f_n-f\|_{C_r}\ge\frac{\varepsilon(r-\rho)}{r}\right\}.$$

Then, by (2.3), it is clear that $S\cap K\subseteq T\cap K$, which yields

$$\sum_{n\in S\cap K}a_{jn}\le\sum_{n\in T\cap K}a_{jn}\ \le\ \sum_{n\in T}a_{jn}\text{ for every }j\in\mathbb{N}.$$

Hence, taking the limit as $j\to\infty$ and using the fact that $\mathrm{st}_A-\lim_n\|f_n-f\|_{C_r}=0$, we get

$$\lim_j\sum_{n\in S\cap K}a_{jn}=0. \tag{2.4}$$

Furthermore, since

$$\sum_{n\in S}a_{jn}\le\sum_{n\in S\cap K}a_{jn}+\sum_{n\in S\cap(\mathbb{N}\setminus K)}a_{jn}\le\sum_{n\in S\cap K}a_{jn}+\sum_{n\in(\mathbb{N}\setminus K)}a_{jn},$$

it follows from (2.1) and (2.4) that

$$\lim_j\sum_{n\in S}a_{jn}=0,$$

which means

$$\mathrm{st}_A-\lim_n\left|\frac{1}{2\pi i}\oint_{C_r}\frac{f_n(z)}{z-w}dz-\frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}dz\right|=0\ \text{ for every }w\in\overline{U}(a,\rho),$$

or equivalently,

$$\mathrm{st}_A-\lim_n\frac{1}{2\pi i}\oint_{C_r}\frac{f_n(z)}{z-w}dz=\mathrm{st}_A-\lim_n f_n(w)=f(w)=\frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}dz$$

for every $w\in\overline{U}(a,\rho)$. Therefore, f is holomorphic on $U(a,\rho)$. Since Ω is open, we clearly see that $f\in\mathcal{H}(\Omega)$.

Now let $w\in\overline{U}(a,\rho)$. Then, for any $n\in K$, we may write that

$$f_n'(w)=\frac{1}{2\pi i}\oint_{C_r}\frac{f_n(z)}{(z-w)^2}dz.$$

Since $\left|\frac{1}{z-w}\right|^2\le\frac{1}{(r-\rho)^2}$ for $z\in C_r$ and $w\in\overline{U}(a,\rho)$, we obtain that

$$\left\|f_n'-f'\right\|_{\overline{U}(a,\rho)}\le\frac{r}{(r-\rho)^2}\,\|f_n-f\|_{C_r} \tag{2.5}$$

holds for every $n \in K$. Hence, applying a similar technique as used above, one can easily obtain from (2.5) and (2.1) that

$$\text{st}_A - \lim_n \left\| f_n' - f' \right\|_{\overline{U}(a,\rho)} = 0. \tag{2.6}$$

Now, for any $a \in \Omega$, we choose $r_a > 0$ such that $\overline{U}(a, r_a) \subset \Omega$. Let $\rho_a = \frac{1}{2} r_a$. In this case, it is clear that $\bigcup_{a \in \Omega} U(a, \rho_a) = \Omega$. Assume that D is any compact subset of Ω. Then, there exist finitely many points $a_1, a_2, \ldots, a_m \in D$ such that $D \subset \bigcup_{i=1}^{m} \overline{U}(a_i, \rho_{a_i})$. Hence,

$$\left\| f_n' - f' \right\|_D \leq \sum_{i=1}^{m} \left\| f_n' - f' \right\|_{\overline{U}(a_i,\rho_i)}, \text{ for each } i = 1, 2, \ldots, r \text{ and } n \in K. \tag{2.7}$$

By (2.6), since

$$\text{st}_A - \lim_n \left\| f_n' - f' \right\|_{\overline{U}(a_i,\rho_i)} = 0 \text{ for each } i = 1, 2, \ldots, m,$$

it follows from (2.7) and (2.1) that

$$\text{st}_A - \lim_n \left\| f_n' - f' \right\|_D = 0.$$

The theorem is proved. □

Proof of Theorem 1.2. Let $a \in \Omega$ and $r > 0$ be so that $\overline{U}(a, r) \subset \Omega$. For any fixed $n \in K$, we have

$$f_n(z) = \sum_{k=0}^{\infty} c_k(f_n)(z - a)^n,$$

where $c_k(f_n) := \frac{f_n^{(k)}(a)}{k!}$, $k = 0, 1, \ldots$ By Cauchy's inequality, there exists a positive number $M = M(r)$ such that

$$c_k(f_n) \leq \frac{M(r)}{r^k} \text{ for every } k = 0, 1, \ldots \text{ and } n \in K. \tag{2.8}$$

For $k = 0$, since $c_0(f_n) \leq M(r)$ for every $n \in K$ with $\delta_A(K) = 1$, the sequence $\{c_0(f_n)\}$ is A-statistically bounded. So, it has an ordinary convergent subsequence, say $\left\{ c_0 \left(f_{n_m^{(1)}} \right) \right\}$, where $\{n_m^{(1)} : m \in \mathbb{N}\}$ is an increasing index set whose terms are chosen from the set K. Notice that each member of this subsequence is holomorphic on Ω. Similarly, by induction, we can construct an index set $\{n_m^{(k)}\}$ such that $\{n_m^{(k)}\} \subset \{n_m^{(k-1)}\}$ $(k \geq 2)$ and $\{c_k(f_{n_m^{(k+1)}})\}$ converges in $\mathbb{C}$ as $m \to \infty$ for each $k = 0, 1, \ldots$ Therefore, the final part of the proof immediately follows from the proof of the classical Montel theorem (see, for instance, [12, pp. 34-35]). □

Proof of Theorem 1.3. By Theorem 1.1, it is clear that $f \in \mathcal{H}(\Omega)$. For the sake of contradiction, assume that f is not identically zero and that there is $a \in \Omega$ with $f(a) = 0$. In this case, we get that f is non-constant. Now choose $r > 0$ such that $\overline{U}(a, r) \subset \Omega$ and $\overline{U}(a, r) \cap \{z \in \Omega : f(z) = 0\} = \{a\}$. So, by the argument principle, we may write that

$$\frac{1}{2\pi i} \oint_{C_r} \frac{f'(z)}{f(z)}\, dz = \text{ord}_a(f) \geq 1.$$

On the other hand, since C_r is a compact subset of Ω, it follows from Theorem 1.1 that

$$\text{st}_A - \lim_n \left\| \frac{f_n'}{f_n} - \frac{f'}{f} \right\|_{C_r} = 0,$$

which implies that

$$\mathrm{st}_A - \lim_n \frac{1}{2\pi i} \oint_{C_r} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \oint_{C_r} \frac{f'(z)}{f(z)}\, dz \neq 0. \tag{2.9}$$

However, since for each $n \in K$, f_n is everywhere nonzero on Ω, the argument principle gives that

$$\frac{1}{2\pi i} \oint_{C_r} \frac{f_n'(z)}{f_n(z)} dz = 0 \text{ for each } n \in K.$$

Using the fact that $\delta_A\{K\} = 1$, we have

$$\mathrm{st}_A - \lim_n \frac{1}{2\pi i} \oint_{C_r} \frac{f_n'(z)}{f_n(z)} dz = 0,$$

which contradicts with (2.9). □

Proof of Theorem 1.4. By Theorem 1.1, we have $f \in \mathcal{H}(\Omega)$. Suppose that f is non-constant and there are points $a, b \in \Omega$ with $a \neq b$ such that $f(a) = f(b) = \beta$. Now choose $r > 0$ such that $\overline{U}(a,r) \cup \overline{U}(b,r) \subset \Omega$ and $\overline{U}(a,r) \cap \overline{U}(b,r) = \phi$. Then, by the principle of analytic continuation, we immediately obtain that $f|_{U(a,r)}$ and $f|_{U(b,r)}$ are non-constant. By Theorem 1.3, for large $n \in K$, the function $f_n - \beta$ has a zero in $U(a,r)$ and one in $U(b,r)$, so that f_n is not injective for $n \in K$. This is a contradiction. □

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