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# ON MEROMORPHIC HARMONIC STARLIKE FUNCTIONS WITH MISSING COEFFICIENTS

Hakan Bostancı[*][†] and Metin Öztürk [‡]

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**Abstract**

In this paper, we introduce a new class of meromorphic harmonic starlike functions with missing coefficients in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. We obtain coefficient inequalities, a distortion theorem and a closure theorem. In addition, we investigate some properties of this class.

**Keywords:** Harmonic functions, Meromorphic functions, Starlike functions.

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## 1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (see [3]). In [5], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping $f$ must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k},$$

[*]Department of Mathematics, Faculty of Science and Art, Dumlupınar University, 43100 Kütahya, Turkey. E-mail: `hakanbostanci16@gmail.com`

[†]Corresponding Author

[‡]Department of Mathematics, Faculty of Science and Art, Uludağ University, 16059 Bursa, Turkey. E-mail: `ometin@uludag.edu.tr`

$0 \le |\beta| < |\alpha|,\ A \in \mathbb{C}$. Also, Jahangiri and Silverman [11], Jahangiri [7] and Murugusundaramoorthy [9, 10] have studied classes of meromorphic harmonic functions.

Let $MH_p$ denote the class of functions harmonic, univalent, sense-preserving and meromorphic in $U^* = \{z : 0 < |z| < 1\}$ and which have the representation

$$f(z) = h(z) + \overline{g(z)} = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} + \overline{\sum_{k=1}^{\infty} b_{p+k}\, z^{p+k}}, \tag{1.1}$$

where $a_{-1} \neq 0,\ p \in \mathbb{N} = \{1, 2, \ldots\}$.

Notice that, if we take $p = 0$, then we are not missing any coefficients. Also, if we substitute $p = 0$ and $a_{-1} = 1$ in the above representation (1.1), then we have

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k\, z^k + \overline{\sum_{k=1}^{\infty} b_k\, z^k}, \tag{1.2}$$

therefore $f \in MH$. The class $MH$ is defined and some subclasses are studied by Bostanci *et. al.* in [2]. Otherwise, if we define the composite function $w(z) = f(1/z)$, then $w(z)$ is a harmonic, sense preserving, univalent mapping in $\tilde{U}$ and $w \in \Sigma''$. This class $\Sigma''$ is investigated by Hengartner and Schober in [5].

A function $f(z) \in MH_p$ is said to be in the subclass $MHS_p^*(\alpha)$ of meromorphic harmonic $\alpha$-starlike functions in $U^*$ if it satisfies the condition

$$\Re\left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > \alpha, \tag{1.3}$$

where $0 \le \alpha < 1$. If $\alpha = 0$ then this class is called the class of meromorphic harmonic starlike functions. We denote the meromorphic harmonic starlike functions by $MHS^*$. This classification (1.3) for harmonic univalent functions was first used by Jahangiri [6].

The class of meromorphic functions has been studied by various authors, including Joshi and Sangle [7], Darwish [5], Uraleggaddi and Somanatha [12], and Aouf and Hossen[4]. In this paper, we define a new operator $H$ for meromorphic harmonic functions. Also, we define the classes $MH_p^n(\alpha)$ and $\overline{MH}_p^n(\alpha)$. Then, we investigate some properties of these classes, such as coefficient estimates and a distortion theorem.

We define the new operator $H$ for harmonic functions as follows:

$$H^0 f(z) = f(z), H^1 f(z) = Hf(z) = \frac{(z^2 h(z))'}{z} - \overline{z^3 \left(\frac{g(z)}{z^2}\right)'},$$

and for $n = 2, \ldots,$

$$H^n f(z) = H(H^{n-1} f(z)).$$

Hence, we obtain for $n = 0, 1, \ldots,$

$$H^n f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} (p+k+2)^n a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} (p+k-2)^n b_{p+k} z^{p+k}}.$$

Using the operator $H$, we now make the following definition:

For $0 \le \alpha < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $MH_p^n(\alpha)$ denotes the class consisting of those functions in $MH_p$ satisfying

$$\Re\left\{ \frac{H^{n+1} f(z)}{H^n f(z)} - 2 \right\} < -\alpha \tag{1.4}$$

for $z \in U^*$.

Also, let $\overline{MH}_p^n(\alpha)$ be the subclass of $MH_p^n(\alpha)$ which consisting of meromorphic harmonic functions of the form

$$f_n(z) = h(z) + (-1)^n\overline{g(z)} = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k}\, z^{p+k}} \tag{1.5}$$

where $a_{-1} > 0, a_{p+k} \geq 0,\ b_{p+k} \geq 0,\ p \in \mathbb{N}$.

## 2. Some results for the class $MH_p$

**2.1. Theorem.** *If $f \in MH_p$, then the diameter $D_f$ of $\mathbb{C} \setminus f(U^*)$ satisfies*

$$D_f \geq 2|a_{-1}|.$$

*This estimate is sharp for $f(z) = a_{-1}z^{-1}$.*

*Proof.* Let $D_f(r)$ be the diameter of $f(|z| = r)$, $0 < r < 1$, and let $D_f^*(r) = \max_{|z|=r} |f(z) - f(-z)|$. Then $D_f(r) \searrow D_f$ as $r \to 1$ and $D_f(r) \geq D_f^*(r)$. Since

$$\begin{aligned}
[D_f^*(r)]^2 &\geq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) - f(re^{-i\theta})|^2 d\theta \\
&= \begin{cases} 4\left[\frac{|a_{-1}|^2}{r^2} + \sum\limits_{k=1}^{\infty}\left(|a_{p+2k-1}|^2 + |b_{p+2k-1}|^2\right) r^{2(p+2k-1)}\right], & p \text{ is odd} \\ 4\left[\frac{|a_{-1}|^2}{r^2} + \sum\limits_{k=1}^{\infty}\left(|a_{p+2k}|^2 + |b_{p+2k}|^2\right) r^{2(p+2k)}\right], & p \text{ is even} \end{cases} \\
&\geq 4|a_{-1}|^2 r^{-2},
\end{aligned}$$

we conclude that $D_f \geq 2|a_{-1}|$. □

**2.2. Corollary.** *If we substitute $p = 0,\ a_{-1} = 1$ and $w(z) = f(1/z)$ in the above theorem, then we have $w \in \Sigma''$ and*

$$D_w \geq 2|1 + b_1|.$$ □

**2.3. Theorem.** *If $f \in MH_p$ has expansion* (1.1) *then*

$$\sum_{k=1}^{\infty}(p+k)(|a_{p+k}|^2 - |b_{p+k}|^2) \leq |a_{-1}|^2.$$

*Equality occurs if and only if $\mathbb{C} \setminus f(U^*)$ has zero area.*

*Proof.* The area of the omitted set is

$$\begin{aligned}
\lim_{r\to 1}\frac{1}{2i}\int_{|z|=r} \bar{f} df &= \lim_{r\to 1}\left[\frac{1}{2i}\int_{|z|=r} \bar{h}h' dz + \frac{1}{2i}\int_{|z|=r} g\overline{g'} d\bar{z}\right] \\
&= \pi\left[-|a_{-1}|^2 + \sum_{k=1}^{\infty}(p+k)(|a_{p+k}|^2 - |b_{p+k}|^2)\right] \\
&\leq 0.
\end{aligned}$$

□

**2.4. Corollary.** *If we substitute $p = 0,\ a_{-1} = 1$ and $w(z) = f(1/z)$ in the above theorem, then we have $w \in \Sigma''$ and*

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2\Re b_1.$$ □

## 3. Coefficient inequalities

**3.1. Theorem.** *If $f(z) = h(z) + \overline{g(z)}$ is of the form* (1.1) *and the condition*

$$\sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha)\,|a_{P+k}| + (p+k-2)^n (p+k-\alpha)\,|b_{p+k}|]\,| \leq (1-\alpha)|a_{-1}| \tag{3.1}$$

*is satisfied then $f(z) \in MH_p^n(\alpha)$.*

*Proof.* Set

$$p(z) = \begin{cases} -\dfrac{H^{n+1}f(z)}{H^n f(z)} + 2 - \alpha, & z \neq 0, \\ 1-\alpha, & z = 0. \end{cases}$$

We must show that if the conditation (3.1) is satisfied then $f(z) \in MH_p^n(\alpha)$. Therefore, it is sufficient to show that $p(z)$ is in the class $PH$ which is the class of harmonic functions with positive real part, or equivalently prove that

$$|p(z)+1| > |p(z)-1|, \ z \in U. \tag{3.2}$$

From (7) we obtain, for $z \in U^*$,

$$\frac{\left|-H^{n+1}f(z) + (2-\alpha+1)H^n f(z)\right|}{|H^n f(z)|} > \frac{\left|-H^{n+1}f(z) + (2-\alpha-1)H^n f(z)\right|}{|H^n f(z)|}. \tag{3.3}$$

Since,

$$\begin{aligned} &\left|-H^{n+1}f(z) + (2-\alpha+1)H^n f(z)\right| \\ &\quad = \left| \frac{(\alpha-2)a_{-1}}{z} + \sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha-1)a_{p+k}\, z^{p+k} \right. \\ &\qquad \left. +(-1)^n \sum_{k=1}^{\infty}(p+k-2)^n(p+k+1-\alpha)\bar{b}_{p+k}\,\bar{z}^{p+k} \right| \\ &\quad \geq \frac{(2-\alpha)|a_{-1}|}{|z|} - \sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha-1)|a_{p+k}||\,z|^{p+k} \\ &\qquad - \sum_{k=1}^{\infty}(p+k-2)^n(p+k+1-\alpha)|\bar{b}_{p+k}||\,\bar{z}|^{p+k} \\ &\quad > |a_{-1}| + \sum_{k=1}^{\infty}(p+k+2)^n|a_{p+k}| - \sum_{k=1}^{\infty}(p+k-2)^n|b_{p+k}|, \end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
&\left|-H^{n+1}f(z)+(2-\alpha-1)H^n f(z)\right| \\
&\qquad = \left|\frac{\alpha a_{-1}}{z}+\sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha+1)a_{p+k}\,z^{p+k}\right. \\
&\qquad\qquad \left.+\,(-1)^n\sum_{k=1}^{\infty}(p+k-2)^n(p+k-1-\alpha)\bar{b}_{p+k}\,\bar{z}^{p+k}\right| \\
&\qquad \le \frac{\alpha|a_{-1}|}{|z|}+\sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha+1)|a_{p+k}||\,z|^{p+k} \\
&\qquad\qquad +\sum_{k=1}^{\infty}(p+k-2)^n(p+k-1-\alpha)|\bar{b}_{p+k}||\,\bar{z}|^{p+k} \\
&\qquad < |a_{-1}|+\sum_{k=1}^{\infty}(p+k+2)^n|a_{p+k}|-\sum_{k=1}^{\infty}(p+k-2)^n|b_{p+k}|,
\end{aligned}
\tag{3.5}
$$

we see that inequality (3.1) holds in $U^*$ by (3.4) and (3.5). The case $z=0$ is trivial. Hence $p(z)\in PH$. So the proof of Theorem 3.1 is complete. □

**3.2. Corollary.** *If we substitute $p=0,\ n=0,\ \alpha=0$ and $a_{-1}=1$ in the above theorem, then we have $\sum_{k=1}^{\infty}k(|a_k|+|b_k|)\le 1$ and $f\in MHS^*$, therefore $f$ harmonic starlike in $U^*$* [2, page 373, Theorem 2.1].

**3.3. Theorem.** *Let the function $f_n(z)$ to be defined by* (1.5). *A necessary and sufficient condition for $f_n(z)\in\overline{MH}_p^n(\alpha)$ is that*

$$
\sum_{k=1}^{\infty}\left[(p+k+2)^n(p+k+\alpha)a_{p+k}+(p+k-2)^n(p+k-\alpha)b_{p+k}\right]\le(1-\alpha)a_{-1}. \tag{3.6}
$$

*The estimate (3.6) is sharp and the equality is attained for the function*

$$
f(z)=\frac{a_{-1}}{z}+\frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}-\frac{(1-\alpha)a_{-1}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k}.
$$

*Proof.* In view of Theorem 3.1, it is sufficient to prove the "only if" part, since $\overline{MH}_p^n(\alpha)\subset MH_p^n(\alpha)$. Assume that $f_n(z)\in\overline{MH}_p^n(\alpha)$. Let $z$ be a complex number. If $\Re(z)>0$ then $\Re(1/z)>0$. Thus from (1.4) we obtain

$$
\Re\left\{-\frac{H^{n+1}f(z)}{H^n f(z)}+(2-\alpha)\right\}>0
$$

and

$$
\begin{aligned}
0 &< \Re\left\{\frac{H^n f(z)}{-H^{n+1}f(z)+(2-\alpha)H^n f(z)}\right\} \\
&\le \left|\frac{H^n f(z)}{-H^{n+1}f(z)+(2-\alpha)H^n f(z)}\right| \\
&= \left|\frac{a_{-1}+\sum_{k=1}^{\infty}(p+k+2)^n a_{p+k}\,z^{p+k+1}+|z|^2\sum_{k=1}^{\infty}(p+k-2)^n b_{p+k}\,\bar{z}^{p+k-1}}{D}\right|,
\end{aligned}
$$

where

$$D = (1-\alpha)a_{-1} - \sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha)a_{p+k}\, z^{p+k+1} + |z|^2\sum_{k=1}^{\infty}(p+k-2)^n(p+k-\alpha)b_{p+k}\, \bar{z}^{n+k-1}.$$

Hence

$$0 \leq \frac{a_{-1} + \sum_{k=1}^{\infty}(p+k+2)^n a_{p+k} + \sum_{k=1}^{\infty}(p+k-2)^n b_{p+k}}{(1-\alpha)a_{-1} - \sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha)a_{p+k} - \sum_{k=1}^{\infty}(p+k-2)^n(p+k-\alpha)b_{p+k}}. \tag{3.7}$$

From (3.7), we must have

$$\sum_{k=1}^{\infty}[(p+k+2)^n(p+k+\alpha)a_{p+k} + (p+k-2)^n(p+k-\alpha)b_{p+k}] \leq (1-\alpha)a_{-1}.$$

Hence, the proof is completed. □

**3.4. Theorem.** *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1,\ p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then* $\overline{MH}_p^n(\alpha_2) \subset \overline{MH}_p^n(\alpha_1)$.

*Proof.* Let the function $f_n(z)$ defined by (1.5) be in the class $\overline{MH}_p^n(\alpha_2)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 3.3, we have

$$\sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha_2)a_{p+k} + (-1)^n(p+k-2)^n(p+k-\alpha_2)b_{p+k} \leq (1-\alpha_2)a_{-1}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha_2)a_{p+k} + (-1)^n(p+k-2)^n(p+k-\alpha_2)b_{p+k} &\\ \leq \frac{(1-\alpha_2)a_{-1}}{p+1-\alpha_2}&\\ \leq a_{-1}.&\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha_1)a_{p+k} + (-1)^n(p+k-2)^n(p+k-\alpha_1)b_{p+k}\\ &\quad = \sum_{k=1}^{\infty}(p+k+2)^n(p+k+\alpha_2)a_{p+k} + (-1)^n(p+k-2)^n(p+k-\alpha_2)b_{p+k}\\ &\qquad + \delta\left(\sum_{k=1}^{\infty}(-1)^n(p+k-2)^n b_{p+k} + (p+k+2)^n a_{p+k}\right)\\ &\quad \leq (1-\alpha_2)a_{-1}.\end{aligned}$$

□

**3.5. Theorem.** *Let* $0 \leq \alpha < 1,\ p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then* $\overline{MH}_p^{n+1}(\alpha) \subset \overline{MH}_p^n(\alpha)$.

*Proof.* The proof of Theorem 3.5 also follows from Theorem 3.3. □

## 4. Distortion theorem

**4.1. Theorem.** *Let the function* $f_n(z)$ *be in the class* $\overline{MH}_p^n(\alpha)$. *Then, for* $0 < |z| = r < 1$, *we have*

$$\frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p} \leq |f_n(z)| \leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p} \tag{4.1}$$

*where equalities hold for the functions*

$$f(z) = \frac{a_{-1}}{z} \pm \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}z, \ z = r$$

*Proof.* In view of Theorem 3.3, for $0 < |z| = r < 1$,

$$\begin{aligned}
|f_n(z)| &= \left|\frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k}\, z^{p+k}}\right| \\
&\leq \frac{a_{-1}}{r} + r^{1+p} \sum_{k=1}^{\infty}(a_{p+k} + b_{p+k}) \\
&\leq \frac{a_{-1}}{r} + \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \sum_{k=1}^{\infty}(p-1)^n(p+1-\alpha)(a_{p+k} + b_{p+k}) \\
&\leq \frac{a_{-1}}{r} + \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \\
&\quad \times \sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k} + (p+k-2)^n(p+k-\alpha)b_{p+k}] \\
&\leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p}
\end{aligned}$$

and

$$\begin{aligned}
|f_n(z)| &= \left|\frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k}\, z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k}\, z^{p+k}}\right| \\
&\geq \frac{a_{-1}}{r} - r^{1+p} \sum_{k=1}^{\infty}(a_{p+k} + b_{p+k}) \\
&\geq \frac{a_{-1}}{r} - \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \sum_{k=1}^{\infty}(p-1)^n(p+1-\alpha)(a_{p+k} + b_{p+k}) \\
&\geq \frac{a_{-1}}{r} - \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \\
&\quad \times \sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k} + (p+k-2)^n(p+k-\alpha)b_{p+k}] \\
&\geq \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p}.
\end{aligned}$$

Thus, inequality (4.1) is obtained. □

## 5. Closure theorem

Let the function $f_{n,j}(z)$ be defined by

$$f_{n,j}(z) = \frac{a_{-1,j}}{z} + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k,\,j} z^{p+k}}, \ j = 1, 2, \ldots, m, \tag{5.1}$$

for $z \in U^*$.

Now we shall prove the following result for the closure of such a function in the class $\overline{MH}_p^n(\alpha)$.

**5.1. Theorem.** *Let the function defined by (5.1) be in the class $\overline{MH}_p^n(\alpha)$. Then the function $F_n(z)$ defined by*

$$F_n(z) = \frac{c_{-1}}{z} + \sum_{k=1}^{\infty} r_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} s_{p+k} z^{p+k}}$$

*is a member of the class $\overline{MH}_p^n(\alpha)$, where*

$$c_{-1} = \frac{1}{m}\sum_{j=1}^{m} a_{-1,j},\ \ r_{p+k} = \frac{1}{m}\sum_{j=1}^{m} a_{p+k,j} \text{ and } s_{p+k} = \frac{1}{m}\sum_{j=1}^{m} b_{p+k,j}.$$

*Proof.* Since $f_{n,j}(z) \in \overline{MH}_p^n(\alpha)$, it follows from Theorem 3.3 that

$$\sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k,j} + (p+k-2)^n(p+k-\alpha)b_{p+k,j}] \le (1-\alpha)a_{-1,j}.$$

Hence,

$$\begin{aligned}
&\sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)r_{P+k} + (p+k-2)^n(p+k-\alpha)s_{p+k}] \\
&\quad = \frac{1}{m}\sum_{j=1}^{m}\left[\sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k,j} + (p+k-2)^n(p+k-\alpha)b_{p+k,j}]\right] \\
&\quad \le (1-\alpha)\frac{1}{m}\sum_{j=1}^{m} a_{-1,j} \\
&\quad = (1-\alpha)c_{-1},
\end{aligned}$$

which implies that $F_n(z) \in \overline{MH}_p^n(\alpha)$. □

**5.2. Theorem.** *The class $\overline{MH}_p^n(\alpha)$ is a convex set.*

*Proof.* Let the function $f_{n,j}(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\overline{MH}_p^n(\alpha)$. It is sufficient to prove that the function

$$H(z) = \lambda f_{n,1}(z) + (1-\lambda) f_{n,2}(z), \qquad 0 \le \lambda \le 1,$$

is also in the class $\overline{MH}_p^n(\alpha)$. Since, for $0 \le \lambda \le 1$,

$$\begin{aligned}
H(z) = \frac{\lambda a_{-1,1} + (1-\lambda)a_{-1,2}}{z} &+ \sum_{k=1}^{\infty} [\lambda a_{p+k,1} + (1-\lambda)a_{p+k,2}]\, z^{p+k} \\
&+ (-1)^n \overline{\sum_{k=1}^{\infty} [\lambda b_{p+k,1} + (1-\lambda)b_{p+k,2}] z^{p+k}},
\end{aligned}$$

with the aid of Theorem 3.3, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\{(p+k+2)^n(p+k+\alpha)[\lambda a_{p+k,1}+(1-\lambda)a_{p+k,2}] \\
&\qquad\qquad +(p+k-2)^n(p+k-\alpha)[\lambda b_{p+k,1}+(1-\lambda)b_{p+k,2}]\} \\
&= \lambda\sum_{k=1}^{\infty}[(p+k+2)^n(p+k+\alpha)a_{p+k,1}+(p+k-2)^n(p+k-\alpha)b_{p+k,1}] \\
&\quad + (1-\lambda)\sum_{k=1}^{\infty}[(p+k+2)^n(p+k+\alpha)a_{p+k,2}+(p+k-2)^n(p+k-\alpha)b_{p+k,2}] \\
&\le (1-\alpha)\{\lambda a_{-1,1}+(1-\lambda)a_{-1,2}\}.
\end{aligned}$$

Hence $H(z)\in\overline{MH}_p^n(\alpha)$. This completes the proof of Theorem 5.2. □

**5.3. Theorem.** *Let, for $p\ge 2$ and $z\in U^*$,*

$$\begin{aligned}
h_p(z) &= g_p(z) = \frac{a_{-1}}{z}, \\
h_{p+k}(z) &= \frac{a_{-1}}{z} + \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}, \text{ and,} \\
g_{p+k}(z) &= \frac{a_{-1}}{z} + \frac{(-1)^n(1-\alpha)a_{-1}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k}.
\end{aligned}$$

*Then $f_n(z)\in\overline{MH}_p^n(\alpha)$ if and only if it can be expressed in the form*

$$f_n(z) = \sum_{k=0}^{\infty}[x_{p+k}h_{p+k}(z)+y_{p+k}g_{p+k}(z)],$$

*where*

$$x_{p+k}\ge 0,\ y_{p+k}\ge 0,\ \text{and}\ \sum_{k=0}^{\infty}(x_{p+k}+y_{p+k})=1.$$

*Proof.* Let $f_n(z)=\sum_{k=0}^{\infty}[x_{p+k}h_{p+k}(z)+y_{p+k}g_{p+k}(z)]$, with

$$x_{p+k}\ge 0,\ y_{p+k}\ge 0 \text{ and } \sum_{k=0}^{\infty}(x_{p+k}+y_{p+k})=1.$$

Then, we have

$$\begin{aligned}
f_n(z) &= \sum_{k=0}^{\infty}[x_{p+k}h_{p+k}(z)+y_{p+k}g_{p+k}(z)] \\
&= x_p h_p(z) + y_p g_p(z) + \sum_{k=1}^{\infty}(x_{p+k}+y_{p+k})\frac{a_{-1}}{z} \\
&\quad + \sum_{k=1}^{\infty}\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}z^{p+k} + \sum_{k=1}^{\infty}\frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k} \\
&= \sum_{k=0}^{\infty}(x_{p+k}+y_{p+k})\frac{a_{-1}}{z} + \sum_{k=1}^{\infty}\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}z^{p+k} \\
&\qquad + \sum_{k=1}^{\infty}\frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k}.
\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}\left[(p+k+2)^n(p+k+\alpha)\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}\right.$$
$$\left.+(p+k-2)^n(p+k-\alpha)\frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}\right]$$
$$=(1-\alpha)a_{-1}\sum_{k=1}^{\infty}(x_{p+k}+y_{p+k})\le(1-\alpha)a_{-1},$$

by Theorem 3.3, $f_n(z)\in\overline{MH}_p^n(\alpha)$.

Conversely, we suppose that $f_n(z)\in\overline{MH}_p^n(\alpha)$ and

$$a_{p+k}=\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)},\ b_{p+k}=\frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}$$

for $k=1,2,\ldots$. Hence, we obtain

$$f_n(z)=h(z)+(-1)^n\overline{g(z)}=\frac{a_{-1}}{z}+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}+(-1)^n\overline{\sum_{k=1}^{\infty}b_{p+k}z^{p+k}}$$
$$=\frac{a_{-1}}{z}+\sum_{k=1}^{\infty}\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}$$
$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k}$$
$$=\sum_{k=0}^{\infty}(x_{p+k}+y_{p+k})\frac{a_{-1}}{z}+\sum_{k=1}^{\infty}\frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}$$
$$+\sum_{k=1}^{\infty}\frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}\bar{z}^{p+k}$$
$$=\frac{a_{-1}}{z}x_p+\frac{a_{-1}}{z}y_p+\sum_{k=1}^{\infty}\left(\frac{a_{-1}}{z}+\frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}\right)x_{p+k}z^{p+k}$$
$$+\sum_{k=1}^{\infty}\left(\frac{a_{-1}}{z}+\frac{(-1)^n(1-\alpha)a_{-1}}{(p+k-2)^n(p+k-\alpha)}\right)y_{p+k}\bar{z}^{p+k}$$
$$=\sum_{k=0}^{\infty}\left[x_{p+k}h_{p+k}(z)+y_{p+k}g_{p+k}(z)\right].$$

This completes the proof of Theorem 5.3. □

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