Hacettepe Journal of Mathematics and Statistics
Volume 38 (1) (2009), 1 – 15

FUZZY BI-Γ-IDEALS IN Γ-SEMIGROUPS

D. R. Prince Williams*, K. B. Latha† and E. Chandrasekeran‡

Received 31 : 07 : 2008 : Accepted 15 : 12 : 2008

Abstract

In this paper, we consider the fuzzification of bi-Γ-ideals in Γ-semigroups, and investigate some of their related properties. Maximal fuzzy bi-Γ-ideals of Γ-semigroups are introduced and their properties discussed. Finally, chain conditions relating to fuzzy bi-Γ-ideals of Γ-semigroups are investigated.

Keywords: Γ-semigroups, Fuzzy Γ-ideals, Fuzzy bi-Γ-ideals, Fuzzy interior Γ-ideals.

2000 AMS Classification: 20 M 99, 20 N 25.

1. Introduction

Following the introduction of fuzzy sets by Zadeh [25], the fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

The study of fuzzy algebraic structures started with the introduction of the concepts of fuzzy subgroup (subgroupoid) and fuzzy (left, right) ideal in the pioneering paper of Rosenfeld [18]. In 1979, Anthony and Sherwood [1] redefined fuzzy subgroups (subgroupoids) using the concept of triangular norm. Das [6] defined level subgroups and used them to study fuzzy groups. Liu [13] introduced such concepts as fuzzy invariant subgroups, fuzzy ideals and in particular, gave a characteristic of a (usual) field by a fuzzy ideal. Mukerjee et.al [15] also worked on characterizing fuzzy subgroups of various groups. Kuroki [10, 11] introduced and studied fuzzy (left, right) ideals and fuzzy bi-ideals in semigroups. Later some basic concepts of fuzzy algebra such as fuzzy (left, right) ideals and fuzzy bi-ideals in a fuzzy semigroup, using a new approach of fuzzy spaces and fuzzy groups was introduced by Dib [7] in 1994.

In [16], Nobusawa introduced the notion of a Γ-ring as a generalization of a ring. Barnes [2] weakened slightly the conditions in the definition of a Γ-ring in the sense of

*Department of Information Technology, College of Applied Sciences, Ministry of Higher Education, Post Box:135, Sohar-311, Sultanate of Oman. E-mail: princeshree1@gmail.com

†Department of Mathematics, Mamallan Institute of Technology, Sriperumbudhur-602105, India. E-mail: kblathamath@yahoo.com

‡Department of Mathematics, Presidency College, Chennai-600 005, India.
E-mail: echandramath@yahoo.co.in

Nobusawa. Barnes [2], Luh [14], Kyono [12] and Ravisankar and Shukla [17] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts of ring theory. Jun *et.al* [8] applied the concept of fuzzy sets to the theory of Γ-rings.

The concept of Γ-nearring, a generalization of both the concepts nearing and Γ-ring, was introduced by Satyanarayana [22]. Later, several authors such as Booth [3] and Satyanarayana [21] studied the ideal theory of Γ-nearrings. Later Jun *et.al* [9] considered the fuzzification of left (resp. right) ideals of Γ-nearrings. Sen and Saha [19, 20, 23] gave a characterization of a Γ-semigroup, studied Γ-groups and Γ-regular semigroups and established a relation between Γ-groups and Γ-regular semigroups. The intuitionistic fuzzification of several types of Γ-ideal in a Γ-semigroup and a discussion of their properties was discussed by Mustafa *et.al* [24]. As we know, Γ-semigroups are a generalization of semigroups. Chinram [4] studied some properties of bi-ideals in semigroups. Recently, Chinram and Jirojkul [5] introduced the concept of a bi-Γ-ideal in a Γ-semigroup and this has motivated us to study the fuzzification of a bi-Γ-ideal in a Γ-semigroup.

In this paper, we consider a fuzzification of the concept of a bi-Γ-ideal in a Γ-semigroup that is different from than in [24], and some properties of such bi-Γ-ideals are investigated. The homomorphic property of fuzzy bi-Γ-ideals is established. The concept of a fuzzy interior Γ-ideal and a fuzzy Γ-ideal are also introduced and some properties are discussed. Later the notion of maximal fuzzy bi-Γ-ideals of Γ-semigroups is introduced, and some properties of such maximal fuzzy bi-Γ-ideals of Γ-semigroups discussed. Finally, chain conditions relating to fuzzy bi-Γ-ideals of Γ-semigroups are discussed.

2. Preliminaries

Let S be a semigroup. By a *subsemigroup of* S we mean anon-empty subset A of S such that $A^2 \subseteq A$.

For the sake of completeness, we now recall some concepts of fuzzy theory.

A mapping $\mu : S \to [0,1]$ is called *fuzzy set* of S and the *complement* of a set μ, denoted by μ' is the fuzzy set in S given by $\mu'(x) = 1 - \mu(x)$ for all $x \in S$.The *level set* of a fuzzy set μ of S is defined as $U(\mu;t) = \{x \in S \mid \mu(x) \geq t\}$.

2.1. Definition. A fuzzy set μ in S is called a *fuzzy subsemigroup* of S if

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$.

A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$.

2.2. Definition. A fuzzy subsemigroup μ of a semigroup S is called a *fuzzy bi-ideal* of S if

$$\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$$

for all x, y and $z \in S$.

2.3. Definition. Let $M = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then M is called a Γ-*semigroup* if it satisfies

(i) $x\gamma y \in M$,
(ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$,

for $x, y, z \in M$ and $\beta, \gamma \in \Gamma$.

2.4. Definition. Let M be a Γ-semigroup. A non-empty subset A of a Γ-semigroup M is said to be a Γ-*subsemigroup* of M if $A\Gamma A \subseteq A$.

2.5. Definition. A *left* (*right*) *ideal* of a Γ-semigroup M is a non-empty subset A of M such that $M\Gamma A \subseteq A (A\Gamma M \subseteq A)$.

If A is both a left and a right ideal of a Γ-semigroup M, then we say that A is a Γ-*ideal* of M.

2.6. Definition. A Γ-semigroup M is called *left-zero* (*right-zero*) if $x\gamma y = x (x\gamma y = y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

2.7. Definition. An element e in a Γ-semigroup M is called an *idempotent* if $e\gamma e = e$ for some $\gamma \in \Gamma$.

2.8. Definition. A Γ-subsemigroup A of a Γ-semigroup M is called an *interior* Γ-*ideal* of S if $M\Gamma A\Gamma M \subseteq A$.

2.9. Definition. Let M be a Γ-semigroup. A sub Γ-semigroup A of M is called a *bi*-Γ-*ideal* of M if $A\Gamma M\Gamma A \subseteq A$.

2.10. Definition. Let M be a Γ-semigroup and M_1 a Γ_1-semigroup. A pair of mappings $f_1 : M \to M_1$ and $f_2 : \Gamma \to \Gamma_1$ is said to be a *homomorphism* from (M, Γ) to (M_1, Γ_1) if $f_1(axb) = f_1(a)f_2(x)f_1(b)$ for all $a, b \in M$ and $x \in \Gamma$.

2.11. Definition. A fuzzy set μ of a Γ-semigroup is called a *fuzzy* Γ-*subsemigroup* of Mif

$$\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in M$ and $\gamma \in \Gamma$.

2.12. Definition. A Γ-semigroup M is said to satisfy the left (right) *ascending chain condition* of left (right) ideals if every strictly increasing sequence $U_1 \subset U_2 \subset U_3 \subset \cdots$ of left (right) ideals of M is of finite length.

2.13. Definition. A Γ-semigroup M is left (resp. right) *Notherian* if M satisfies the left (right) ascending chain condition of left (resp. right) ideals.

Below, let E_S denote the set of all idempotents in a Γ-semigroup S and χ_A the characteristic function of A.

3. Fuzzy bi-Γ-ideals

In what follows, M will denote a Γ-semigroup unless otherwise specified.

Now, we introduce a notion of fuzzy bi-Γ-ideal of M, which is different from [24].

3.1. Definition. A fuzzy set μ of M is called a *fuzzy bi*-Γ-*ideal* of M if

(i) $(\forall x, y \in M, \gamma \in \Gamma)(\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\})$,
(ii) $(\forall x, y, z \in M, \alpha, \beta \in \Gamma)(\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\})$.

3.2. Example. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\gamma, \alpha, \beta\}$ be the non-empty set of binary operations defined below:

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

Clearly M is a Γ-semigroup. Moreover the fuzzy set $\mu : M \to [0, 1]$ defined by $\mu(0) = 0.6, \mu(a) = 0.7, \mu(b) = 0.8, \mu(c) = 0.9$ is a fuzzy bi-Γ-ideal of M.

3.3. Lemma. *If B is a bi-Γ-ideal of M then for any $0 < t < 1$, there exists a fuzzy bi-Γ-ideal μ of M such that $\mu_t = B$.*

Proof. Let $\mu \to [0,1]$ be defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in B, \\ 0, & \text{if } x \notin B, \end{cases}$$

where t is a fixed number in $(0,1)$. Then, clearly $\mu_t = B$.

Now suppose that B is a bi-Γ-ideal of M. For all $x, y, \in B$ and $\gamma \in \Gamma$ such that $x\gamma y \in B$,, we have

$$\mu(x\gamma y) \geq t = min\{\mu(x), \mu(y)\}.$$

Also, for all $x, y, z \in B$ and $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta z \in B$, we have

$$\mu(x\alpha y\beta z) \geq t = min\{\mu(x), \mu(z)\}.$$

Thus μ is a fuzzy bi-Γ-ideal of M. □

3.4. Lemma. *Let B be a non-empty subset of M. Then B is a bi-Γ-ideal of M if and only if χ_B is a fuzzy bi-Γ-ideal of M.*

Proof. Let $x, y \in B$ and $\gamma \in \Gamma$. From the hypothesis, $x\gamma y \in B$.

(i) If $x, y \in B$, then $\chi_B(x) = 1$ and $\chi_B(y) = 1$. In this case

$$\chi_B(x\gamma y) = 1 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(ii) If $x \in B, y \notin B$, then $\chi_B(x) = 1$ and $\chi_B(y) = 0$. Thus,

$$\chi_B(x\gamma y) = 0 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(iii) If $x \notin B, y \in B$, then $\chi_B(x) = 0$ and $\chi_B(y) = 1$. Thus,

$$\chi_B(x\gamma y) = 0 \geq \min\{\chi_B(x), \chi_B(y)\}.$$

(iv) If $x \notin B$, $y \notin B$, then $\chi_B(x) = 0$ and $\chi_B(y) = 0$. Thus,

$$\chi_B(x\gamma y) \geq 0 = \min\{\chi_B(x), \chi_B(y)\}.$$

Thus (i) of Definition 3.1 holds good.

Let $x, y, z \in B$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x\alpha y\beta z \in B$.

(i) If $x, z \in B$, then $\chi_B(x) = 1$ and $\chi_B(z) = 1$. Thus,

$$\chi_B(x\alpha y\beta z) = 1 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(ii) If $x \in B, z \notin B$, then $\chi_B(x) = 1$ and $\chi_B(z) = 0$. Thus,

$$\chi_B(x\alpha y\beta z) = 0 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(iii) If $x \notin B, z \in B$,then $\chi_B(x) = 0$ and $\chi_B(z) = 1$. Thus,

$$\chi_B(x\alpha y\beta z) = 0 \geq \min\{\chi_B(x), \chi_B(z)\}.$$

(iv) If, $x \notin B$, $z \notin B$, then $\chi_B(x) = 0$ and $\chi_B(z) = 0$. Thus,

$$\chi_B(x\alpha y\beta z) \geq 0 = \min\{\chi_B(x), \chi_B(z)\}.$$

Thus (ii) of Definition 3.1 holds good.

Conversely, Suppose χ_B is a fuzzy bi-Γ-ideal of M. Then by Lemma 3.3, χ_B is two-valued, Hence B is a bi-Γ-ideal of M. This completes the proof. □

The following theorem proves that an intersection of a family of fuzzy bi-Γ-ideals is also a fuzzy bi-Γ-ideal.

3.5. Theorem. *If $\{A_i\}_{i\in\wedge}$ is a family of fuzzy bi-Γ-ideals of M then $\cap A_i$ is a fuzzy bi-Γ-ideals of M, where $\cap A_i = \{\Lambda\mu_i\}$ and $\Lambda\mu_i(x) = \inf\{\mu_i(x) \mid i \in \Lambda,\ x \in M\}$.*

Proof. Let $x, y \in M$. Then we have

$$\begin{aligned}
\text{(i)} \quad \Lambda\mu_i(x\gamma y) &= \inf\{\min\{\mu_i(x), \mu_i(y)\} \mid i \in \Lambda, x, y \in M\} \\
&= \min\{\{\inf(\mu_i(x)), \inf(\mu_i(y))\} \mid i \in \Lambda, x, y \in M\} \\
&= \min\{\{\inf(\mu_i(x) \mid i \in \Lambda, x \in M)\}, \{\inf(\mu(y) \mid i \in \Lambda, y \in M)\}\} \\
&= \min\{\Lambda\mu_i(x), \Lambda\mu_i(y)\}.
\end{aligned}$$

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\text{(ii)} \quad \Lambda\mu_i(x\alpha y\beta z) &= \inf\{\min\{\mu_i(x), \mu_i(z)\} \mid i \in \Lambda, x, z \in M\} \\
&= \min\{\{\inf(\mu_i(x)), \inf(\mu_i(z))\} \mid i \in \Lambda, x, z \in M\} \\
&= \min\{\{\inf(\mu_i(x) \mid i \in \Lambda, x \in M)\}, \{\inf(\mu(z) \mid i \in \Lambda, z \in M)\}\} \\
&= \min\{\Lambda\mu_i(x), \Lambda\mu_i(z)\}.
\end{aligned}$$

Hence, $\cap A_i$ is a fuzzy bi-Γ-ideal of M. □

The following theorem proves that the complement of a fuzzy bi-Γ-ideal is also a fuzzy bi-Γ-ideal.

3.6. Theorem. *If μ is a fuzzy bi-Γ-ideal of M then μ' is also a fuzzy bi-Γ-ideal of M.*

Proof. (i) Let $x, y \in M$ and $\gamma \in \Gamma$. We have:

$$\begin{aligned}
\mu'(x\gamma y) &= 1 - \mu(x\gamma y) \\
&= 1 - \min\{\mu(x), \mu(y)\} \\
&= \min\{1 - \mu(x), 1 - \mu(y)\} \\
&= \min\{\mu'(x), \mu'(y)\}.
\end{aligned}$$

(ii) Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We have:

$$\begin{aligned}
\mu'(x\alpha y\beta z) &= 1 - \mu(x\alpha y\beta z) \\
&= 1 - \min\{\mu(x), \mu(z)\} \\
&= \min\{1 - \mu(x), 1 - \mu(z)\} \\
&= \min\{\mu'(x), \mu'(z)\}.
\end{aligned}$$

Therefore, μ' is also a fuzzy bi-Γ-ideal of M. □

The following theorem gives the relation between fuzzy bi-Γ-ideal and bi-Γ-ideals.

3.7. Theorem. *A fuzzy set μ in a Γ-ideal of M is a fuzzy bi-Γ-ideal of M if and only if the level set $U(\mu; t) = \{x \in M \mid \mu(x) \geq t\}$ is a bi-Γ-ideal of M when it is non-empty.*

Proof. Let μ be a fuzzy bi-Γ-ideal of M. Then $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$.

$$\begin{aligned}
x, y \in U(\mu; t),\ \gamma \in \Gamma \implies\ & \mu(x) \geq t, \mu(y) \geq t \\
& \mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\} \geq t \\
& \mu(x\gamma y) \geq t \\
\implies\ & x\gamma y \in U(\mu; t).
\end{aligned}$$

Also,

$$\mu(x\alpha y\beta y) \geq \min\{\mu(x), \mu(z)\}.$$

$$
\begin{aligned}
x, y \in U(\mu;t),\ \alpha,\ \beta \in \Gamma \implies & \mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t \\
& \mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\} \geq t \\
& \mu(x\alpha y\beta z) \geq t \\
\implies & x\alpha y\beta z \in U(\mu;t).
\end{aligned}
$$

Thus, $U(\mu;t)$ is a bi-Γ-ideal of M.

Conversely, if $U(\mu;t)$ is a bi-Γ-ideal of M let $t = \min\{\mu(x), \mu(y)\}$. Then:

$$
\begin{aligned}
x, y \in U(\mu;t),\ \gamma \in \Gamma \implies & x\gamma y \in U(\mu;t) \\
\implies & \mu(x\gamma y) \geq t \\
\implies & \mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}.
\end{aligned}
$$

Next, define $t = \min\{\mu(x), \mu(z)\}$. Then:

$$
\begin{aligned}
x, y, z \in U(\mu;t),\ \alpha, \beta \in \Gamma \implies & x\alpha y\beta z \in U(\mu;t) \\
\implies & \mu(x\alpha y\beta z) \geq t \\
\implies & \mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}.
\end{aligned}
$$

Consequently, μ is a fuzzy bi-Γ-ideal of M. □

Now we consider completely regularity of fuzzy bi-Γ-ideals.

3.8. Theorem. *Let μ be a fuzzy bi-Γ-ideal of M. If M is completely regular, then $\mu(a) = \mu(a\alpha a)$ for all $a \in M$ and $\alpha \in \Gamma$.*

Proof. Straightforward. □

Let f be a mapping from a set X to Y, and μ be fuzzy set on Y. Then the *preimage* of μ under f, denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu(x)) = \mu(f(x))$, for all $x \in X$.

The following theorem gives the homomorphic property of fuzzy bi-Γ-ideals of a Γ-semigroup.

3.9. Theorem. *Let the pair of mappings $f : M \to M_1$, $h : \Gamma \to \Gamma_1$ be a homomorphism of Γ-semigroups. If μ is a fuzzy bi-Γ-ideal of M_1, then the preimage $f^{-1}(\mu)$ of μ under f is a fuzzy bi-Γ-ideal of M.*

Proof. (i) Let $x, y \in M$ and $\gamma \in \Gamma$. Then we have

$$
\begin{aligned}
f^{-1}(\mu)(x\gamma y) &= \mu(f(x\gamma y)) \\
&= \mu(f(x)h(\gamma)f(z)) \\
&= \min\{\mu(f(x)),\ \mu(f(y))\} \\
&= \min\{f^{-1}(\mu(x)),\ f^{-1}(\mu(y))\}.
\end{aligned}
$$

(ii) Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then:

$$
\begin{aligned}
f^{-1}(\mu)(x\alpha y\beta z) &= \mu(f(x\alpha y\beta z)) \\
&= \mu(f(x)h(\alpha)f(y)h(\beta)f(z)) \\
&\geq \min\{\mu(f(x)),\ \mu(f(z))\} \\
&= \min\{f^{-1}(\mu(x)),\ f^{-1}(\mu(z))\}.
\end{aligned}
$$

Therefore $f^{-1}(\mu)$ is a fuzzy bi-Γ-ideal of M. □

3.10. Definition. A fuzzy subset μ of M is called a *fuzzy* Γ*-left ideal* (resp. *fuzzy* Γ*-right ideal*) of M if

$$\mu(x\alpha y\beta z) \geq \mu(z) \text{ (resp. } \mu(x\alpha y\beta z) \geq \mu(x))$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

3.11. Definition. A fuzzy set μ of M is called a *fuzzy* Γ*-ideal* of M if it is both a fuzzy Γ-left ideal and a fuzzy right Γ-right ideal of M.

3.12. Example. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be the non-empty set of binary operations defined below:

α	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Clearly M is a Γ-semigroup. Moreover the fuzzy set $\mu : M \to [0,1]$ defined by $\mu(0) = 0.7, \mu(a) = 0.6, \mu(b) = \mu(c) = 0.3$ is a fuzzy Γ-ideal of M.

3.13. Proposition. *If B is a left-zero Γ-subsemigroup of M and μ a fuzzy Γ-left ideal of M, then $\mu(x) = \mu(z)$ for all $x, y, z \in M$.*

Proof. Let $x, y, z \in B$. Since B is left-zero, $x\alpha y\beta z = x$ and $z\beta y\alpha x = z$ for all $\alpha, \beta \in \Gamma$.

In this case, from the hypothesis, we have that

$$\mu(x) = \mu(x\alpha y\beta z) \geq \mu(z),$$
$$\mu(z) = \mu(z\beta y\alpha x) \geq \mu(x).$$

Thus we obtain $\mu(x) = \mu(z)$ for all $x, z \in M$. □

3.14. Lemma. *If B is a Γ-left ideal of M, then χ_B is a fuzzy Γ-left ideal of M.*

Proof. Let $x, z \in B$ and $\alpha, \beta \in \Gamma$. Since B is a Γ-left ideal of M, $x\alpha y\beta z \in M$

(i) If $z \in M$, then $\chi_B(z) = 1$. It follows that

$$\chi_B(x\alpha y\beta z) = 1 = \chi_B(z).$$

(ii) If $z \notin M$, then $\chi_B(z) = 0$, hence

$$\chi_B(x\alpha y\beta z) \geq 0 = \chi_B(z).$$

Consequently, χ_B is a fuzzy Γ-left ideal of M.

□

3.15. Theorem. *Let μ be a fuzzy Γ-left ideal of M. If E_B is a left-zero Γ-subsemigroup of M, then $\mu(e) = \mu(e_2)$ for all $e, e_2 \in E_B$.*

Proof. Let $e, e_2 \in E_B$. From the hypothesis, $e\alpha e_1\beta e_2 = e$ and $e_2\beta e_1\alpha e = e_2$ for all $e_1 \in E_B$ and $\alpha, \beta \in \Gamma$. Thus, since μ is a fuzzy Γ-left ideal of M, we get that

$$\mu(e) = \mu(e\alpha e_1\beta e_2) \geq \mu(e_2),$$
$$\mu(e_2) = \mu(e_2\beta e_1\alpha e) \geq \mu(e).$$

Hence we have $\mu(e) = \mu(e_2)$ for all $e, e_2 \in E_B$. This completes the proof. □

3.16. Definition. A fuzzy subset μ is called a *fuzzy interior* Γ*-ideal* of M if

$$\mu(x\alpha y\beta z) \geq \mu(y)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

3.17. Theorem. *If B is an interior Γ-ideal of M, then χ_B is a fuzzy interior Γ-ideal of M.*

Proof. Since B is a non-empty subset of M. Let $x, y, z \in B$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x\alpha y\beta z \in B$

(i) If $y \in B$, then $\chi_B(y) = 1$. Thus,

$$\chi_B(x\alpha y\beta z) = 1 = \chi_B(y)$$

ii If $y \notin B$, then $\chi_B(y) = 0$. Thus,

$$\chi_B(x\alpha y\beta z) \geq 0 = \chi_B(y).$$

Hence χ_B is a fuzzy interior Γ-ideal of M. □

3.18. Theorem. *Every fuzzy Γ-ideal of M is a fuzzy interior Γ-ideal of M.*

Proof. Straightforward. □

On the other hand, the following example shows that a fuzzy interior Γ-ideal μ in M need not be a fuzzy Γ-ideal of M.

3.19. Example. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be the non-empty set of binary operations defined below:

α	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Clearly M is a Γ-semigroup. Moreover the fuzzy set $\mu : M \to [0, 1]$ defined by $\mu(0) = 0.9, \mu(a) = 0.7, \mu(b) = 0.6, \mu(c) = 0.1$ is easily seen to be a fuzzy interior Γ-ideal of M. But μ is not a fuzzy Γ-ideal of M, since $\mu(b\alpha c\beta c) = \mu(c) = 0.1 \ngeq 0.6 = \mu(b)$.

3.20. Theorem. *If M is regular, then every fuzzy interior Γ-ideal of M is a fuzzy Γ-ideal of M.*

Proof. Let μ be a fuzzy interior Γ-ideal of M and $x, y, z \in M$. In this case, because of M is regular, there exists $y, y' \in M$ and $\alpha, \alpha', \beta, \beta' \in \Gamma$ such that $x = x\alpha y\beta x$ and $y = z\beta' y'\alpha' z$. Thus,

$$\begin{aligned} \mu(x\alpha' y\beta' z) &= \mu(x\alpha' z\beta' y'\alpha' z\beta' z) \\ &= \mu(x\alpha' z\beta'(y'\alpha' z\beta' z)) \\ &\geq \mu(z). \end{aligned}$$

Similarly, we can show that μ is a fuzzy Γ-right ideal of M. This completes the proof. □

Combining Theorem 3.18 and Theorem 3.20, if M is regular it is clear that the concepts of fuzzy Γ-ideals and fuzzy interior Γ-ideals coincide.

4. Normal fuzzy bi-Γ-ideals

4.1. Definition. A fuzzy bi-Γ-ideal μ of a Γ-semigroup M is said to be a *normal* if $\mu(0) = 1$.

4.2. Example. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\gamma, \alpha, \beta\}$ be the non-empty set of binary operations defined below:

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

Clearly M is a Γ-semigroup. Moreover the fuzzy set $\mu : M \to [0,1]$ defined by $\mu(0) = 1 = \mu(a) = \mu(b) = \mu(c)$ is easily seen to be a fuzzy bi-Γ-ideal of M with $\mu(0) = 1$. Thus μ is a normal fuzzy bi-Γ-ideal of M.

4.3. Theorem. *Given a fuzzy bi-Γ-ideal μ of a Γ-semigroup M, let $\mu^\star$ be the fuzzy set in M defined by $\mu^\star(x) = \mu(x) + 1 - \mu(0)$ for all $x \in M$. Then $\mu^\star$ is a normal fuzzy bi-Γ-ideal of M which contains μ.*

Proof. For all $x, y \in M$ we have $\mu^\star(0) = \mu(0) + 1 - \mu(0) = 1$. Also, for all $x, y \in M$ and $\gamma \in \Gamma$ we have

$$\begin{aligned}\mu^\star(x\gamma y) &= \mu(x\gamma y) + 1 - \mu(0)\\ &\geq \min\{\mu(x), \mu(y)\} + 1 - \mu(0)\\ &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\}\\ &= \min\{\mu^\star(x), \mu^\star(y)\}.\end{aligned}$$

Now let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We have,

$$\begin{aligned}\mu^\star(x\alpha y\beta z) &= \mu(x\alpha y\beta z) + 1 - \mu(0)\\ &\geq \min\{\mu(x), \mu(z)\} + 1 - \mu(0)\\ &= \min\{\mu(x) + 1 - \mu(0), \mu(z) + 1 - \mu(0)\}\\ &= \min\{\mu^\star(x), \mu^\star(z)\}.\end{aligned}$$

Therefore, $\mu^\star$ is a normal fuzzy bi-Γ-ideal of M, and obviously $\mu \subseteq \mu^\star$. □

4.4. Corollary. *Let μ and $\mu^\star$ be as in Theorem 4.3. If there exists $x \in M$ such that $\mu^\star(x) = 0$, then $\mu(x) = 0$.*

4.5. Theorem. *Let μ be a fuzzy bi-Γ-ideal of a Γ-semigroup M and let $f : [0, \mu(0)] \to [0,1]$ be an increasing function. Then the fuzzy set $\mu_f : M \to [0,1]$ defined by $\mu_f(x) = f[\mu(x)]$ is a fuzzy bi-Γ-ideal of M. In particular, if $f[\mu(0)] = 1$, then μ_f is normal; if $f(t) \geq t$ for all $t \in [0, \mu(0)]$, then $\mu \subseteq \mu_f$.*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. We have,

$$\begin{aligned}\mu_f(x\gamma y) &= f(\mu(x\gamma y))\\ &\geq f(\min\{\mu(x), \mu(y)\})\\ &= \min\{f(\mu(x)), f(\mu(y))\}\\ &= \min\{\mu_f(x), \mu_f(y)\}.\end{aligned}$$

If $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ then

$$\begin{aligned}\mu_f(x\alpha y\beta z) &= f[\mu(x\alpha y\beta z)]\\ &\geq f(\min\{\mu(x), \mu(z)\})\\ &= \min\{f(\mu(x)), f(\mu(z))\}\\ &= \min\{\mu_f(x), \mu_f(z)\}.\end{aligned}$$

Therefore, μ_f is a fuzzy bi-Γ-ideal of M. If $f[\mu(0)] = 1$, then $\mu_f(0) = 1$. Thus μ_f is normal.

Assume that $f(t) = f[\mu(x)] \geq \mu(x)$, for any $x \in M$, which gives $\mu \subseteq \mu_f$. This completes the proof. □

Let $\mathcal{N}(M)$ denote the set of all normal fuzzy bi-Γ-ideals of M. Note that $\mathcal{N}(M)$ is a poset under set inclusion.

4.6. Theorem. *Let $\mu \in \mathcal{N}(M)$ be a non-constant maximal element of $(\mathcal{N}(M), \subseteq)$. Then μ takes only the two values 0 and 1.*

Proof. Since μ is normal, we have $\mu(0) = 1$. Let $\mu(x) \neq 1$ for some $x \in M$. We claim that $\mu(x) = 0$. If not, then there exists $x_0 \in M$ such that $0 < \mu(x_0) < 1$.

Define a fuzzy set ν on M, by setting $\nu(x) = \left(\frac{\mu(x)+\mu(x_0)}{2}\right)$ for all $x \in M$. Then, clearly ν is wellûdefined and for all $x, y \in M$ and $\gamma \in \Gamma$ we have

$$\begin{aligned}\nu(x\gamma y) &= \left(\frac{\mu(x\gamma y) + \mu(x_0)}{2}\right)\\ &\geq \left(\frac{\min\{\mu(x), \mu(y)\} + \mu(x_0)}{2}\right)\\ &= \min\left\{\frac{\mu(x) + \mu(x_0)}{2}, \frac{\mu(y) + \mu(x_0)}{2}\right\}\\ &= \min\{\nu(x), \nu(y)\}.\end{aligned}$$

Moreover, for $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ we have:

$$\begin{aligned}\nu(x\alpha y\beta z) &= \left(\frac{\mu(x\alpha y\beta z) + \mu(x_0)}{2}\right)\\ &\geq \left(\frac{\min\{\mu(x), \mu(z)\} + \mu(x_0)}{2}\right)\\ &= \min\left\{\frac{\mu(x) + \mu(x_0)}{2}, \frac{\mu(z) + \mu(x_0)}{2}\right\}\\ &= \min\{\nu(x), \nu(z)\}.\end{aligned}$$

Therefore, ν is a fuzzy bi-Γ-ideal of M. By Theorem 4.3, $\nu^\star$ is a maximal fuzzy bi-Γ-ideal of M. Note that

$$\begin{aligned}\nu^\star(x_0) &= \nu(x_0) + 1 - \nu(0)\\ &= \frac{1}{2}(\mu(x_0) + \mu(x_0)) + 1 - \frac{1}{2}(\mu(0) + \mu(x_0))\\ &= \frac{1}{2}(\mu(x_0)) + 1 - \frac{1}{2}(\mu(0))\\ &= \frac{1}{2}(\mu(x_0) + 1)\\ &= \nu(x_0),\end{aligned}$$

and $\nu(x_0) < 1 = \nu^\star(0)$.

Hence $\nu^\star$ is non-constant, and μ is not a maximal element of $\mathcal{N}(M)$. This is a contradiction. □

4.7. Definition. A non-constant fuzzy bi-Γ-ideal μ of M is called *maximal* if $\mu^\star$ is a maximal element of $\mathcal{N}(M)$.

4.8. Theorem. *If a fuzzy bi-Γ-ideal μ of M is maximal, then*

(i) *μ is normal,*
(ii) *μ takes only the values 0 and 1,*
(iii) *$\chi_{U^0} = \mu$,*
(iv) *μ^0 is a maximal bi-Γ-ideal of M.*

Proof. Let μ be a maximal bi-Γ-ideal of M. Then $\mu^\star$ is a non-constant maximal element of the poset $(\mathcal{N}(M), \subseteq)$. It follows from Theorem 4.6 that $\mu^\star$ takes only the values 0 and 1. Note that $\mu^\star(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^\star(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$.

By Corollary 4.4, we have $\mu(x) = 0$ and so $\mu(0) = 1$. Hence μ is normal and $\mu^\star = \mu$.

This proves (i) and (ii).

(iii) Obvious.

(iv) It is clear that we can prove $\mu^0 = \{x \in M \mid \mu(0) = 1\}$ is a bi-Γ-ideal. Obviously $\mu^0 \neq S$ because μ takes just two values. Let A be a bi-Γ-ideal containing μ^0. Then $\mu_{U^0} \subseteq \mu_A$ and in consequence, $\mu = \mu_{U^0} \subseteq \mu_A$. Since μ is normal, μ_A also is normal and takes only two values 0 and 1. But, by the assumption, μ is maximal, so $\mu = \mu_A$ or $\mu = \omega$, where $\omega(x) = 1$ for all $x \in M$. In the last case $\mu^0 = \mu$, which is impossible. So $\mu^0 = \mu_A$. That is, $\mu_A = \chi_A$. Therefore $\mu^0 = A$. □

4.9. Definition. A normal fuzzy bi-Γ-ideal μ of a Γ-semigroup M is said to be *completely normal* if there exists $x \in M$ such that $\mu(x) = 0$.

4.10. Example. Let $M = \{0, a, b, c\}$ and $\Gamma = \{\gamma, \alpha, \beta\}$ the non-empty set of binary operations defined below:

α	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

β	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	0	b	0
c	0	0	0	c

γ	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

Clearly M is a Γ-semigroup. Moreover the fuzzy set $\mu : M \to [0,1]$ defined by $\mu(0) = 1, \mu(a) = \mu(b) = \mu(c) = 0$ is a normal fuzzy bi-Γ-ideal of M, and $\mu(x) = 0$ for all $x(\neq 0) \in M$. Thus μ is a completely normal fuzzy bi-Γ-ideal of M.

Denote by $\xi(M)$ the set of all completely normal fuzzy bi-Γ-ideals of M. We note that $\xi(M) \subseteq \mathcal{N}(M)$ and that the restriction of the partial ordering $\subseteq$ of $\mathcal{N}(M)$ gives a partial ordering of $\xi(M)$.

4.11. Proposition. *Any non-constant maximal element of $(\mathcal{N}(M), \subseteq)$ is also a maximal elements of $(\xi(M), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(M), \subseteq)$. By Theorem 4.8, μ takes only the values 0 and 1, and so $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in M$. Hence $\mu \in \xi(M)$. Assume that there exists $\nu \in \xi(M)$ and that $\mu \subseteq \nu$. It follows that $\mu \subseteq \nu$ in $\mathcal{N}(M)$. Since μ is maximal in $(\mathcal{N}(M), \subseteq)$ and ν is non-constant, therefore $\mu = \nu$. Thus μ is a maximal element of $(\xi(M), \subseteq)$, which ends the proof. □

4.12. Theorem. *Every maximal fuzzy bi-Γ-ideal of a Γ-semigroup $M$is completely normal.*

Proof. Let μ be a maximal fuzzy bi-Γ-ideal of M. Then by Theorem 4.8, μ is normal and $\mu = \mu^\star$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in M$. Hence μ is completely normal, which ends the proof. □

5. Chain conditions

We know that the intersection of all bi-Γ-ideal of a Γ-semigroup $M$is also a bi-Γ-ideal of M. Let Λ be a totally ordered set and let $\{A_j \mid j \in \Lambda\}$ be a collection of bi-Γ-ideal of a Γ-semigroup M such that for all $i, j \in \Lambda,\ j < i$ if and only if $A_i \subset A_j$. Then, $\bigcap\limits_{j<i} A_j$ is a bi-Γ-ideal of M.

5.1. Theorem. *Let $\{A_j \mid j \in \Lambda \subseteq [0,1]\}$ be a collection of bi-Γ-ideal of a Γ-semigroup M such that*

(i) $M = \bigcup_{j\in\Lambda} A_j$
(ii) $j < i$ *if and only if* $A_i \subset A_j$ *for all* $i, j \in \Lambda$.

Then the fuzzy set μ in M defined by $\mu(x) = \sup\{j \in \Lambda \mid x \in A_j\}$ for all $x \in M$ is a fuzzy bi-Γ-ideal of M.

Proof. For any $i \in [0,1]$, we consider the following two cases:

$$i = \sup\{j \in \Lambda \mid j < i\},\ i \neq \sup\{j \in \Lambda \mid j < i\}.$$

For the first case, we know that, for all $x, y \in A_j$ and $\gamma \in \Gamma$, $x\gamma y \in U(\mu; i)$ if and only if $x\gamma y \in A_j$ for all $j < i$, if and only if $x\gamma y \in \bigcap\limits_{j<i} A_j$. Also, for all $x, y, z \in A_j$ and $\alpha, \beta \in \Gamma$, $x\alpha y\beta z \in U(\mu; i)$ if and only if $x\alpha y\beta z \in A_j$ for all $j < i$, if and only if $x\alpha y\beta z \in \bigcap\limits_{j<i} A_j$.

Hence, $U(\mu; i) = \bigcap\limits_{j<i} A_j$, which is a bi-$\Gamma$-ideal of M. The second case implies that there exists $\epsilon > 0$ such that $(i - \epsilon, i) \cap \Lambda = \phi$.

We claim that $U(\mu; i) = \bigcap\limits_{j<i} A_j$. If $x\gamma y \in \bigcap\limits_{j<i} A_j$, $x\gamma y \in A_j$ for some $j < i,\ \gamma \in \Gamma$. It follows that $\mu(x\gamma y) \geq j \geq i$. Hence $x\gamma y \in U(\mu; i)$, showing that if $x\gamma y \in A_j$, $j \leq i - \epsilon$ and so $x\gamma y \notin U(\mu; i)$.

Also, if $x\alpha y\beta z \in \bigcap\limits_{j<i} A_j$ then $x\alpha y\beta z \in A_j$ for some $j < i, \alpha, \beta \in \Gamma$. It follows that $\mu(x\alpha y\beta z) \geq j \geq i$. Hence $x\alpha y\beta z \in U(\mu; i)$, and so $x\alpha y\beta z \notin U(\mu; i)$. Therefore $U(\mu; i) \subseteq \bigcap\limits_{j<i} A_j$ and so $U(\mu; i) = \bigcap\limits_{j<i} A_j$. Hence μ is a fuzzy bi-Γ-ideal of M. □

5.2. Theorem. *Let $\{A_n \mid n \in \Lambda\}$ be a family of bi-Γ-ideal of a Γ-semigroup M which is nested, that is $M = A_1 \supset A_2 \supset \cdots$.*

Let μ be the fuzzy in M defined by

$$\mu(x) = \begin{cases} \dfrac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 1, 2, 3, \ldots \\ 1 & \text{if } x \in \bigcap\limits_{n=1}^{\infty} A_n, \end{cases}$$

for all $x \in M$. Then μ is a fuzzy bi-Γ-ideal of M.

Proof. Let $x, y \in M$, $\gamma \in \Gamma$. Suppose that $x \in A_k \setminus A_{k+1}$ and $y \in A_r / A_{r+1}$ for $k = 1, 2, 3, \ldots,\ r = 1, 2, 3 \ldots$.

Without loss of generality, we assume that $k \leq r$.

Then $x\gamma y \in A_k$ and so $\mu(x\gamma y) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(y)\}$.

If $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x\gamma y \in \bigcap_{n=1}^{\infty} A_n$ and thus $\mu(x\gamma y) = 1 = \min\{\mu(x), \mu(y)\}$.

If $x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exists $i \in \Lambda$ such that $y \in A_i \setminus A_{i+1}$. It follows that $x\gamma y \in \bigcap_{n=1}^{\infty} A_n$, and thus $\mu(x\gamma y) \geq \frac{i}{i+1} = \min\{\mu(x), \mu(y)\}$.

Similarly, we know that $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$, whenever $x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \in \bigcap_{n=1}^{\infty} A_n$. Now, let $x, y, z \in M, \alpha,\ \beta \in \Gamma$. If $x \in A_k \setminus A_{k+1}$, $y \in A_r \setminus A_{r+1}$, and $z \in A_s \setminus A_{s+1}$, for $k = 1, 2, 3 \ldots;\ r = 1, 2, 3 \ldots;\ s = 1, 2, 3 \ldots$.

Without loss of generality, we may assume that $k \leq r \leq s$, whence $\mu(x\alpha y\beta z) \geq \frac{k}{k+1} = \min\{\mu(x), \mu(z)\}$.

If $x, y, z \in \bigcap_{n=1}^{\infty} A_n$ and $\alpha, \beta \in \Gamma$, then $x\alpha y\beta z \in \bigcap_{n=1}^{\infty} A_n$, and thus $\mu(x\alpha y\beta z) = 1 = \min\{\mu(x), \mu(z)\}$. If $x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$ and $z \notin \bigcap_{n=1}^{\infty} A_n$ then there exists $i \in \Lambda$ such that $y, z \in A_i / A_{i+1}$. It follows that $x\alpha y\beta z \in A_i$ so that $\mu(x\alpha y\beta z) \geq \frac{i}{i+1} = \min\{\mu(x), \mu(z)\}$.

Similarly, we know that for all these cases

(i) $x \in \bigcap_{n=1}^{\infty} A_n,\ y \in \bigcap_{n=1}^{\infty} A_n$ and $z \notin \bigcap_{n=1}^{\infty} A_n$,

(ii) $x \notin \bigcap_{n=1}^{\infty} A_n,\ y \notin \bigcap_{n=1}^{\infty} A_n$ and $z \in \bigcap_{n=1}^{\infty} A_n$,

(iii) $x \notin \bigcap_{n=1}^{\infty} A_n,\ y \in \bigcap_{n=1}^{\infty} A_n$ and $z \in \bigcap_{n=1}^{\infty} A_n$,

(iv) $x \notin \bigcap_{n=1}^{\infty} A_n,\ y \in \bigcap_{n=1}^{\infty} A_n$ and $z \notin \bigcap_{n=1}^{\infty} A_n$.

Thus $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$. Consequently, μ is a fuzzy bi-Γ-ideal of M. This complete the proof. □

Let $\mu : M \to [0, 1]$ be a fuzzy set. The smallest fuzzy bi-Γ-ideal containing μ is called the *fuzzy bi-Γ-ideal generated by* μ, and μ is said to be *n-valued* if $\mu(M)$ is a finite set of n elements. When no specific n is intended, we call μ a *finite-valued* fuzzy set.

5.3. Theorem. *A fuzzy bi-Γ-ideal λ of a Γ-semigroup M is finite-valued if and only if it is generated by a finite-valued fuzzy set μ in M.*

Proof. If $\lambda : M \to [0, 1]$ is a finite–valued fuzzy bi-Γ-ideal of M, then one may choose $\mu = \lambda$. Consequently, assume that $\mu : M \to [0, 1]$ is a n-valued fuzzy set with n distinct values $t_1, t_2, \cdots t_n$, where $t_1 > t_2 > \cdots > t_n$. Let G^i be the inverse image of t_i under μ, that is $G^i = \mu^{-1}(t_i)$. Obviously, $\bigcup_{i=1}^{j} G^i \subseteq \bigcup_{i=1}^{r} G^i$ when $j < r$. Let A^j be the bi-Γ-ideal of M generated by the set $\bigcup_{i=1}^{j} G^i$. Then we have the following chain of bi-Γ-ideals:

$A^1 \subseteq A^2 \subseteq \cdots \subseteq A^n = M$. Define a fuzzy set $\lambda : M \to [0,1]$ by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in A^1 \\ t_j & \text{if } x \in A^j \setminus A^{j-1}; j = 2, 3 \ldots n. \end{cases}$$

We claim that λ is the fuzzy bi-Γ-ideal of M generated by μ. Let $x, y \in M,\ \gamma \in \Gamma$ and let i and $j$be the smallest integers such that $x \in A^i$ and $y \in A^j$. We assume that $i > j$ without loss of generality. Then $x\gamma y \in A^i$ and so

$$\lambda(x\gamma y) \geq t_i = \min\{\lambda(x), \lambda(y)\}.$$

Similarly, if $x, y \in A^j \setminus A^{j-1}$ and $\gamma \in \Gamma$, we get

$$\lambda(x\gamma y) \geq t_j = \min\{\lambda(x), \lambda(y)\}.$$

Now let $x, y, z \in M,\ \alpha, \beta \in \Gamma$. If $x \in A^i$, $y \in A^j$ and $z \in A^k$ for some $i > j > k$ then $x, y, z \in A^i$ and so $x\alpha y\beta z \in A^i$ as A^i is a bi-Γ-ideal of M. Thus

$$\lambda(x\alpha y\beta z) \geq t_i = \min\{t_i, t_k\} = \min\{\lambda(x), \lambda(z)\}.$$

Similarly,if $x, y, z \in A^j \setminus A^{j-1}$ and $\gamma \in \Gamma$, we get

$$\lambda(x\alpha y\beta z) \geq t_j = \min\{t_j, t_j\} = \min\{\lambda(x), \lambda(z)\}.$$

Therefore λ is a fuzzy bi-Γ-ideal of M. If $x\gamma y \in M,\ \gamma \in \Gamma$ and $\mu(x\gamma y) = t_i$, then $x\gamma y \in G^i$ and so $x\gamma y \in A^i$. But we now get $\lambda(x\gamma y) \geq t_i = \mu(x\gamma y)$. Also, if $x\alpha y\beta z \in M,\ \alpha, \beta \in \Gamma$ and $\mu(x\alpha y\beta z) = t_i$, then $x\alpha y\beta z \in G^i$ and so $x\alpha y\beta z \in A^i$. But we then get $\lambda(x\alpha y\beta z) \geq t_i = \mu(x\alpha y\beta z)$.

Consequently, $\mu \subseteq \lambda$. Let δ be any fuzzy bi-Γ-ideal of M containing μ. Then,

$$\bigcup_{i=1}^{j} G^i = U(\mu; t_j) \subseteq U(\delta; t_j),$$

and thus $A^j \subseteq U(\delta; t_j)$. Hence $\lambda \subseteq \delta$ and so λ is generated by μ. Note that $|\mathrm{Im}\mu| = n = |\mathrm{Im}\lambda|$. This completes the proof. □

A Γ-semigroup M is said to Γ-*Notherian* if it satisfies the ascending chain condition on Γ-ideals of M.

5.4. Theorem. *If M is a Γ-Notherian, then every fuzzy bi-Γ-ideal of M is finite valued.*

Proof. Let $\mu : M \to [0,1]$ be a fuzzy bi-Γ-ideal of M which is not finite-valued. Then, there exists an infinite sequence of distinct numbers $\mu(0) = t_1 > t_2 > \cdots > t_n > \cdots$, where $t_i = \mu(x_i)$ for some $x_i \in M$. This sequence induces an infinite sequence of distinct bi-Γ-ideals of M:

$$U(\mu; t_1) \subset U(\mu; t_2) \subset \cdots \subset U(\mu; t_n) \subset \cdots.$$

This is a contradiction. □

Combining Theorems 5.3 & 5.4 we have the following corollary.

5.5. Corollary. *If M is a Γ-Notherian, then every fuzzy bi-Γ-ideal of M is generated by a fuzzy set in M.*

Acknowledgment The authors are very thankful to the learned referee for his invaluable suggestions for the improvement of this paper.

References

[1] Anthony, J. M. and Sherwood, H. *Fuzzy groups redefined*, J.Math. Anal. Appl. **69**, 124–130, 1979.

[2] Barnes, W. E. *On the* Γ*-rings of Nobusawa*, Pacific J. Math. **18**, 411–422, 1966.

[3] Booth, G. L. *A note on* Γ*-nearrings*, Stud. Sci. Hungarica **23**, 471–475, 1988.

[4] Chinram R. *Generalized transformation semigroups whose sets of quasi-ideals and bi–ideals coincide*, Kyungpook Math. J. **45**, 161–166, 2005.

[5] Chinram, R. and Jirojkul, C. *On bi-*Γ*-ideals in* Γ*-semigroups*, Songklanakarin J. Sci. Technol. **29** (1), 231–234, 2007.

[6] Das, P. S. *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84**, 264 -269, 1981.

[7] Dib, K. A. *On Fuzzy spaces and Fuzzy group theory*, Inform. Sci. **80**, 253–282, 1994.

[8] Jun, Y. B. and Lee, C. Y. *Fuzzy* Γ*-rings*, Pusan Kyongnam Math. J. (presently, East Asian Math. J.) **8** (2), 163–170, 1992.

[9] Jun, Y. B., Sapanci, M. and Ozturk, M. A. *Fuzzy ideals in gamma near-rings*, Tr. J. of Mathematics **22**, 449–459, 1998.

[10] Kuroki, N. *Fuzzy bi–ideals in semigroups*, Comment. Math. Univ. St. Paul. **28**, 17–21, 1980.

[11] Kuroki, N. *On Fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems **5**, 203–215, 1981.

[12] Kyuno, S. *On the radicals of* Γ*-rings*, Osaka J. Math. **12**, 639, 1975.

[13] Liu, W. *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems **8**, 133–139, 1982.

[14] Luh, J. *On the theory of simple* Γ*-rings*, Michigan Math. J. **16**, 65, 1969.

[15] Mukerjee, N. P. and Bhattacharya, P. *Fuzzy groups: some group theoretic analogs*, Inform. Sci. **39**, 247–268, 1986.

[16] Nobusawa, N. *On generalization of the ring theory*, Osaka J. Math. **1**, 81–89, 1964.

[17] Ravisankar, T. S. and Shukla, U. S. *Structure of* Γ*-rings*, Pacific J. Math,**80** (2), 537, 1979.

[18] Rosenfeld, A. *Fuzzy groups*, J. Math. Anal. Appl. **35**, 512–517, 1971.

[19] Saha, N. K. *On* Γ*-semigroup–II*, Bull. Calcutta Math. Soc. **79** (6), 331–335, 1987.

[20] Saha, N. K. *On* Γ*-semigroup–III*, Bull. Calcutta Math. Soc. **80** (1), 1–12, 1988.

[21] Satyanarayana, Bh. *Contributions to Near–ring Theory* (Doctoral Thesis, Nagarjuna University, 1984).

[22] Satyanarayana, Bh. *A note on* Γ*-Near-rings*, Indian J. Mathematics, bf41 (3), 427–433, 1999.

[23] Sen, M. K. and Saha, N. K., *On* Γ*-semigroup–I*, Bull. Calcutta Math. Soc. **78** (3), 180–186, 1986.

[24] Uckan, M., Mehmet, A. O. and Jun, Y. B. *Intuitionistic fuzzy sets in* Γ*-semigroups*, Bull. Korean Math. Soc. **44** (2), 359–367, 2007.

[25] Zadeh, L. A. *Fuzzy sets*, Information and Control, **8**, 338–353, 1965.