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GENERALIZED DERIVATIONS AS HOMOMORPHISMS OR AS ANTI-HOMOMORPHISMS IN A PRIME RING

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Abstract

Let R be a prime ring. Suppose that θ, ϕ are endomorphisms of R. An additive mapping $F : R \to R$ is called a generalized (θ, ϕ)-derivation if there exists a (θ, ϕ)-derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ holds for all $x, y \in R$. Let J be a nonzero Jordan ideal of R. In the present paper we begin by proving the following: If F is a generalized (θ, ϕ)-derivation on R which acts as a homomorphism or as an anti- homomorphism on J, then either $d = 0$ or $J \subseteq Z(R)$.

Keywords: Jordan ideals, Torsion free rings, Derivations, Generalized derivations, Generalized (θ, ϕ)-derivations, (θ, ϕ)-derivations.

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1. Introduction

Throughout R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies that $a = 0$). For any $x, y \in R$ we shall write $[x, y] = xy - yx$ and $x \circ y = xy + yx$. An additive subgroup J of R is said to be a Jordan ideal of R if $x \circ r \in J$ for all $x \in R$ and $r \in J$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x, y \in R$. Let θ, ϕ be endomorphisms of R. An additive mapping $d : R \to R$ is called a (θ, ϕ)-derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is called a left (θ, ϕ)-derivation if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$, holds for all $x, y \in R$. An example of a (θ, ϕ)-derivation on a ring R when R has a nontrivial central idempotent e is the mapping $d : R \to R$ such that $d(x) = ex$, $\theta = I_R$ (or d), and $\phi(x) = (1 - e)x$ (formally). Here d is a not a derivation on R, for $d(ee) = eee \neq 2eee = (ee)e + e(ee) = d(e)e + ed(e)$. In any ring

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R with endomorphism θ if we let $d = I_R - \theta$, then d is a (θ, I_R)- derivation, but not a derivation on R. An additive mapping $F : R \to R$ is called a generalized (θ, ϕ)-derivation on R if there exists a (θ, ϕ)-derivation $d : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ holds for all $x, y \in R$. Clearly concept of a generalized (θ, ϕ)-derivation includes the concepts of (θ, ϕ)- derivations ($F = d$), of derivations ($F = d$ and $\theta = \phi = I_R$) and of generalized derivations ($\theta = \phi = I_R$, [6]). Hence it would be interesting if one could extend the results concerning these notions to generalized (θ, ϕ)- derivations.

Bell and Kappe [4] proved that if d is a derivation of a prime ring R which acts as a homomorphism, or as an anti-homomorphism on a nonzero ideal I of R, then $d = 0$ on R. Recently Asma et al [1] obtained the result in the setting of Lie ideals of a prime ring.

Further, Yenigul and Argac [7] proved the above result for α-derivations in prime rings. Ashraf et al. [2] obtained the result for (σ, τ)-derivations in prime rings.

The purpose of this paper is to extend the mentioned results for generalized (θ, ϕ)-derivations on a Jordan ideal of a prime ring.

2. Main Results

2.1. Theorem. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal and a subring of R. Suppose θ is an automorphism of R and $F : R \to R$ is a generalized (θ, θ)-derivation with associated (θ, θ)-derivation d.*

(i) *If F acts as a homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.*
(ii) *If F acts as an anti-homomorphism on J, then either $d = 0$ on R or $J \subseteq Z(R)$.*

Proof. We begin with the following lemmas which are essential for developing the proof of our theorem. The proofs of Lemma 2.2 - 2.4 follow immediately from Herstein's Theorem on Jordan ideals of prime rings [5, Theorem 1.1], and that of lemma 2.5 from [3, Lemma 2].

2.2. Lemma. *Let R be a prime ring and J be a nonzero Jordan ideal of R. If $a \in R$ and $aJ = (0)$ or $Ja = (0)$, then $a = 0$.* □

2.3. Lemma. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If $aJb = (0)$, then either $a = 0$ or $b = 0$.* □

2.4. Lemma. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.* □

2.5. Lemma. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Suppose θ, ϕ are automorphisms of R. If R admits a (θ, ϕ)-derivation d such that $d(J) = (0)$, then either $d = 0$ or $J \subseteq Z(R)$.* □

Going to the proof of Theorem 2.1, suppose that $J \not\subseteq Z(R)$.

(i) If F acts as a homomorphism on J, then we have

(2.1) $$F(uv) = F(u)\theta(v) + \theta(u)d(v) = F(u)F(v), \text{for all } u, v \in J.$$

Replacing v by vw in (2.1), we get

$$F(u)\theta(v)\theta(w) + \theta(u)(d(v)\theta(w) + \theta(v)d(w)) = F(u)(F(v)\theta(w) + \theta(v)d(w)),$$

for all $u, v, w \in J$. Using (2.1), the above relation yields that $(F(u) - \theta(u))\theta(v)d(w) = 0$, for all $u, v, w \in J$. That is, $\theta^{-1}(F(u) - \theta(u))v\theta^{-1}(d(w)) = 0$, for all $u, v, w \in J$ and hence $\theta^{-1}(F(u) - \theta(u))J\theta^{-1}(d(w)) = (0)$, for all $u, w \in J$. Now Lemma 2.3 implies that either $F(u) - \theta(u) = 0$ or $d(w) = 0$. If $F(u) - \theta(u) = 0$, for all $u \in J$, then the relation (2.1) implies that $\theta(u)d(v) = 0$, for all $u, v \in J$. Now replace u by uw, to get $\theta(u)\theta(w)d(v) = 0$, for all $u, v, w \in J$. This implies that $uw\theta^{-1}(d(v)) = 0$ and hence $uJ\theta^{-1}(d(v)) = (0)$, for all $u, v \in J$. Again by Lemma 2.3, we have either $u = 0$ or $d(v) = 0$. Since J is a nonzero Jordan ideal, we find that $d(v) = 0$, for all $v \in J$. Hence Lemma 2.5 completes the proof.

(ii) If F acts as an anti-homomorphism on J, then we have

$$F(uv) = F(u)\theta(v) + \theta(u)d(v) = F(v)F(u), \text{ for all } u, v \in J. \tag{2.2}$$

Replacing u by uv in (2.2), we get

$$\theta(u)\theta(v)d(v) = F(v)\theta(u)d(v), \text{ for all } u, v \in J. \tag{2.3}$$

Substituting wu in place of u, we have $\theta(w)\theta(u)\theta(v)d(v) = F(v)\theta(w)\theta(u)d(v)$, for all $u, v \in J$. Multiplying (2.3) on the left by $\theta(w)$, we get $[F(v), \theta(w)]\theta(u)d(v) = 0$, for all $u, v, w \in J$. This implies that $\theta^{-1}([F(v), \theta(w)])u\theta^{-1}(d(v)) = 0$, for all $u, v, w \in J$. Thus, using Lemma 2.3, either $d(v) = 0$ or $[F(v), \theta(w)] = 0$ for all $v, w \in J$. If $[F(v), \theta(w)] = 0$ for all $w, v \in J$, then replacing v by vw in the above relation, we get $\theta(v)[d(w), \theta(w)] + [\theta(v), \theta(w)]d(w) = 0$, for all $v, w \in J$. Now replace v by uv to get $[\theta(u), \theta(w)]\theta(v)d(w) = 0$, for all $v, u, w \in J$. This gives that $[u, w]v\theta^{-1}(d(w)) = 0$, for all $v, u, w \in J$. Again by Lemma 2.3, for each $w \in J$, either $[u, w] = 0$ or $d(w) = 0$. Hence by using Braur's trick, we find that either $[u, w] = 0$, for all $u, w \in U$ or $d(w) = 0$, for all $w \in J$. If $[u, w] = 0$, for all $u, w \in J$, then by Lemma 2.4, J is central, a contradiction. On the other hand, if $d(w) = 0$, for all $w \in J$, then by Lemma 2.5 we get the required result. □

2.6. Theorem. *Let R be a semiprime ring and θ an automorphism on R. Suppose $F : R \to R$ is a generalized (θ, θ)-derivation with associated (θ, θ)-derivation d. If F acts as a homomorphism on R, then $d = 0$.*

Proof. If F acts as a homomorphism on R, then we have $F(xy) = F(x)F(y)$. This implies that

$$F(x)\theta(y) + \theta(x)d(y) = F(x)F(y), \text{ for all } x, y \in R. \tag{2.4}$$

Replacing y by yz, we get

$$F(x)\theta(y)\theta(z) + \theta(x)d(y)\theta(z) + \theta(x)\theta(y)d(z) = F(x)F(y)\theta(z) + F(x)\theta(y)d(z), \quad \text{for all } x, y \in R. \tag{2.5}$$

Multiplying (2.4) on the right by $\theta(z)$, we obtain

$$F(x)\theta(y)\theta(z) + \theta(x)d(y)\theta(z) = F(x)F(y)\theta(z), \text{ for all } x, y \in R. \tag{2.6}$$

Now Comparing (2.5) and (2.6), we have

$$\theta(x)\theta(y)d(z) = F(x)\theta(y)d(z), \text{ for all } x, y, z \in R. \tag{2.7}$$

Substituting xz for x in (2.7), we obtain

$$\theta(x)\theta(z)\theta(y)d(z) = F(x)\theta(z)\theta(y)d(z) + \theta(x)d(z)\theta(y)d(z), \text{ for all } x, y, z \in R. \tag{2.8}$$

Replacing y by zy in (2.7), we have

$$\theta(x)\theta(z)\theta(y)d(z) = F(x)\theta(z)\theta(y)d(z), \text{ for all } x, y \in R. \tag{2.9}$$

Comparing (2.8) and (2.9), we find that $\theta(x)d(z)\theta(y)d(z) = 0$, for all $x, y, z \in R$. Substituting yx for y we obtain $\theta(x)d(z)\theta(y)\theta(x)d(z) = 0$, for all $x, y, z \in R$, that is $\theta(x)d(z)R\theta(x)d(z) = (0)$, for all $x, z \in R$. The fact that R is semiprime yields that $\theta(x)d(z) = 0$, for all $x, z \in R$. Thus, we have $d(z)\theta(x)d(z) = 0$, for all $x, z \in R$, that is $d(z)Rd(z) = (0)$, $x, z \in R$. Again, since R is semiprime we obtain the required result. □

2.7. Theorem. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal and a subring of R. Suppose that θ, ϕ are automorphisms of R, and that $d : R \to R$ is a left (θ, ϕ)-derivation of R.*

(*i*) *If d acts as a homomorphism on J, then $d = 0$ on R.*
(*ii*) *If d acts as an anti-homomorphism on J, then $d = 0$ on R.*

Proof. (*i*) If d acts as a homomorphism, then we have

$$d(uv) = d(u)d(v) = \theta(u)d(v) + \phi(v)d(u), \text{ for all } u, v \in J. \tag{2.10}$$

Substituting vw for v in (2.10), we find that $d(u)d(v)d(w) = \theta(u)d(v)d(w)+\phi(v)\phi(w)d(u)$, for all $u, v, w \in J$. Multiplying (2.10) on the right by $d(w)$, we obtain $d(u)d(v)d(w) = \theta(u)d(v)d(w)+\phi(v)d(u)d(w)$ for all $u, v, w \in J$. Hence we have $\phi(v)\{d(u)d(w)-\phi(w)d(u)\} = 0$, for all $u, v, w \in J$. Now using (2.10) we find that $\phi(v)\theta(u)d(w) = 0$, for all $u, v, w \in J$, that is, $v\phi^{-1}(\theta(u)d(w)) = 0$, for all $u, v, w \in J$. An application of Lemma 2.2 yields that $\phi^{-1}(\theta(u)d(w)) = 0$ i.e., $\theta(u)d(w) = 0$, for all $u, w \in J$. Thus, $u\theta^{-1}(d(w)) = 0$, for all $u, w \in J$. Again Lemma 2.2 yields that

$$d(w) = 0, \text{ for all } w \in J. \tag{2.11}$$

Replacing w by $wr + rw$ in (2.11), we obtain

$$\theta(w)d(r) + \phi(w)d(r) = 0, \text{ for all } w \in J,\ r \in R. \tag{2.12}$$

Replace w by uw in (2.12), to get $\theta(u)\theta(w)d(r)+\phi(u)\phi(w)d(r) = 0$ for all $u, w \in J,\ r \in R$. Multiplying (2.12) on the left by $\theta(u)$, we obtain $\theta(u)\theta(w)d(r) + \theta(u)\phi(w)d(r) = 0$ for all $u, w \in J,\ r \in R$. Hence we have $\{\theta(u) - \phi(u)\}\phi(w)d(r) = 0$, for all $u, w \in J,\ r \in R$, that is $\phi^{-1}\{\theta(u) - \phi(u)\}J\phi^{-1}d(r) = 0$, for all $u, w \in J, r \in R$. Now an application of Lemma 2.3 yields that either $\theta(u) - \phi(u) = 0$ or $d(r) = 0$, for all $u \in J$ and $r \in R$. If $\theta(u) = \phi(u)$, for all $u \in J$, then the relation (2.12) implies that $2\theta(u)d(r) = 0$, for all $u \in J$ and $r \in R$. Since R is 2-torsion free, $\theta(u)d(r) = 0$, i.e., $u\theta^{-1}(d(r)) = 0$, for all $u \in J$ and $r \in R$. Lemma 2.2 yields that $\theta^{-1}(d(r)) = 0$ i.e., $d(r) = 0$, for all $r \in R$. Hence, in both the cases $d = 0$.

(*ii*) If d acts as an anti-homomorphism on J, then

$$d(uv) = d(v)d(u) = \theta(u)d(v) + \phi(v)d(u), \text{ for all } u, v \in J. \tag{2.13}$$

Replacing u by u^2 in (2.13), we have $d(v)d(u)d(u) = \theta(u)\theta(u)d(v) + \phi(v)d(u)d(u)$, for all $u, v \in J$. Multiplying (2.13) by $d(u)$ on the right, we get $d(v)d(u)d(u) = \theta(u)d(v)d(u) + \phi(v)d(u)d(u)$, for all $u, v \in J$. Hence we obtain $\theta(u)\{d(v)d(u) - \theta(u)d(v)\} = 0$, for all $u, v \in J$. Using (2.13), we obtain $\theta(u)\phi(v)d(u) = 0$, that is, $\phi^{-1}(\theta(u))J\phi^{-1}(d(u)) = (0)$, for all $u \in J$. An application of Lemma 2.3 yields that either $\theta(u) = 0$ or $d(u) = 0$, that is $u = 0$ or $d(u) = 0$, for all $u \in J$. But $u = 0$ yields that $d(u) = 0$, for all $u \in J$. Using similar arguments to those used to get $d = 0$ from (2.7), we get the required result. □

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